ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1999

SOME ASPECTS OF DENTABILITY IN BITOPOLOGICAL AND LOCALLY CONVEX SPACES

MISHEL FUNDO

ABSTRACT. This paper is a continuation of the study we presented in [**6**]. A modified version of Namioka's argument is reconsidered to obtain an extended form of a result of Namika-Asplund; this leads to the improvement of several theorems and to a generalized version of the Dunford Pettis theorem [**2**]. Moreover, two versions of Rieffel's converse theorem are discussed. It is shown that the first one holds true in locally convex spaces, but not generally in the spaces with two topologies; the second leads to a new characterization of the Radon-Nikodym property in real Banach spaces and in their duals.

1. Introduction. The main results are contained in Sections 2 and 3. Section 2 deals with sufficient conditions of τ-dentability in bitopological spaces [**13**]. Corollaries in locally convex spaces are obtained. Analogous questions have been treated in a different context by Rieffel [**15**], Maynard [**12**], Lindenstrauss [**11**], Troyanski [**17**], Bourgain [**1**], Namioka [**13**], etc.

Section 3 deals with the problem of equivalence between the dentability of a bounded set and the dentabilities of its closed convex hull and its closed equilibrated convex hull; the aim is to find the possibly larger extension of such an equivalence. This is an important tool in the study of the geometric aspects of the Radon-Nikodym property.

In this paper the same notations and definition as in [**6**] are used. Thus we recall some of them by paying special attention to the notion of bitopological spaces introduced by Namioka [**13**].

We consider a real vector space E with two topologies r_0 and r such that $r < r_0$, and we suppose that the pairs $(E, r_0) = E$ and $(E, r) = E_r$ are Hausdorff locally convex spaces (hlcs); their topological duals are denoted by E' and E'_r, respectively, and the system of neighborhoods of the origin in E by $\gamma(0)$. The weak-compact sets in E_r will be called

Received by the editors on February 5, 1996, and in revised form on April 8, 1997.

r-weak-compacts, while by $(\overline{C}^{rw}(A))\overline{C}^{r}(A)$ will be denoted the (weak) closed convex hull of a set A in the space E_r. By $\overline{A}^{rw}$ is denoted the weak-closure of the set A in the space E_r. A bounded set in the space $E_r(E)$ will be called an r-bounded (bounded) set and analogously if we replace here *bounded* by *closed*. All the locally convex spaces (lcs) in the paper over the real vector space E.

Definition 1. An r-bounded set $B \subset E$ is called *r-dentable* in the space E if

$$(\forall V \in \gamma(0))(\exists x \in B)(x \notin \overline{C}^{r}(B\backslash x + V).$$

If, instead of r, we have r_0, it will be called a dentable set. It is clear that each bounded r-dentable set is dentable.

Definition 2. A set $A \subset E$ has the *Radon-Nikodym property* (RNP) if each bounded subset of it is dentable.

The topological dual E' of an hlcs is called *w^*-dentable* if each equicontinuous subset of it is w^*-dentable.

Definition 3. The triple (E, r_0, r) is a *bitopological space* if there is a local base of the origin for the space E consisting of r-closed sets.

For this definition it is not necessary that $r < r_0$, see [**13**]. Here we consider only the case $r < r_0$.

Two classical examples of bitopological spaces are the triples $(E, T, \sigma(E, E'))$ and $(E', \beta(E', E), W^*)$, in which (E, T) is an hlcs with $\sigma(E, E')$ for its weak topology, while $\beta(E', E)$ and W^* are the strong and the weak-star topologies in the topological dual, respectively.

2. The main theorem and other results. Theorem 1 below is the main result. It is the extension to the class of r-weak compact convex sets of a Namioka-Asplund result, see [**7**] or Corollary 2. Its proof follows the method of the proofs of Namioka's Proposition 2.2 in [**13**], or of the lemma in [**14**], and of the similar propositions of Maynard [**12**] and of Bourgain [**1**] in Banach-spaces.

Theorem 1. *If the triple (E, r_0, r) is a bitopological space, with*

$r < r_0$, then each r-weak compact convex set[1]*, bounded and separable in the space E, is r-dentable.*

Proof. Let $V_0 \in \gamma(0)$ be an arbitrary neighborhood of the origin in the space E and K an r-weak compact convex set, bounded and separable in E. We denote by $\{x_n\}_{n\in N}$ a countable set, in E or in K, which is dense in K, for the topology $r_0 : K \subset \overline{\{x_n\}}$. By the bitopological property of the triple (E, r_0, r), there is an r-closed convex equilibrated neighborhood V of the origin in E for which $V+V+V+V = 4V \subset V_0$. Let A be the set of extreme points of K; then $\overline{A}^{r_w}$ is an r-weak compact set which is, for this reason, a Baire-space for the weak topology of E_r induced on $\overline{A}^{r_w}$. On the other side the inclusion $K \subset \overline{\{x_n\}}$ implies the equality:

$$\overline{A}^{r_w} = \bigcup_{n=1}^{\infty} [\overline{A}^{r_w} \cap (x_n + V)].$$

The set V is a weakly closed set in E_r since it is an r-closed convex one. Thus all the sets $\overline{A}^{r_w} \cap (x_n + V)$ are weakly closed sets in E_r, hence in $\overline{A}^{r_w}$. Consequently, for some $m \in \mathbf{N}$, the corresponding set $B_m = \overline{A}^{r_w} \cap (x_m + V)$ has nonempty interior; thus, we can suppose the existence of a weakly open set N in the space E_r with its complement N^c a finite union of some r-weakly closed convex sets, such that $N \cap \overline{A}^{r_w} \subset B_m$. In this case the set $K \backslash N = K \cap N^c$ is a finite union of r-weakly compact convex sets and its convex hull $K_1 = C(K\backslash N)$ equals $\overline{C}^{r_w}(K\backslash N) = \overline{C}(K\backslash N)$ is an r-weakly compact convex. By the fact that $N \cap \overline{A}^{r_w} \neq \varnothing$ and the fact that N is a weakly open set in the space E_r, there exists the point $x \in E$ such that $x \in N \cap A$. From the inclusion $N \cap A \subset x_m + V$, it follows that the r-weakly compact convex set $K_2 = \overline{C}^{r_w}(N \cap A) = \overline{C}^{r}(N \cap A)$ is $2V$-small. It is clear that $K_1, K_2 \subset K$, $A \subset K_1 \cup K_2$ and $x \in K_2$, but $x \notin K_1$ because the point x is an extreme point of the set K. By the Krein-Milman theorem, we have

$$\begin{aligned} K &= \overline{C}^{r_w}(K_1 \cup K_2) = C(K_1 \cup K_2) \\ &= \{\lambda x_1 + (1-\lambda)x_2 : x_i \in K_i, 0 \le \lambda \le 1\}, \end{aligned}$$

where $C(K_1 \cup K_2)$ is the convex hull of $K_1 \cup K_2$. As the set K is bounded in the space E, there exists an $\alpha > 1$ such that $K/\alpha \subset V/2$. It is very easy to verify that the set:

$$C = \{\lambda x_1 + (1-\lambda)x_2 : x_i \in K_i, 1/\alpha \le \lambda \le 1\}$$

is an r-weakly compact convex set; furthermore, $x \notin C$ because $x \notin K_1$ and the point x is an extreme point for the set K.

So $x \in N \backslash C = N_1$ where N_1 is weakly open in the space E_r. By the choice of the set C and of the number α and by the fact that the set K_2 is $2V$-small, it is easy to verify that the set K/C is $4V$-small, or yet V_0-small, and so is the set $N_1 \cap K$ because of the inclusion $N_1 \cap K \subset K \backslash C$.

As above with the set N, we can choose the neighborhood N_1 of x smaller and such that its complement N_1^c is a finite union of r-weak closed convex sets. Consequently,

$$C(K \cap N_1^c) = C(K \backslash N_1) = \overline{C}^{rw}(K \backslash N_1) = \overline{C}^{r}(K \backslash N_1).$$

But $N_1 \cap K \subset x + V_0$ and $x \notin C(K \backslash N_1)$ because $x \in N_1$ and x is an extreme point of K. Thus

$$x \notin \overline{C}^{rw}(K \backslash N_1) \supset \overline{C}^{r}(K \backslash (x + V_0)),$$

and accordingly the set K is r-dentable, Definition 1. □

Corollary 1. *In the bitopological space (E, r_0, r) each bounded subset of an r-weakly compact, convex and separable for the topology r_0 set A, is r-dentable; furthermore, the set A has the* (RNP).

Proof. Let B be an arbitrary bounded subset of A. By Proposition 1 in [**6**], to show that B is r-dentable, we show that its r-closed convex hull $\overline{C}^{r}(B)$ has the same property. Since $\overline{C}^{r}(B) = \overline{C}^{rw}(B) \subset A$, the set $\overline{C}^{r}(B)$ is an r-weakly compact convex, bounded (by the bitopological property) and separable in the space E set. Therefore $\overline{C}^{r}(B)$ is r-dentable by Theorem 1. As an r-dentable set, B is dentable, too. □

Remark 1. As an r-compact set is an r-weak compact one, from Corollary 1 it follows that an r-compact convex and separable for the topology r_0 set in the bitopological space (E, r_0, r) has the (RNP). For the same reason Theorem 1 holds if we replace the r-weak compact convex set by an r-compact convex one.

Corollary 2 (Asplund-Namioka [**7**]). *Each separable weak-compact convex set in an* hlcs *is a dentable set and has the* (RNP).

Proof. Corollary 1 is applied for $r = r_0$ where r_0 is the topology of the given hlcs E; in this case, the r-weak topology is the weak topology $\sigma(E, E')$ so each weak compact is bounded. □

Corollary 3. *Each w^*-compact convex and separable* (*for the strong topology*) *set in the dual E' of an* hlcs *has the* (RNP). *If E' is the dual of a real Banach space, then Corollary* 4 *in* [**3**] *follows, which states*: *A w^*-compact convex set in E', for which the set of its extreme points is norm separable, has the* (RNP).

Proof. For the first part of the theorem, Remark 1 is applied for $r = w^*$ and $r_0 = \beta(E', E)$. The second part is an immediate corollary of the first part and of Haydon's theorem [**9**]. □

The next corollary is an improvement of Theorem 3.5 of [**7**]. Here we do not need the dual E' to be a Frechet-space; also its proof is easier and it is neither based on Theorem 2.3 in [**13**] nor on Choquet's theorem in [**10**].

We remember that an lcs is called a quasiseparable space if each bounded subset of it is separable.

Corollary 4. *If the topological dual E' of an* hlcs E *is quasiseparable* (*for the strong topology*), *then it is w^*-dentable.*

Proof. Theorem 1 is applied and the second part of Remark 1 to the bitopological space $(E', \beta(E', E), w^*)$ where $r_0 = \beta(E', E)$ and $r = w^*$. The w^*-closed convex hull of an equicontinuous set in E' is a w^*-compact convex equicontinuous one. Consequently, it is bounded and separable for the strong topology in E' and hence w^*-dentable. □

Remark 2. The classes of w^*-weak compact and of w^*-compact sets are the same, because $(E', w^*)^* = E$.

Corollary 5 below is an improved form of Corollary 8.3 in [**7**] because here it is proved without needing the dual E' to be a Frechet space.

This form is also a good generalization of the Dunford-Pettis theorem [**2, 4**] and in another direction in comparison to the interesting one given by Bourgin in [**3**].

We remember that an hlcs is called a quasibarreled space if each closed convex equilibrated subset of it, which absorbs all the bounded sets, is a neighborhood of the origin.

An hlcs is a quasibarreled space if and only if each (strong) bounded set in its topological dual is an equicontinuous set.

Corollary 5. *Let* (E,T) *be an* hlcs *with its topological dual* E' *a quasiseparable space* (*for the strong topology*)*. If each bounded set in the* hlcs $(E',\beta(E',E))$ *is a relatively* w^**-compact one* (*or, especially, if the* hlcs (E,T) *is a quasibarreled space*)*, then the space* $(E',\beta(E',E))$ *has the* (RNP)*.*

Proof. Let A' be an arbitrary bounded set in the space $(E',\beta(E',E))$ and $K=\overline{C}^{W^*}(A')$ its w^*-closed convex hull. By the bitopological property of the triple $(E',\beta(E',E),w^*)$, the set K is bounded. Consequently, it is separable in $(E,\beta(E',E))$ and w^*-compact by the conditions of the theorem. It follows from Theorem 1 and Remark 1 that the set K is w^*-dentable; thus, the set A' is w^*-dentable, therefore dentable. $\square$

Corollary 6 (The Dunford-Pettis theorem). *The separable dual of a real Banach space has the* (RNP)*.*

Proof. Each Banach-space is quasibarreled (barreled) and each separable space is a quasiseparable one. Corollary 5 is applied. $\square$

2.2. Based on the above and on that of [**6**], we will give by Theorems 2 and 3 below the direct and easier proofs of Theorems 3.6 and 2.9 in [**7**], respectively; furthermore, Theorem 2 is an improved form of Theorem 3.6 in [**7**], because we replace "the dual Frechet space" by a "quasimetrizable dual space." The basic tool in the proofs of Theorems 2 and 3 is the known proposition which states: "The weak-compact convex sets in a quasimetrizable l.c.s. have the (R.N.P.)." This proposition is an easy corollary of Corollary 2 and of the generalized form of Maynard's lemma as it is stated in [**7, 8**].

Theorem 2. *If the topological dual E' of a semi-reflexive* hlcs E *is quasimetrizable (for the strong topology), then it is w^*-dentable.*

Proof. By the semi-reflexivity of E we have that $E''=(E',\beta(E',E))'=E$, thus the weak-star and the weak topologies in E' are the same. For each equicontinuous set A' in E', its w^*-closed convex hull $\overline{C}^{w}(A')$ is a w^*-compact convex set, and hence it is dentable as a weak-compact convex set in the quasimetrizable lcs $(E',\beta(E',E))$. Since in this case $\overline{C}^{w^*}(M')=\overline{C}^{w}(M')=\overline{C}(M')$ for each set $M'\subset E'$, the set $\overline{C}^{w^*}(A')$ is w^*-dentable (Definition 1) and therefore the set A' is w^*-dentable too (Rieffel's Proposition 1 in [**6**]). □

Theorem 3. *Each weak-compact convex set in a quasimetrizable* lcs *is the closed convex hull of its denting points.*

Proof. Let B be a weakly compact convex set in the quasimetrizable lcs E, and let us suppose that $0\in B$; otherwise, we take $B_1=B-a$ where $a\in B$. Considering the hlcs $(E,\sigma(E,E'))$ and the continuity of the maps $x\to -x$ and $(x,y)\to x+y$, we conclude that the equilibrated convex set $A=B-B$ is a weak compact set. This set contains the set $B(0\in B)$. Therefore, the set A is subset-dentable and consequently subset $\sigma(E,E')$-dentable, Definition 1. As the weak compact set B is complete in the uniform structure of the space $(E,\sigma(E,E'))$, by Theorem 1 in [**6**] and by Proposition 2 in [**6**], which in this case states that the $E'_{\sigma(E,E')}$-strongly exposed points are $\sigma(E,E')$-denting points and hence denting ones, we conclude that set B is the weak closed, equivalently closed, convex hull of its denting points. □

Remark 3. The method of the proof for Theorem 3 cannot be applied for the more general Theorem 2.2 in [**8**], in which is replaced the weakly compact convex set B by a complete convex set C with the (RNP) because, in the last case, the difference set $C-C$ may not be a Radon-Nikodym set, see [**16**].

3. The converse of Rieffel's theorem[2]**.** In [**6**] the next (extended) version of Rieffel's theorem, [**2**, Corollary 2.3.3], is given.

Proposition 1. *If the r-closed convex hull* $\overline{C}^r(A)$ *of the r-bounded set A is r-dentable, then so is A.*

It is not difficult to also see the validity of the similar proposition for the "equilibrated" case as follows:

Proposition $1'$**.** *If the r-closed convex equilibrated hull, denoted by* $\overline{C}^r$-equil (A)*, of the r-bounded set A is r-dentable, then A has the same property.*

In the real Banach spaces, the converse of Rieffel's theorem is true, see the note in Corollary 2.3.3 in [**2**], but generally this is not true in spaces with two topologies. For this reason in point i) the possibilities of the validity of the converse of Proposition 1 are studied, and the same for Proposition $1'$ in ii).

i) It will be shown now by Counterexample 1 that the converse of Proposition 1 does not hold true in spaces with two topologies. By Theorem 4 and its corresponding Lemma 1, the validity of such a converse in lcs is proved.

Counterexample 1. An r-bounded and r-dentable set with its r-closed convex hull which is not r-dentable.

Let $A = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \le 1\}$ be the unit ball in the Hilbert Euclidean space $\mathbf{R}^2$. We denote by r_0 its norm-topology and by r the locally convex topology defined by the semi-norm $p : \mathbf{R}^2 \to \mathbf{R}^+$, $p((x,y)) = |f(x,y)|$, where $f(x,y) = x - y$. Then $r < r_0$ and A has the (RNP) as a subset of a reflexive Banach-space. In the space E_r the sets $V_\varepsilon = \{(x,y) \in \mathbf{R}^2 : -\varepsilon < x - y < \varepsilon\}$ are a fundamental system of neighborhoods of the origin, so that the set A is r-bounded. We will prove that the set A is r-dentable. For each neighborhood V of the origin in $(\mathbf{R}^2, r_0)$ there exists $0 < \varepsilon < 1/2$ such that the ball $B(0,\varepsilon) \subset A$. Let M be the point $(-1/\sqrt{2}, 1/\sqrt{2})$ of the boundary of A; then $A \backslash (M + V) \subset A \backslash B(M,\varepsilon)$, where $B(M,\varepsilon) = M + B(0,\varepsilon)$.

The convex hull $C(A\backslash B(M,\varepsilon))$ is contained in the r-closed set $D = \{(x,y) \in \mathbf{R}^2 : \alpha \le x - y \le \sqrt{2}\}$, where $x - y = \alpha$ is the line defined by the points of the intersection of the boundaries of the sets A and $B(M,E)$. Consequently, $\overline{C}^r(A\backslash B(M,E)) \subset D$ and since $M \notin D$, then $M \notin \overline{C}^r(A\backslash B(M,E))$. Thus,

$$(\forall V \in \gamma(0))(\exists M \in A)(M \notin \overline{C}^r(A\backslash M + V))$$

which means the set A is r-dentable.

Let us show now that the r-closed convex hull $\overline{C}^r(A)$ is not r-dentable. The r-closed set $F = \{(x,y) \in \mathbf{R}^2 : -\sqrt{2} \le x - y \le \sqrt{2}\}$ contains the set A, so $\overline{C}^r(A) \subset F$; in fact, $\overline{C}^r(A) = F$. Indeed, for each $\varepsilon > 0$ and each $N \in F$, $(N + V_\varepsilon) \cap A \ne \varnothing$, for all $\varepsilon > 0$, so $N \in \overline{A}^r \subset \overline{C}^r(A)$.

Let $B(0,1/2)$ be the open ball with the ray $1/2$ in the space $(\mathbf{R}^2, r_0)$. For each $N \in F = \overline{C}^r(A)$, we have $\overline{C}^r(F\backslash B(N,1/2)) = F$, so

$$(\forall N \in F)(N \in \overline{C}^r(F\backslash B(N,1/2)),$$

which shows that the set $F = \overline{C}^r(A)$ is not r-dentable. □

To prove Theorem 4 we need first to extend Asplund-Namioka-Bourgain's Theorem 3.4.1 of [**2**] in the lcs as follows:

Lemma 1. *Let (E,T) be an lcs, V a convex equilibrated neighborhood of its origin and J, K_0, K_1 three bounded-closed convex sets, such that*

1) $J \subset \overline{C}(K_0 \cup K_1)$,

2) $K_0 \subset J$, $K_0 - K_0 \subset V/2$,

3) $J\backslash K_1 \ne \varnothing$, ($\varnothing$, *the empty set*).

Then there exists a V-small slice, which contains some point of the set K_0.

Proof. As in the proof of Theorem 3.4.1 [**2**], consider the convex set

$$C_r = \{x \in E : x = (1-\lambda)x_0 + \lambda x_1, (x_0, x_1, \lambda) \in K_0 \times K_1 \times [r,1]),$$

where $r \in [0,1]$. Then $C_1 = K_1$ and $J \subset \overline{C}_0$.

From condition 3) and the separation theorem, there exists an $x' \in E'$ such that $x'(K_1) < \sup x'(J)$. As in Theorem 3.4.1 in [**2**], we find that $K_0 \backslash \overline{C}_r \neq \varnothing$. The set $C_0 \backslash \overline{C}_r$ is dense in the sets $\overline{C}_0 \backslash \overline{C}_r \supset J \backslash \overline{C}_r$. There also exists $r \in (0;1)$ such that $r(K_1 - K_0) \subset V/8$, because the set $K_1 - K_0$ is bounded. For this r, the set $J \backslash \overline{C}_r$ is a V-small set. Indeed, for $w \in J \backslash \overline{C}_r$ there is an $x \in \overline{C}_0 \backslash \overline{C}_r$ for which $x \in w + V/8$ or $w - x \in V/8$. If $x_0 \in K_0$ and $x_1 \in K_1$ are two points such that $x = (1-\lambda)x_0 + \lambda x_1$, then $0 \leq \lambda < r$ and

$$w - x_0 = w - x + x - x_0 = (w - x) + \lambda(x_1 - x_0) \in V/4.$$

Consequently, if $w, w' \in J \backslash \overline{C}_r$, denoting by x_0 and x_0' their corresponding points as above, we find:

$$\begin{aligned} w - w' &= w - x_0 - (w' - x_0') + x_0' - x_0 \\ &\subset V/4 + V/4 + K_0 - K_0 \subset V. \end{aligned}$$

Let y_0 be an arbitrary point of the set $K_0 \backslash \overline{C}_r \subset J \backslash \overline{C}_r$. By the separation theorem there exists $x' \in E'$ and a real number β for which $x'(y_0) > \beta$ and $\sup x'(\overline{C}_r) \leq \beta$. Then the slice $T = T(x', \alpha, J)$, the definition in [**6**, page 1], with $\alpha = \sup x'(J) - \beta > 0$ contains the point $y_0 \in K_0$ and is a V-small set because it is contained in the set $J \backslash \overline{C}_r$. □

In the proof of the theorem below, we apply Definition 1 of the dentability given in [**6**].

Theorem 4. *In a locally convex space the dentability of a bounded set A implies the dentability of its closed convex hull.*

Proof. For each arbitrary neighborhood V_0 of the origin in the given lcs E, there is a closed convex equilibrated neighborhood V of the origin such that $V \subset V_0$. Supposing that A is dentable, we can find a $V/4$ small slice $T_0 = T_0(x_0', \alpha_0, A)$ of $A : T_0 - T_0 \subset V/4$. Let x_0 be a point of T_0; then $T_0 \subset x_0 + V/4$ and consequently $K_0 = \overline{C}(T_0) \subset x_0 + V/4$, or $K_0 - K_0 \subset V/2$. We also have $K_0 \subset \overline{C}(A)$ because $T_0 \subset A$. Denoting $K_1 = \overline{C}(A \backslash T_0)$ and $J = \overline{C}(A)$, we have that $J \subset \overline{C}(K_0 \cup K_1)$. On the other hand, from $T_0 = \{x \in A : x_0'(x) > \sup x_0'(A) - \alpha = \beta\}$, it follows that $A\backslash T_0 = \{x \in A : x_0'(x) \le \beta\} \subset \{x \in E : x_0'(x) \le \beta\} = B$, where B is a closed convex set. Consequently, $K_1 = \overline{C}(A\backslash T_0) \subset B$, so for each $y \in T_0$, $y \notin K_1$ because $x_0'(y) > \beta$. For such a point $y, y \in K_0 \subset J$, so $J\backslash K_1 \ne \varnothing$. By Lemma 1 applied to the sets K_0, K_1 and J, there exists a V-small slice $T(x', \alpha, J)$ of J which is also a V_0-small one. Thus the set $J = \overline{C}(A)$ is a dentable set. □

Based on a similar lemma to Lemma 1 and on the bitopological nature of the triple $(E', \beta(E', E), w^*)$, the same proof as in the above theorem leads us to the following version of the converse of Rieffel's theorem for the case of w^*-dentability:

Theorem 5. *If the equicontinuous set A' in the topological dual E' of an* hlcs (E, T) *is w^*-dentable, then its w^*-closed convex hull has the same property.*

In a barreled hlcs this theorem gives us, as a corollary, a more "proper" form of the converse Rieffel's theorem for the case of w^*-dentability:

Proposition 2. *In the topological dual E' of a barreled* hlcs*, the w^*-closed convex hull of a w^*-bounded and a w^*-dentable set is also w^*-dentable.*

Proof. We apply Theorem 5 observing that, in a barreled hlcs, each w^*-bounded subset of its topological dual is an equicontinuous set and vice versa. □

ii) It is important to observe that such extensions as in i) are not possible for the converse of Proposition $1'$; in this case, we have the following characterization:

Proposition 3. *In a real Banach space E the two following facts are equivalent*:

1) *The space E has the* (RNP)*.*

2) *The closed convex equilibrated hull of each bounded dentable set in*

E is dentable.

Proof. We need to prove only implication 2) $\Rightarrow$ 1). If this is not true, then there exists a real Banach space E with property 2) which does not have the (RNP). Let A be a bounded, not dentable subset of E. The closed convex hull $B = \overline{C}(A)$ of A is also not dentable by Rieffel's theorem. We can suppose that $0 \notin B$, otherwise we consider the no dentable bounded closed convex set $a + B$, in which $\|a\| > \sup\{\|x\| : x \in B\}$. The closed convex equilibrated hull D of the set B is not dentable by Rieffel's theorem for the equilibrated case, Proposition 1$'$. On the other hand, the set $B_1 = B \cup \{0\}$ is dentable (because the origin 0 is a strongly exposed point of it) and its closed convex equilibrated hull $\overline{C}$-equil (B_1) is the set D. This leads to a contradiction with property 2). Thus 2) $\Rightarrow$ 1). $\square$

Remark 4. Instead of the set B_1 in the proof of the above proposition, we can take its closed convex hull B_2 which also has the origin 0 as a strongly exposed point.

In the dual Banach spaces, corresponding to Proposition 3 is the following

Proposition 4. *In the dual E' of a real Banach space E the two following facts are equivalent*:

1) *the space E' has the* (RNP).

2) *The w^*-closed convex equilibrated hull of each w^*-bounded and w^*-dentable set in E' is also w^*-dentable.*

Proof. We have that 1) $\Rightarrow$ 2). Indeed, if $A' \subset E'$ is a w^*-bounded and w^*-dentable set, then its w^*-closed convex equilibrated hull $\overline{C}^{w^*}$-equil $(A') = D'$ is a bounded, equivalently w^*-bounded, set and, consequently, a dentable one by the (RNP) of the dual space E'. Then the set D' is w^*-dentable by Proposition 3.4 in [**5**].

The implication 2 $\Rightarrow$ 1) is proved similarly as the corresponding one to Proposition 3. We only observe that if property 2) is true and 1) fails, then there exists a bounded, equivalently w^*-bounded, not dentable set

$A' \subset E'$, which also is not w^*-dentable; the origin 0 is a w^*-strongly exposed point for the set $B'_1 = B' \cup \{0\}$, where $B' = \overline{C}^w(A')$ and $0 \notin B'$. □

Remark 5. In the proof of 2) $\Rightarrow$ 1), we have applied Propositions 1 and 1′ for $r = w^*$.

Remark 6. r and r_0, $r < r_0$, dentabilities are the same in the cases when r is the weak topology in the lc space (E, r_0) or when r is the weak-star topology and r_0 the strong topology in a real dual Banach space E', Proposition 3.4 in [**5**]; but this is not true in the general case. For example, the square $K = \{(x, y) \in \mathbf{R}^2 : |x| + |y| \leq 1\}$ in the triple $(\mathbf{R}^2, r_0, r)$ of Counterexample 1 is dentable and r-bounded as a bounded set in the space $(\mathbf{R}^2, r_0)$, but it is not r-dentable. Indeed, denoting by $B(0, 1/4)$ the open 1/4-ball in $(\mathbf{R}^2, r_0)$ and $B(M, 1/4) = M + B(0, 1/4)$, $M \in \mathbf{R}^2$, it is easy to see that

$$(\forall\, M \in K)(M \in \overline{C}^r(K \backslash B(M, 1/4))).$$

ENDNOTES

1. Here and in all other theorems and propositions we consider that the sets are nonempty.

2. In this section the triple (E, r_0, r) is not generally a bitopological space.

REFERENCES

1. J. Bourgain, *Strongly exposed points in weakly compact convex sets in Banach spaces*, Proc. Amer. Math. Soc. **58** (1976), 197–200.

2. R.D. Bourgin, *Geometric aspects of convex sets with the Radon-Nakodym property*, Springer-Verlag, Berlin, 1983.

3. ———, *Weak* compact convex sets with separable extremal subsets have the Radon-Nikodym property*, Amer. Math. Soc. **69** (1978), 81–84.

4. J. Diestel, *Geometry of Banach spaces–selected topics*, Lecture Notes in Math., Springer Verlag, Berlin, (1975).

5. S. Fitzpatrick, *Separably related sets and the Radon-Nikodym property*, Illinois J. Math. **29** (1985), 235–236.

6. M. Fundo, *The dentability in the spaces with two topologies*, Rocky Mountain J. Math. **27** (1997), 1117–1130.

7. S. Gjinushi, *Espaces de Fréchet fortement differentiables*, These du dipolome de Docteur de 3^e cicle Paris **6** (1977), 20–22, 29–31, 71.

8. ———, *Bashkësi të kufizuara konvekse që kanë vetinë e Radon-Nakodym*, Bul. Shk. Natyrore. **3** (1979), 11–12, 14–15.

9. R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford Ser. (2) **27** (1976), 379–385.

10. Kothe. 6. *Topological vector spaces* I, Springer-Verlag, New York, 1968.

11. J. Lindenstrauss, *Weakly compact sets–their topological properties and the Banach spaces they generate*, Ann. of Math. **69** (1972), 235–273.

12. H.B. Maynard, *A geometrical characterisation of Banach spaces with the Radon-Nikodym property*, Trans. Amer. Math. Soc. **185** (1973), 493–500.

13. I. Namioka, *Neighborhoods of extreme points*, Israel J. Math. **5** (1967), 145–152.

14. I. Namioka and E. Asplund, *A geometric proof of Ryll-Nardzewski's fixed point theorem*, Bull. Amer. Math. Soc. **73** (1967), 443–445.

15. M.A. Rieffel, *Dentable subsets of Banach spaces with application to a Radon-Nikodym theorem*, Tompson Book, Washington, DC, 1967.

16. W. Schachermayer, *The sum of two Radon-Nikodym-sets need not be a Radon-Nikodym-set*, Proc. Amer. Math. Soc. **95** (1985), 51, 53–54.

17. S.L. Troyanski, *On locally uniformly convex and differentiable norms in certain non separable Banach spaces*, Stud. Math. **38** (1971), 173–180.

Department of Civil Engineering, Aristotle University, GR-54006, Thessaloniki, Greece