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THE DIOPHANTINE EQUATION $ax^5 + by^5 + cz^5 = au^5 + bv^5 + cw^5$

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ABSTRACT. A parametric solution of the Diophantine equation $ax^5 + by^5 + cz^5 = au^5 + bv^5 + cw^5$ is obtained when a, b and c are distinct nonzero integers such that $a+b+c=0$.

While several parametric solutions of the Diophantine equation

$$x^5 + y^5 + z^5 = u^5 + v^5 + w^5$$

are known [**1, 2, 3, 4, 5**], the Diophantine equation

$$ax^5 + by^5 + cz^5 = au^5 + bv^5 + cw^5 \tag{1}$$

has not been considered earlier. In this paper we give a parametric solution of (1) when a, b, c are integers such that

$$a + b + c = 0. \tag{2}$$

The solution reduces to a trivial one when $abc(a-b)(b-c)(c-a) = 0$. Thus, nontrivial solutions of (1) are obtained only when a, b, c are distinct nonzero integers satisfying the relation (2).

To solve (1), we write

$$\begin{aligned} x &= p\theta + \alpha, & y &= q\theta + \alpha, & z &= r\theta + \alpha, \\ u &= -b\theta + \beta, & v &= a\theta + \beta, & w &= \beta \end{aligned} \tag{3}$$

where p, q, r, α and β are arbitrary, and we will take $\alpha\beta \neq 0$. Substituting these values in (1), we get the following fifth degree equation in θ:

$$\begin{aligned} a\{(p\theta + \alpha)^5 - (-b\theta + \beta)^5\} + b\{(q\theta + \alpha)^5 - (a\theta + \beta)^5\} \\ + c\{(r\theta + \alpha)^5 - \beta^5\} = 0. \end{aligned} \tag{4}$$

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In view of (2), we note that the constant term in (4) is zero. We will choose p, q, r, α and β such that the coefficients of θ, θ^2 and θ^3 in (4) also become zero. Equation (4) will then be readily solved to give a nonzero rational root which will lead to a solution of (1).

Equating to zero the coefficient of θ in (4), we get

$$ap + bq + cr = 0.$$

Using (2), we get

$$r = (ap + bq)(a + b)^{-1}. \tag{5}$$

Using this value of r, and equating to zero the coefficient of θ^2 in (4), we get

$$\alpha^3(p - q)^2 = \beta^3(a + b)^2$$

or

$$\left(\frac{\beta}{\alpha}\right)^3 = \left(\frac{p - q}{a + b}\right)^2 = t^6 \tag{6}$$

where t is arbitrary. This gives

$$\beta = \alpha t^2$$

and

$$p = q + (a + b)t^3. \tag{7}$$

Finally, equating to zero the coefficient of θ^3 in (4), we get

$$(2t^5 - 1)a + (t^5 + 1)b + 3t^2 q = 0. \tag{8}$$

The equations (5), (7) and (8) are readily solved for p, q and r and we take

$$\alpha = (a - b)\{(2a^2 + 5ab + 2b^2)t^{10} - (17a^2 + 20ab + 17b^2)t^5 + 17a^2 + 20ab + 17b^2\}.$$

Now (4) is readily solvable and we get

$$\theta = 45(a^2 + ab + b^2)(t^7 - t^5)$$

as a root of (4). Substituting the values of p, q, r, α, β and θ in (3), we get the following solution of (1):

$$\begin{aligned}
x &= (a+2b)\{(17a^2+14ab+14b^2)t^{10} \\
&\qquad -(17a^2+14ab+14b^2)t^5+2a^2-ab-b^2\}, \\
y &= -(2a+b)\{(14a^2+14ab+17b^2)t^{10} \\
&\qquad -(14a^2+14ab+17b^2)t^5-a^2-ab+2b^2\}, \\
z &= (a-b)\{(17a^2+20ab+17b^2)t^{10} \\
&\qquad -(17a^2+20ab+17b^2)t^5+2a^2+5ab+2b^2\}, \\
u &= (a+2b)\{(2a^2-ab-b^2)t^{12}-(17a^2+14ab+14b^2)t^7 \\
&\qquad +(17a^2+14ab+14b^2)t^2\}, \\
v &= (2a+b)\{(a^2+ab-2b^2)t^{12}+(14a^2+14ab+17b^2)t^7 \\
&\qquad -(14a^2+14ab+17b^2)t^2\}, \\
w &= (a-b)\{(2a^2+5ab+2b^2)t^{12}-(17a^2+20ab+17b^2)t^7 \\
&\qquad +(17a^2+20ab+17b^2)t^2\}.
\end{aligned}$$

When $a=1$ and $b=2$, we get, after making appropriate changes of sign, a parametric solution of the equation

$$X^5+2Y^5+3Z^5=U^5+2V^5+3W^5$$

which is as follows:

$$\begin{aligned}
&X=101t^{10}-101t^5-4, \quad Y=4t^{12}-88t^7+88t^2, \quad Z=25t^{10}-25t^5+4, \\
&U=-4t^{12}-101t^7+101t^2, \quad V=88t^{10}-88t^5+4, \quad W=4t^{12}-25t^7+25t^2.
\end{aligned}$$

As a numerical example, when $t=-3/2$, we get the solution

$$\begin{aligned}
&6745229^5+2.2273841^5+3.1674721^5 \\
&\qquad = 1468359^5+2.5884696^5+3.1026441^5.
\end{aligned}$$

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