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GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A SEMILINEAR PARABOLIC SYSTEM

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ABSTRACT. We discuss the initial-boundary value problem $(u_i)_t = \Delta u_i + f_i(u_1, \dots, u_m)$, with $u_i|_{\partial\Omega} = 0$ and $u_i(x,0) = \phi_i(x)$, $i = 1, \dots, m$, in a bounded domain $\Omega \in R^n$, with $n \geq 1$ and $m \geq 1$. Under suitable assumptions on the nonlinear terms f_i we will prove that, if $0 \leq \phi_i < \lambda\psi_i$ with $\lambda < 1$, then the solutions are global, while if $\phi_i > \lambda\psi_i$ with $\lambda > 1$, then the solutions must blow up in finite time, where the ψ_i are positive solutions of $\Delta\psi_i + f_i(\psi_1, \dots, \psi_m) = 0$ with $\psi_i|_{\partial\Omega} = 0$.

In this paper we study the initial boundary-value problem

$$
\begin{aligned}
\mathbf{u}_t &= \Delta\mathbf{u} + \mathbf{f}(\mathbf{u}), && x \in \Omega, t > 0,\\
\mathbf{u}(x,t) &= \mathbf{0}, && x \in \partial\Omega, t > 0,\\
\mathbf{u}(x,0) &= \boldsymbol{\phi}(x), && x \in \Omega,
\end{aligned} \tag{1}
$$

where Ω is a bounded domain in R^n with smooth boundary and $n \geq 1$, and $\mathbf{u} = (u_1, \dots, u_m)$, $\mathbf{f} = (f_1, \dots, f_m)$ are vectors with $m \geq 1$. It is well-known that, for some small initial values, the solution may exist globally, while for some large initial values the solution may blow-up in finite time if the nonlinear term $\mathbf{f}(\mathbf{u})$ increases superlinearly, see [**2**–**6, 8, 14**] and [**17**]. For a large class of nonlinearities, considerably less is known as to when solutions exist globally or blow-up in finite time.

If $m = 1$ and $f(u) = |u|^{p-1}u$ with $p > 1$, Levine [**14**] proved that solutions of (1) must blow-up in finite time, if $\phi(x)$ is large enough in the sense that its "energy,"

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 - \frac{1}{p+1}\|\phi\|_{p+1}^{p+1}$$

is negative. Weissler [**18**] proved that for $n = 1$ blow-up occurs only at the point $x = 0$ if $\phi = k\psi$ and p, k are sufficiently large, where ψ

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is a positive solution of $\psi_{xx} + \psi^p = 0$, $\psi|_{\partial\Omega} = 0$ with $\Omega = (-1, 1)$. In [**19**], he also proved that, for $n \leq 2$ or $p < (n+2)/(n-2)$ there exists a constant $c > 0$ such that

$$u(x,t) \leq c(T-t)^{-1/(p-1)},$$

if $\phi \geq 0$ is radially symmetric and $\Delta\phi + \phi^p \geq 0$. For more general cases, see [**4**].

If $m = 2$ and $\mathbf{f}(\mathbf{u}) = (u_1^\beta u_2, -u_1^\beta u_2)$ with $\beta \geq 1$, Hollis, Martin and Pierre [**13**] have shown that solutions of (1) exist globally. If $\mathbf{f}(\mathbf{u}) = ((1-u_2)g(u_1), (1-u_2)g(u_1))$ with $g(u_1) = e^{u_1}$, Bebernes and Lacey [**2**] have shown that solutions of (1) blow up in finite time. Later, in [**3**] they extended their results to more general $g(u_1)$. If $\mathbf{f}(\mathbf{u}) = (u_2^p, u_1^q)$, Escobedo and Herrero [**8**] have proved that all solutions of (1) are global if $pq < 1$, while if $pq > 1$, both global solutions and solutions that blow-up in finite time can occur, depending on the initial values. Later Caristi and Mitidieri [**5**] obtained the following estimates:

$$u_1(x,t) \leq c(T-t)^{-(p+1)/(pq-1)}, \quad u_2(x,t) \leq c(T-t)^{-(q+1)/(pq-1)},$$

if $pq > 1$, where T is the blow-up time.

Lu and Sleeman [**15**] gave several sufficient conditions to get the blow-up property for the one-dimensional parabolic system with $m = 2$

$$\frac{\partial u_i}{\partial t} = \alpha_i \frac{\partial^2 u_i}{\partial x^2} + f_i(u_1, u_2), \quad -a < x < a, \quad \alpha_i > 0, \quad i = 1, 2.$$

However, in order to get blow-up solutions, many authors, see [**9**], for example, need to assume that $\partial\mathbf{u}/\partial t \geq \mathbf{0}$ or $\Delta\boldsymbol{\phi} + \mathbf{f}(\boldsymbol{\phi}) \geq \mathbf{0}$, for $n > 1$ (where $\mathbf{u} > \mathbf{0}$ means $u_i > 0$ for all i). So there is a gap between the global solutions and the blow-up solutions. For the Cauchy problem,

$$\begin{aligned} u_t &= \Delta u + u^p, & x \in R^n, \quad t > 0, \\ u(x,0) &= \phi(x), & x \in R^n, \end{aligned}$$

with $p \geq (n+2)/(n-2)$ and $n \geq 3$, Gui, Ni and Wang [**11**] have obtained the following perfect results:

(i) if $\phi \leq u_\alpha$ and $\phi \not\equiv u_\alpha$ for some α, then $\|u\|_\infty \to 0$ as $t \to \infty$;

(ii) if $\phi \geq u_\alpha$ and $\phi \not\equiv u_\alpha$ for some α, then u must blow-up in finite time, where u_α is a positive solution of $\Delta\psi + \psi^p = 0$, $\psi(0) = \alpha$.

The purpose of this paper is to fill in this gap for the initial-boundary problem (1). We assume that

(i) $\boldsymbol{\psi}(x)$ is a positive solution of $\Delta\boldsymbol{\psi} + \mathbf{f}(\boldsymbol{\psi}) = \mathbf{0}$, with $\boldsymbol{\psi}|_{\partial\Omega} = \mathbf{0}$.

(ii) $\boldsymbol{\phi}(x) \in C^\alpha(\overline{\Omega}, R^m)$, with $\boldsymbol{\phi}|_{\partial\Omega} = \mathbf{0}$ for $0 < \alpha < 1$.

(iii) $\mathbf{f} : R_+^m \to R_+^m$ is locally Lipschitz continuous, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and

$$f_i(\mathbf{u})/u_i > f_i(\mathbf{v})/v_i \quad \text{for any} \quad \mathbf{u} > \mathbf{v} \geq \mathbf{0} \quad \text{and} \quad i = 1, \dots, m.$$

(iv) $f_i(\mathbf{u})/u_i^\sigma \geq c_0 > 0$ for some $\sigma > 1$ and all $\mathbf{u} > \mathbf{0}$, $i = 1, \dots, m$.

Our result is:

Theorem. *If $0 \leq \phi_i < \lambda\psi_i$ for all i, with $\lambda < 1$, and the assumptions* (i)—(iii) *hold, then the solution* $\mathbf{u}$ *of* (1) *is global with exponential decay. If $\phi_i > \lambda\psi_i$ for all i, with $\lambda > 1$, and the assumptions* (i)–(iv) *hold, then the solution* $\mathbf{u}$ *of* (1) *must blow-up in finite time.*

Proof. First we prove global existence. Since $\boldsymbol{\phi} \in C_0^\alpha(\overline{\Omega})$, from standard parabolic PDE theory, there exists a unique solution $\mathbf{u}(x,t) \in C(\overline{\Omega} \times [0,\tau]) \cap C^{2,1}(\overline{\Omega} \times (0,\tau])$ for some $\tau > 0$, see [**1** or **10**] and $\mathbf{u} \geq \mathbf{0}$, see [**17**]. Set $v_i(x,t) = (\lambda + ct)\psi_i(x) - u_i(x,t)$ with c large enough such that

$$|(\lambda + 1)\frac{f_i(\boldsymbol{\psi})}{\psi_i} - \frac{f_i(\mathbf{u})}{\psi_i}| < c \quad \text{in } \Omega \times (0,\tau).$$

Then

$$\begin{aligned}(v_i)_t - \Delta v_i &= c\psi_i - (u_i)_t - (\lambda + ct)\Delta\psi_i + \Delta u_i \\ &= \psi_i\left[c + (\lambda + ct)\frac{f_i(\boldsymbol{\psi})}{\psi_i} - \frac{f_i(\mathbf{u})}{\psi_i}\right] > 0,\end{aligned}$$

for $t \leq \min(1/c, \tau)$. By the maximum principle, $v_i \geq 0$. Choose t_1 sufficiently small such that $\lambda + ct_1 \leq 1$. Then $(\lambda + ct)\psi_i(x) \geq u_i(x,t)$ for $t \leq t_1$. Now, for any number n, set

$$g_i^n(t) = \int_\Omega u_i^{n+2}(x,t)\psi_i^{-n}(x)\, d\Omega, \quad \text{for } t \in [0, t_1]. \tag{2}$$

Then $g_i^n(t)$ is well defined, and $g_i^n(t) \in C[0, t_1]$, because $u_i(x, t) \in C(\overline{\Omega} \times [0, \tau])$. Differentiating (2), substituting in the equations (1) and integrating by parts (notice that the boundary values are always zero), we have

$$\begin{aligned}
\frac{d}{dt} g_i^n(t) &= (n+2) \int_\Omega u_i^{n+1} \psi_i^{-n} (\Delta u_i + f_i(\mathbf{u}))\, d\Omega \\
&= (n+2) \int_\Omega u_i^{n+1} \psi_i^{-n} f_i(\mathbf{u})\, d\Omega \\
&\quad + (n+2) \bigg\{ n \int_\Omega u_i^{n+1} \psi_i^{-n-1} \nabla u_i \nabla \psi_i \, d\Omega \\
&\qquad\qquad - (n+1) \int_\Omega u_i^n \psi_i^{-n} |\nabla u_i|^2 \, d\Omega \bigg\} \\
&= (n+2) \int_\Omega u_i^{n+2} \psi_i^{-n} \frac{f_i(\mathbf{u})}{u_i} \, d\Omega \\
&\quad - (n+1)(n+2) \int_\Omega u_i^n \psi_i^{-n-2} |\psi_i \nabla u_i - u_i \nabla \psi_i|^2 \, d\Omega \\
&\quad - (n+2)^2 \int_\Omega u_i^{n+1} \psi_i^{-n-1} \nabla u_i \nabla \psi_i \, d\Omega \qquad (3) \\
&\quad + (n+1)(n+2) \int_\Omega u_i^{n+2} \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
&\leq (n+2) \int_\Omega u_i^{n+2} \psi_i^{-n} \frac{f_i(\mathbf{u})}{u_i} \, d\Omega \\
&\quad + (n+1)(n+2) \int_\Omega u_i^{n+2} \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
&\quad - (n+2) \bigg\{ (n+1) \int_\Omega u_i^{n+2} \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
&\qquad\qquad - \int_\Omega u_i^{n+2} \psi_i^{-n-1} \Delta \psi_i \, d\Omega \bigg\} \\
&= -(n+2) \int_\Omega u_i^{n+2} \psi_i^{-n} \left(\frac{f_i(\boldsymbol{\psi})}{\psi_i} - \frac{f_i(\mathbf{u})}{u_i} \right) d\Omega \leq 0,
\end{aligned}$$

for $t \leq t_1$. Thus, we get

$$g_i^n(t) \leq g_i^n(0) \quad \text{for } t \in (0, t_1].$$

Taking the nth roots and letting $n \to \infty$, we have

$$(4) \qquad \frac{u_i(x,t)}{\psi_i(x)} \leq \sup_\Omega \frac{u_i}{\psi_i} \leq \sup_\Omega \frac{\phi_i}{\psi_i} \leq \lambda, \quad \text{for } t \in (0, t_1].$$

We can extend u_i from t_1 to t_2 by the same method since $u_i(x,t)/\psi_i(x) \le \lambda$. Hence, (4) holds for all $t > 0$ for which the solution u_i exists. By [**1**] or [**12**], we can extend u_i to ∞ as long as u_i is bounded.

Now we prove that $u_i(x,t)$ decays exponentially. For any $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$. Since $\lambda < 1$, by assumption (iii), there exists a $c_\varepsilon > 0$ such that

$$\frac{f_i(\boldsymbol{\psi})}{\psi_i} - \frac{f_i(\mathbf{u})}{u_i} = \frac{f_i(\boldsymbol{\psi})}{\psi_i}\left(1 - \frac{f_i(\mathbf{u})/u_i}{f_i(\boldsymbol{\psi})/\psi_i}\right) \ge c_\varepsilon \tag{5}$$

for $x \in \Omega - \Omega_\varepsilon$. Using (3), we define $g^n_{\varepsilon,i}(t)$ by

$$\frac{d}{dt}g_i^n(t) \le -(n+2)c_\varepsilon \int_{\Omega-\Omega_\varepsilon} u_i^{n+2}\psi_i^{-n}\, d\Omega \equiv -(n+2)c_\varepsilon g^n_{\varepsilon,i}(t).$$

Then

$$g^n_{\varepsilon,i}(t) < g^n_i(t) \le g^n_i(0) - (n+2)c_\varepsilon \int_0^t g^n_{\varepsilon,i}(\tau)\, d\tau,$$

which implies that, for fixed n,

$$\frac{1}{\max \psi_i^n} \int_{\Omega-\Omega_\varepsilon} u_i^{n+2}\, d\Omega < g^n_{\varepsilon,i}(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

Since $u_i(x,t) = 0$ on the boundary, we have $\int_\Omega u_i^{n+2}\, d\Omega \to 0$ as $t \to \infty$. If $\lim_{\psi_i \to 0} f_i(\boldsymbol{\psi})/\psi_i = 0$ by [**12**], $u_i(x,t)$ decays exponentially. If $f_i(\boldsymbol{\psi})/\psi_i \ge c_0 > 0$ for any $x \in \Omega$, applying (5) to (3), we obtain

$$\frac{d}{dt}g_i^n(t) \le -c(n+2)g_i^n(t)$$

which also implies that $u_i(x,t)$ decays exponentially.

Now we prove the blow-up property. Similar to the argument above, we set

$$h_i^n(t) = \int_\Omega \psi_i^{n+2}(x) u_i^{-n}(x,t)\, d\Omega.$$

Then

$$\begin{aligned}\frac{d}{dt}h_i^n(t) \le & -n\int_\Omega \psi_i^{n+2}u_i^{-n}\left(\frac{f_i(\mathbf{u})}{u_i} - \frac{f_i(\boldsymbol{\psi})}{\psi_i}\right) d\Omega \\ & - n(n+1)\int_\Omega \psi_i^n u_i^{-n-2}|\psi_i \nabla u_i - u_i \nabla \psi_i|^2\, d\Omega \le 0.\end{aligned} \tag{6}$$

So $u_i(x,t) \geq \lambda\psi_i(x)$ for $\lambda > 1$ and all $t > 0$ such that $u(x,t)$ exists. By assumption (iii), there exists $c_1 > 0$ such that

$$1 - \frac{f_i(\boldsymbol{\psi})/\psi_i}{f_i(\mathbf{u})/u_i} \geq c_1.$$

From (6) with $n = \sigma - 1$ and assumption (iv), we get

$$\begin{aligned} \frac{d}{dt} h_i^{\sigma-1}(t) &\leq -(\sigma-1)\int_\Omega \psi_i^{\sigma+1}\frac{f_i(\mathbf{u})}{u_i^\sigma}\left(1 - \frac{f_i(\boldsymbol{\psi})/\psi_i}{f_i(\mathbf{u})/u_i}\right) d\Omega \\ &\leq -c\int_\Omega \psi_i^{\sigma+1}\, d\Omega. \end{aligned}$$

Hence

$$0 \leq h_i^{\sigma-1}(t) \leq h_i^{\sigma-1}(0) - ct\int_\Omega \psi_i^{\sigma+1}\, d\Omega, \quad \text{or} \quad t \leq \frac{h_i^{\sigma-1}(0)}{c\int_\Omega \psi_i^{\sigma+1}\, d\Omega},$$

which means t cannot increase to infinity. The proof is complete. □

Example. It is easy to see that, if $f_1 = (1+u_2)u_1^2$ and $f_2 = (1+u_1)u_2^2$, then $\mathbf{f} = (f_1, f_2)$ satisfies the conditions (iii) and (iv) and that $\boldsymbol{\psi} = (\psi_1, \psi_2)$ can be chosen such that $\psi_1 = \psi_2$ and ψ_i is a positive solution of $\Delta\psi_i + (1+\psi_i)\psi_i^2 = 0$ with $\psi_i|_{\partial\Omega} = 0$.

Remark . For general $\mathbf{f}$, the system $\Delta\boldsymbol{\psi} + \mathbf{f}(\boldsymbol{\psi}) = \mathbf{0}$ with $\boldsymbol{\psi}|_{\partial\Omega} = \mathbf{0}$ might have no positive solutions, see, for example, [**16**, Theorem 4.1] or the work of [**7**]. However, when $n = 1$, such solutions always exist for many kinds of f.

Appendix

We need some results from the theory of analytic semigroups, see [**1**] and [**12**]. Suppose the Laplace operator Δ is the infinitesimal generator of an analytic semigroup $\{e^{t\Delta} \mid 0 \leq t < \infty\}$. Then there exist positive constants c_1, c_2 and δ, independent of t, such that

$$\|e^{t\Delta}\| \leq c_1 e^{-t\delta}, \qquad \|\Delta e^{t\Delta}\| \leq c_2 e^{-t\delta}/t, \tag{7}$$

where $\|\cdot\|$ is the norm of $X = L^p$ for $p \geq 2$. This implies the existence of the integral

$$\Delta^{-\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty \tau^{\mu-1} e^{\tau\Delta}\, d\tau, \tag{8}$$

for every $\mu > 0$, where $\Gamma(x)$ is the gamma function. It follows that each $\Delta^{-\mu}$ is an injective continuous endomorphism of X. Hence $\Delta^{\mu} = [\Delta^{-\mu}]^{-1}$ is a closed bijective linear operator in X. If $\phi \in D(\Delta^{\mu})$, the domain of Δ^{μ}, then

$$\begin{aligned} \|\Delta^{\mu} e^{t\Delta}\phi\| &\leq c e^{-t\delta} \|\Delta^{\mu}\phi\|, \\ \|\Delta^{\mu} e^{t\Delta}\phi\| &\leq c t^{-\mu} e^{-t\delta} \|\phi\|. \end{aligned} \tag{9}$$

Now set $X = [L^p(\Omega)]^m$ with the L^p norm $\|\cdot\|$ and $D(\Delta) = [W_0^{2,p}(\Omega)]^m$. We have the following lemma:

Lemma. *Suppose that $n/(2p) < \beta < 1/2$. Then $X_\beta = (D(\Delta^{\beta}), \|\cdot\|_\beta)$ is embedded into $[C^{\mu}(\overline{\Omega})]^m$ with $0 < \mu < 2\beta - n/p$.*

Proof. The proof is similar to that of [**1**]. It follows from Friedman [**10**, Theorem (I.10.1)] that

$$\|u\|_{C^{\mu}(\overline{\Omega})} \leq c \|u\|_{W^{2,p}}^{\nu} \|u\|_{L^p}^{1-\nu},$$

where $\nu = \mu/2 + n/(4p) < \beta$. If we let $u = \Delta^{-\beta} v$ with $v \in [L^p(\Omega)]^m$, we obtain from (7)–(9),

$$\begin{aligned} \|\Delta^{-\beta} v\|_{C^{\mu}(\overline{\Omega})} &\leq \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} \|e^{\tau\Delta} v\|_{C^{\mu}(\overline{\Omega})}\, d\tau \\ &\leq c \int_0^\infty \tau^{\beta-1} \|\Delta e^{\tau\Delta} v\|_{L^p}^{\nu} \|e^{\tau\Delta} v\|_{L^p}^{1-\nu}\, d\tau \\ &\leq c \int_0^\infty \tau^{\beta-\nu-1} e^{-\delta\tau}\, d\tau \|v\|_{L^p}. \end{aligned}$$

Hence $\|u\|_{C^{\mu}(\overline{\Omega})} \leq c\|\Delta^{\beta} u\| = c\|u\|_\beta$, and the assertion follows. □

Since $\boldsymbol{\phi} \in C_0^{\alpha}(\overline{\Omega})$, we have $\boldsymbol{\phi} \in D(\Delta^{\alpha})$. From [**12**], $\mathbf{u}(t) \in C([0,\tau], D(\Delta^{\alpha}))$, which implies that $\mathbf{u}(x,t) \in C(\Omega \times [0,\tau])$ by the lemma with $\beta = \alpha$.

REFERENCES

1. H. Amann, *Nonlinear analysis: A collection of papers in honor of Erich Rothe*, Academic Press, New York, 1978, 1–29.

2. J. Bebernes and A. Lacey, *Finite time blowup for a particular parabolic system*, SIAM J. Math. Anal. **21** (1990), 1415–1425.

3. ———, *Finite time blowup for semilinear reactive-diffusive systems*, J. Differential Equations **95** (1992), 105–129.

4. H. Bellout, *A criterion for blow-up of solutions to semilinear heat equations*, SIAM J. Math. Anal. **18** (1987), 722–727.

5. G. Caristi and E. Mitidieri, *Blow-up estimates of positive solutions of a parabolic system*, J. Differential Equations **113** (1994), 265–271.

6. S.-H. Chen, *Large time existence and behavior of solution for systems of nonlinear reaction-diffusion equations*, Acta Math. Sci. (English ed.), **8** (1988), 35–44.

7. S.-H. Chen and G.-Z. Lu, *Existence and nonexistence of positive radial solutions for a class of semilinear elliptic systems*, to appear in Nonlinear Analysis.

8. M. Escobedo and M.A. Herrero, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl. (4) **165** (1993), 315–336.

9. M. Escobedo and H. Levine, *Critical blow-up and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Rational Mech. Anal. **129** (1995), 47–58.

10. A. Friedman, *Partial differential equations*, Holt, New York, 1969.

11. C. Gui, W.-M. Ni and X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation in R^n*, Comm. Pure Appl. Math. **45** (1992), 1153–1181.

12. D. Henry, *Geometric theory of semilinear parabolic equations*, Springer-Verlag, Berlin, 1981.

13. S. Hollis, R. Martin and M. Pierre, *Global existence and boundedness in reaction-diffusion systems*, SIAM J. Math. Anal. **18** (1987), 744–761.

14. H.A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$*, Arch. Rational Mech. Anal. **51** (1973), 371–385.

15. Lu Gang and B.D. Sleeman, *Non-existence of global solutions to systems of semi-linear parabolic equations*, J. Differential Equations **104** (1993), 147–168.

16. E. Mitidieri, *Nonexistence of positive solutions of semilinear elliptic systems in R^n*, Differential Integral Equations **9** (1996), 465–479.

17. C.V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

18. F.B. Weissler, *Single point blow-up for a semilinear initial value problem*, J. Differential Equations **55** (1984), 204–224.

19. ———, *An L^∞ blow-up estimate for a nonlinear heat equation*, Comm. Pure Appl. Math. **38** (1985), 291–295.

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