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ALMOST SURE CONVERGENCE AND DECOMPOSITION OF MULTIVALUED RANDOM PROCESSES

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ABSTRACT. This paper examines multi-valued random processes (random sets) with values in a separable Banach space. Several results on the almost sure convergence and decomposition properties of various classes of random processes are established. Special consideration is given to multi-valued submartingales, uniform amarts, weak sequential amarts and amarts of infinite order. In the process some results concerning sequences of vector-valued random variables are also proved.

1. Introduction. Multi-valued random processes and more specifically multi-valued discrete time martingales, were first introduced in the early seventies by Van Cutsem [**35, 36**] in connection with problems of stochastic optimization. Since then the subject has attracted the interest of many mathematicians and further contributions were made from both the theoretical and applied viewpoints. The development of the theory can be traced in the works Neveu [**26**], Daures [**11**], Hiai and Umegaki [**18**], Coste [**10**], Luu [**23, 24**], Bagchi [**3, 4, 5**], Papageorgiou [**28, 30, 31**], de Korvin and Kleyle [**20**], Hess [**17**] and Wang and Xue [**38**]. Applications to stochastic optimization, mathematical economics and information systems can be found in the works of Arstein and Hart [**1**], Salinetti and Wetts [**33**], Papageorgiou [**27**] and Yovits et al. [**40**].

In this paper we make some further contributions to the convergence and decomposition theories of set-valued random processes (random sets). We consider a variety of multi-valued random processes starting with submartingales. Via a "weak* compactness" result for multifunctions, we prove a submartingale convergence theorem which extends the result of Neveu [**26**].

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In Section 4 we look at multi-valued uniform amarts. With the help of a convergence theorem for vector-valued functions, we prove a general convergence theorem for multi-valued uniform amarts. In case the range space is a separable dual Banach space, we obtain a second generalization of the result of Neveu [**26**].

In Section 5 we turn our attention to multi-valued weak sequential amarts and, in analogy with the single-valued case, we prove a convergence and a decomposition theorem for such multi-valued processes.

Finally, in Section 6, we examine amarts of infinite order (which include amarts) and establish for them a Riesz decomposition type result.

But first in the next section we fix our notation and terminology and recall some basic notions and facts from set-valued analysis and the theory of Banach space-valued, discrete time stochastic processes.

2. Preliminaries. Let (Ω, Σ, μ) be a complete probability space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex)}\}$$

and

$$P_{wk(c)}(X) = \{A \subseteq X : \text{ nonempty, } w\text{-compact, (convex)}\}.$$

Given a set $A \subseteq 2^X \backslash \{\varnothing\}$, we also define:

$$|A| = \sup\{\|x\| : x \in A\} \quad \text{(the } \textit{norm} \text{ of } A),$$
$$\sigma(x^*, A) = \sup[(x^*, x) : x \in A\} \quad \text{for } x^* \in X^*$$
$$\text{(the } \textit{support function} \text{ of } A), \text{ and}$$
$$d(z, A) = \inf[\|z - x\| : x \in A\} \quad \text{for } z \in X$$
$$\text{(the } \textit{distance function} \text{ from } A).$$

A multi-function (set-valued function) $F : \Omega \to P_f(X)$ is said to be *measurable*, if, for all $z \in X$, the R_+-valued function $\omega \to d(z, F(\omega)) = \inf\{\|x - z\| : x \in F(\omega)\}$ is measurable. In fact, this definition of measurability of $F(\cdot)$ turns out to be equivalent to the existence of a sequence $f_n : \Omega \to X$, $n \geq 1$, of measurable functions such that $F(\omega) = \overline{\{f_n(\omega)\}}_{n\geq 1}$ for every $\omega \in \Omega$. Moreover, since we have assumed Σ to be μ-complete, these definitions of measurability are equivalent to saying that $\mathrm{Gr}F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X, *graph measurability.* In the absence of completeness with respect to $\mu(\cdot)$ of Σ, we can only say that measurability implies graph measurability, while the converse is not in general true. Using this equivalence of measurability and graph measurability and the fact that the support function of a weakly compact set in X is m-continuous (m- being the Mackey topology $m(X^*, X)$), we can easily check that a multi-function $F : \Omega \to P_{wkc}(X)$ is measurable if and only if for every $x^* \in X^*$, $\omega \to \sigma(x^*, F(\omega))$ is measurable. Recall that, since X is separable, X^* is separable for the Mackey topology $m(X^*, X)$, see, for example, Wilansky [**39**, p. 144]. Details on the measurability properties of a multi-function can be found in the survey paper of Wagner [**37**].

We also use S_F^1 to denote the set of measurable selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^1(\Omega, X)$. In general, this set may be empty. However, a straightforward application of Aumann's selection theorem, see Wagner [**37**, p. 873], shows that, for a graph measurable multi-function $F : \Omega \to 2^X\backslash\{\varnothing\}$, S_F^1 is nonempty if and only if $\omega \to \inf\{\|z\| : z \in F(\omega)\} \in L^1(\Omega)$. In particular, if $\omega \to |F(\omega)|$ belongs in $L^1(\Omega)$, such a multi-function is usually called "integrably bounded," then $S_F^1 \neq \varnothing$. Using S_F^1, we can define a set-valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)\, d\mu(\omega) = \{\int_\Omega f(\omega)\, d\mu(\omega) : f \in S_F^1\}$. The vector-valued integrals are understood in the sense of Bochner.

Now let $F : \Omega \to P_f(X)$ be a measurable multi-function with $S_F^1 \neq \varnothing$. Following Hiai and Umegaki [**18**] we define the conditional expectation of $F(\cdot)$ with respect to Σ_0, (a complete sub-σ-field of Σ), to be the Σ_0-measurable multi-function $E^{\Sigma_0}F : \Omega \to P_f(X)$ satisfying $S^1_{E^{\Sigma_0}F}(\Sigma_0) = \mathrm{cl}\{E^{\Sigma_0}f : f \in S_F^1\}$, the closure taken in $L^1(\Omega, X)$. To see that this is a well-defined notion, note that the set $K = \{E^{\Sigma_0}f : f \in S_F^1\}$ is Σ_0-*decomposable*, i.e., if $(A, g_1, g_2) \in \Sigma_0 \times K \times K$, then $\chi_A g_1 + \chi_{A^c} g_2 \in K$. In particular, then $\mathrm{cl}\,K$ is Σ_0-decomposable and so, invoking Theorem 3.1, [**18**, p. 158], we get a unique (up to μ-null sets) Σ_0-measurable multi-function $E^{\Sigma_0}F : \Omega \to P_f(X)$ for which we have $S^1_{E^{\Sigma_0}F}(\Sigma_0) = \mathrm{cl}\,K$. If $F(\cdot)$ is convexvalued, respectively integrably bounded, then so is $E^{\Sigma_0}F(\cdot)$. The set-valued conditional expectation behaves much like the ordinary vector-valued conditional expectation. So, in particular, if $F(\cdot)$ is Σ_0-measurable, then $E^{\Sigma_0}F(\omega) = F(\omega)$ μ-almost everywhere; if Σ_0', Σ_0 are complete sub-σ-fields of Σ and $\Sigma_0' \subseteq \Sigma_0$, then $E^{\Sigma_0'}(E^{\Sigma_0}F) = E^{\Sigma_0'}F$ μ-almost everywhere; and if $A \in \Sigma_0$, cl $\int_A^{(\Sigma_0)} E^{\Sigma_0}F(\omega)\,d\mu(\omega) =$ cl $\int_A F(\omega)\,d\mu(\omega)$ where here $\int_A^{(\Sigma_0)} E^{\Sigma_0}F(\omega)\,d\mu(\omega) = \{\int_A g(\omega)\,d\mu(\omega) : g \in L^1(\Sigma_0, X), g(\omega) \in E^{\Sigma_0}F(\omega)$ μ-almost everywhere$\}$. Moreover, if $F(\cdot)$ is $P_{fc}(X)$-valued, then cl $\int_A E^{\Sigma_0}F(\omega)\,d\mu(\omega) =$ cl $\int_A F(\omega)\,d\mu(\omega)$. For details we refer to Hiai and Umegaki [**18**].

Let $\{\Sigma_n\}_{n\geq 1}$ be an increasing sequence of complete sub-σ-fields of Σ and assume that $\Sigma = \sigma(\cup_{n\geq 1}\Sigma_n)$. A sequence of multi-functions $F_n : \Omega \to P_f(X)$, $n \geq 1$, is said to be *adapted to* Σ_n if, for every $n \geq 1$, $F_n(\cdot)$ is Σ_n-measurable. An adapted sequence $\{F_n, \Sigma_n\}_{n\geq 1}$ as above is said to be multi-valued *martingale*, respectively *submartingale*, *supermartingale*, if $E^{\Sigma_n}F_{n+1}(\omega) = F_n(\omega)$ μ-almost everywhere, respectively $E^{\Sigma_n}F_{n+1}(\omega) \supseteq F_n(\omega)$ μ-almost everywhere $E^{\Sigma_n}F_{n+1}(\omega) \subseteq F_n(\Omega)$ μ-almost everywhere, for every $n \geq 1$. A function $\tau : \Omega \to N \cup \{+\infty\}$ is said to be a *stopping time* with respect to $\{\Sigma_n\}_{n\geq 1}$ if, for every $n \geq 1$, $\{\tau = n\} = \{\omega \in \Omega : \tau(\omega) = n\} \in \Sigma_n$. The set of all stopping times is denoted by T^*.

We can partially order T^* in the obvious pointwise way, i.e., $\tau_1 \leq \tau_2$ if and only if $\tau_1(\omega) \leq \tau_2(\omega)$ for every $\omega \in \Omega$. By T we will denote the subset of T^* consisting of all bounded stopping times. So $\tau \in T$ if and only if $\tau \in T^*$ and $\tau(\Omega)$ is a finite set in N. The order induced on T by T^* has the property that N is cofinal in T. Given $\tau \in T$ we define $A_\tau = \{A \in \Sigma : A \cap \{\tau = n\} \in \Sigma_n$ for every $n \geq 1\}$. Then $\{\Sigma_\tau\}_{\tau\in T}$ is an increasing net of complete sub-σ-fields of Σ. If Σ_n denotes the totality of events observed before the deterministic time $n \geq 1$, then Σ_τ consists of the events observed before the random time τ. For $t \in T$ we define $F_\tau(\omega) = F_{\tau(\omega)}(\omega) = \sum_{n=\min\tau}^{n=\max\tau} F_n(\omega)\chi_{\{\tau=n\}}(\omega)$.

It is clear that $F_\tau : \Sigma \to P_f(X)$ is Σ_τ-measurable. In analogy with the vector-valued case, see, for example Egghe [**15**], we introduce the following multi-valued discrete-time random processes.

So let $\{F_n, \Sigma_n\}_{n\geq 1}$ be an adapted sequence. We define:

(a) $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued *uniform amart* if

$$\lim_{\sigma\in T}\sup_{\tau\in T(\sigma)}\Delta(F_\sigma, E^{\Sigma_\sigma}F_\tau) = 0$$

where $\Delta(F_\sigma E^{\Sigma_\sigma}F_\tau) = \int_\Omega h(F_\sigma(\omega),\, E^{\Sigma_\sigma}F_\tau(\omega)\, d\mu(\omega)$ and $T(\sigma) = \{\tau \in T : \tau \geq \sigma\}$.

(b) $\{F_n, \Sigma_n\}_{n\geq 1}$ is said to be a multi-valued *amart* if the net $\{\mathrm{cl}\int_\Omega F_\tau\, d\mu\}_{\tau\in T}$ is h-convergent, i.e., convergent in $P_f(X)$ for the Hausdorff generalized metric on $P_f(X)$, see below. Since $(P_f(X), h)$ is complete, there is a $C \in P_f(X)$ such that $h(\mathrm{cl}\int_\Omega F_\tau\, d\mu, C) \to 0$ for $\tau \in T$.

(c) $\{F_n, \Sigma_n\}_{n\geq 1}$ is said to be a *multi-valued weak sequential amart* (WS-amart for short), if, for every increasing sequence $\{\tau_n\}_{n\geq 1}$ in T (not necessarily cofinal) we have, for every $x^* \in X^*$ the sequence $\{\sigma(x^*, \int_\Omega F_{\tau_n}\, d\mu)\}_{n\geq 1}$ converges.

For $l \in N$, let T^l be the set of all bounded stopping times having at most l-values. Clearly T^l is a directed set filtering to the right with the order that it inherits from T. Also, if $l_1 \leq l_2$, then $T^{l_1} \subseteq T^{l_2}$ and $T = \cup_{l\geq 1}T^l$.

We also define

(d) $\{F_n, \Sigma_n\}_{n\geq 1}$ is said to be a multi-valued *amart of order* l, if the net $\{\mathrm{cl}\int_\Omega F_\tau\, d\mu\}_{\tau\in T^i}$ h-converges.

We will say that $\{F_n, \Sigma_n\}_{n\geq 1}$ is an *amart of finite order*, if it is an amart of order l for every $l \in N$.

Let UAM, respectively AM, WSAM, AM^l, AM$^\infty$, denote the adapted families $\{F_n, \Sigma_n\}_{n\geq 1}$, which are multi-valued *uniform amarts*, respectively amarts, weak sequential amarts, amarts of order l and amarts of infinite order. It is clear that UAM $\subseteq$ AM $\subseteq$ WSAM, AM$^\infty = \cap_{l\geq 1}$ AM^l and UAM $\subseteq$ AM $\subseteq$ AM$^\infty \subseteq$ AM^l for every $l \geq 1$. The inclusions can be strict. In fact, the inclusions AM $\subseteq$ AM$^\infty \subseteq$ AM^l can be strict even in the single-valued and R-valued case.

For an analysis of the corresponding single-valued notions we refer to Egghe [**15**], who treats uniform amarts, amarts and weak sequential amarts as well as several other types of stochastic processes, and to Luu [**22**], who introduced amarts of order $l \geq 1$ and of infinite order.

In what follows by $\mathcal{L}^1_f(\Sigma, X)$ we will denote the set of all equivalence classes of integrably bounded multi-functions $F : \Omega \to P_f(X)$, where two multi-functions F_1 and F_2 are considered to be identical if and only if $F_1(\omega) = F_2(\omega)$, μ almost everywhere. Equipped with the metric $\Delta(F, G) = \int_\Omega h(F(\omega), G(\omega))\, d\mu(\omega)$, $\mathcal{L}^1_f(\Sigma, X)$ becomes a complete metric space. Similarly, we can define $\mathcal{L}^1_{fc}(\Sigma, X)$ and $\mathcal{L}^1_{wkc}(\Sigma, X)$. Note that $\mathcal{L}^1_{fc}(\Sigma, X)$ is a closed subspace of the metric space $(\mathcal{L}^1_f(\Sigma, X), \Delta)$, hence $(\mathcal{L}^1_{fc}(\Sigma, X), \Delta)$, itself a complete metric space. Also, for X^*-valued we can define in a similar way the space $\mathcal{L}^1_{w^*kc}(\Sigma, X)$.

Now we will introduce the modes of set convergence that we will be using in the sequel. So let $\{A_n\}_{n\geq 1} \subseteq 2^X \backslash \{\varnothing\}$ and define

$$\begin{aligned}\underline{\lim}\, A_n &= \{x \in X : \lim d(x, A_n) = 0\} \\ &= \{x \in X : x = \lim x_n, x_n \in A_n, n \geq 1\}\end{aligned}$$

and

$$\begin{aligned}w - \overline{\lim}\, A_n = \{x \in X : x = w - \lim x_{n_k},& \\ x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}.&\end{aligned}$$

Here w-denotes the weak topology on the Banach space X. Evidently we always have $\underline{\lim}\, A_n \subseteq w - \overline{\lim}\, A_n$. We say that A_n's convergence to A in the *Mosco sense*, denoted by $A_n \xrightarrow{M} A$, if and only if $\underline{\lim}\, A_n = w - \overline{\lim}\, A_n = A$. We say that A_n's convergence to A *weakly* (or scalarly), denoted by $A_n \xrightarrow{w} A$ if and only if, for every $x^* \in X^*$, $\sigma(x^*, A_n) \to \sigma(x^*, A)$. We say that the A_n's convergence to A in the *Wijsman sense*, denoted by $A_n \xrightarrow{W} A$ if and only if, for every $x \in X$, $d(x, A_n) \to d(x, A)$.

Recall that on $P_f(X)$ we can define a generalized metric, known in the literature as the *Hausdorff metric* (or *Hausdorff distance of A and B*), by setting for $A, B \in P_f(X)$,

$$h(A, B) = \max\left[\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right]$$

with $d(a, B) = \inf\{\|a - b\| : b \in B\}$ and $d(b, A) = \inf\{\|b - a\| : a \in A\}$.

It is well known that $(P_f(X), h)$ is a complete metric space and $(P_k(X), h)$ is a closed and separable subspace of it. We know that

$h(\cdot,\cdot)$ is a pseudo-metric on 2^X with $\varnothing$ an isolated point. Moreover, if $A, B \in P_f(X)$ and are also bounded, then

$$h(A,B) = \sup[|\sigma(x^*, A) - \sigma(x^*, B)| : \|x^*\| \leq 1].$$

We say that A_n's converge to A in the "Hausdorff sense," denoted by $A_n \xrightarrow{h} A$ if and only if $h(A_n, A) \to 0$. In this paper we will be dealing with sequences in $P_{fc}(X)$. Within this context h-convergence implies W-convergence and, if the sets are also bounded, h-convergence implies M-convergence and w-convergence. In general M-convergence and w-convergence are disjoint notions. In fact, M-convergence of sets corresponds to a variational convergence of their support functions, known as the *epigraphical* or *Mosco convergence of functions*, see Attouch [**2**] for analysis in the context of Banach spaces and Zabell [**41**] for extensions to locally convex spaces. In reflexive Banach spaces M-convergence implies W-convergence, while the converse is true if, in addition, X is strictly convex and has the Kadec-Klee property, i.e., if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ in X, see Attouch [**2**, p. 322] or Tsukada [**34**, p. 305]. Finally, if X is finite dimensional and the sets are bounded, i.e., belong in $P_{kc}(X)$, then the above four modes of set-convergence coincide. Note that in this case the Mosco convergence reduces to the classical Kuratowski convergence of sets.

Finally a Hausdorff topological space Z is said to be *Polish* if it is homeomorphic to a separable complete metric space. A *Souslin* space is a Hausdorff topological space V which is the continuous image of a Polish space. So a Souslin space is always separable, but need not be metrizable. Consider, for example, an infinite dimensional separable Banach space X furnished with the weak topology. More generally, if X is a separable Banach, then $X^*_{w^*}$, the dual X^* endowed with the w^*-topology, is Souslin. To see that, let $\overline{B}^*_1 = \{x^* \in X^* : \|x^*\| \leq 1\}$. As is well known, $\overline{B}^*_1$ equipped with the relative w^*-topology is compact metrizable, in particular Souslin. Since $X^*_{w^*} = \cup_{n\geq 1}\overline{B}^*_1$ and countable unions of Souslin subspaces of a Hausdorff topological space is Souslin, we conclude that $X^*_{w^*}$ is indeed Souslin.

Two comparable Souslin topologies generate the same Borel σ-field. Finally, a Souslin subspace of a Polish space is also called *analytic*.

3. Submartingales. Throughout the rest of this paper (Ω, Σ, μ)

is a complete probability space, $\{\Sigma_n\}_{n\geq 1}$ is an increasing sequence of complete sub-σ-fields of Σ such that $\Sigma = \sigma(\cup_{n\geq 1}\Sigma_n)$ and X is a separable Banach space. Additional hypotheses will be introduced as needed.

In this section first we prove a multi-valued Dunford-Pettis type theorem (similar to the ones established in [**31**]) and then, as a consequence of it, we get a submartingale convergence theorem, extending an earlier result due to Neveu [**26**] which dealt with multi-valued martingales. A similar extension of Neveu's result to w^*-amarts can be found in Bagchi [**3**].

In analogy with the weak convergence of sets introduced in Section 2, we say that a sequence $\{F_n\}_{n\geq 1} \subseteq \mathcal{L}^1_{w^*kc}(X^*)$ converges to $F \in \mathcal{L}^1_{w^*kc}(X^*)$ in the w^*-sense in $\mathcal{L}^1_{w^*kc}(X^*)$ denoted by $F_n \overset{w^*}{\to} F$ in $\mathcal{L}^1_{w^*kc}(X^*)$ if and only if, for every $u \in L^\infty(\Omega, X)$, we have $\int_\Omega \sigma(u(\omega), F_n(\omega))\, d\mu(\omega) \to \int_\Omega \sigma(u(\omega), F(\omega))\, d\mu(\omega)$ as $n\to\infty$. Since

$$\sigma(u, S^1_{F_n}) = \int_\Omega \sigma(u(\omega), F_n(\omega))\, d\mu(\omega)$$

and

$$\sigma(u, S^1_F) = \int_\Omega \sigma(u(\omega), F(\omega))\, d\mu(\omega)$$

we see that this notion is a w^*-variant of the weak mode of convergence of sets in $L^1(\Omega, X^*)$.

Theorem 3.1. *If X^* is separable, $\{F_n\}_{n\geq 1} \subseteq \mathcal{L}^1_{w^*kc}(X^*)$ and $\sup_{n\geq 1} |F_n(\cdot)| = \vartheta(\cdot) \in L^1(\Omega)$, then there exists a subsequence $\{F_{n_m} = F_m\}_{m\geq 1}$ of $\{F_n\}_{n\geq 1}$ and $F \in \mathcal{L}^1_{w^*kc}(X^*)$ such that $F_m \overset{w^*}{\to} F$ in $\mathcal{L}^1_{w^*kc}(X^*)$.*

Proof. For every $n \geq 1$, let $f^k_n : \Omega \to X^*$, $k \geq 1$, be a sequence of measurable functions such that $F_n(\omega) = \overline{\{f^k_n(\omega)\}}_{k\geq 1}$ for every $\omega \in \Omega$, cf. Section 2.

Fix $k \geq 1$. We know that $\mathrm{Gr} f^k_n \in \Sigma_n \times B(X)$, $n \geq 1$, and so by Lemma 3, [**21**, p. 110], of Levin, we can find $\Sigma'_n \subseteq \Sigma_n$, a sub-$\sigma$-field which is countably generated and such that $\mathrm{Gr} f_n^k \in \Sigma_n' \times B(X)$, $n \geq 1$. Let $\Sigma' = \sigma(\cup_{n\geq 1}\Sigma_n')$. Evidently, Σ' is a countably generated sub-σ-field of Σ. Let $\Im$ be a countable field generating Σ'. Then by a standard Cantor diagonal process we can find a subsequence $\{F_{n_m} = F_m\}_{m\geq 1}$ of $\{F_n\}_{n\geq 1}$ such that, for every $A \in \Im$ and every $k \geq 1$, we have $\int_A f_n^k(\omega)\, d\mu(\omega) \xrightarrow{w^*} m_k(A)$ in X^* as $n \to \infty$.

Since, by hypothesis, $\omega \to \vartheta(\omega) = \sup_{n\geq 1} |F_n(\omega)|$ belongs in $L^1(\Omega)$, given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that if $C \in \Sigma$ with $\mu(C) \leq \delta$, then $\int_C \vartheta(\omega)\, d\mu(\omega) \leq \varepsilon$. Letting $A \in \Sigma'$, we can find $A' \in \Im$ such that $\mu(A\Delta A') \leq \delta$. Then

$$\left\| \int_A f_m^k(\omega)\, d\mu(\omega) - \int_{A'} f_m^k(\omega)\, d\mu(\omega) \right\| \leq \int_{A\Delta A'} \vartheta(\omega)\, d\mu(\omega) \leq \varepsilon.$$

Therefore, for every $A \in \Sigma'$ and every $k \geq 1$,

$$w^* - \lim_{m\to\infty} \int_A f_m^k(\omega)\, d\mu(\omega) = m_k(A)$$

exists.

Moreover, from Egghe [**15**, p. 100], we know that for every $k \geq 1$, $m_k : \Sigma' \to X^*$ is a vector measure, which clearly is absolutely continuous with respect to μ. In addition, for every $k \geq 1$, $|m_k|(\Omega) \leq \sup_{n\geq 1} \int_\Omega |F_n(\omega)|\, d\mu(\omega)$ and so $m_k(\cdot)$, $k \geq 1$ is of bounded variation.

Since X^* is separable, it has the RNP (Dunford-Pettis theorem; see Diestel and Uhl [**12**, p. 79]). So we can find $f_k' \in L^1(\Sigma', X^*)$ such that $m_k(A) = \int_A f_k'(\omega)\, d\mu(\omega)$ for every $A \in \Sigma'$ and $k \geq 1$.

Exploiting the fact that every element $L^\infty(\Sigma', X)$ is the uniform limit of countably-valued functions, see Diestel and Uhl [**12**, p. 42], by an easy density argument we can show that $\langle f_m^k, u\rangle \to \langle f_m', u\rangle$ as $m \to \infty$, with $\langle \cdot, \cdot\rangle$ denoting the duality brackets for the pair $(L^1(\Sigma', X^*), L^\infty(\Sigma', X))$, i.e., $\langle g, v\rangle = \int_\Omega (g(\omega), v(\omega))\, d\mu(\omega)$ for every $g \in L^1(\Sigma', X^*)$ and every $v \in L^\infty(\Sigma', X)$.

Now let $A \in \Sigma$. Then for every $k \geq 1$ we have

$$\begin{aligned}\int_A f_m^k(\omega)\,d\mu(\omega) &= \int_\Omega \chi_A(\omega) f_m^k(\omega)\,d\mu(\omega)\\ &= \int_\Omega E^{\Sigma'}(\chi_A(\omega) f_m^k(\omega))\,d\mu(\omega)\\ &= \int_\Omega (E^{\Sigma'}(\chi_A(\omega)) f_m^k(\omega))\,d\mu(\omega).\end{aligned}$$

For every $x \in X$, $(E^{\Sigma'}\chi_A(\cdot))x \in L^\infty(\Sigma', X)$ and so

$$\begin{aligned}&\left(\int_\Omega (E^{\Sigma'}\chi_A(\omega)) f_m^k(\omega)\,d\mu(\omega), x\right)\\ &\qquad\qquad = \left(\int_\Omega (f_m^k(\omega), (E^{\Sigma'}\chi_A(\omega))x\right) d\mu(\omega)\\ &\qquad\qquad = \langle f_m^k, (E^{\Sigma'}\chi_A)x\rangle.\end{aligned}$$

By what we proved earlier, we have

$$\langle f_m^k, (E^{\Sigma'}\chi_A)x\rangle \longrightarrow \langle f_k', (E^{\Sigma'}\chi_A)x\rangle \quad \text{as } m \to \infty.$$

So we get that, for every $k \geq 1$ and every $A \in \Sigma$,

$$w^* - \lim_{m\to\infty} \int_A f_m^k(\omega)\,d\mu(\omega) = m_k(A)$$

exists and, as before, $m_k(A) = \int_A f_k(\omega)\,d\mu(\omega)$ for every $A \in \Sigma$ and some $f_k \in L^1(\Sigma, X^*)$.

Evidently, $E^{\Sigma'} f_k = f_k'$, $k \geq 1$. Moreover, for every $u \in L^\infty(\Sigma, X)$ we have $\langle f_m^k, u\rangle \to \langle f_k, u\rangle$ with $\langle\cdot,\cdot\rangle$ now denoting the duality brackets for the pair $(L^1(\Sigma, X^*), L^\infty(\Sigma, X))$. □

Set $= \overline{\text{conv}}^{w^*}\{f_k(\omega)\}_{k\geq 1}$, $\omega \in \Omega$. By modifying this on a μ-null set, we have that $F(\omega)$ is w^*-compact, convex in X^*. Also, since a bounded set in X^* is metrizable, we see that $\text{Gr}F \in \Sigma \times B(X^*_{w^*})$. But, since X^* is separable and $X^*_{w^*}$ is Souslin, $B(X^*_{w^*}) = B(X^*)$, cf. Section 2. So $\omega \to F(\omega)$ is graph measurable, thus measurable, cf. Section 2. Hence, $F \in \mathcal{L}^1_{w^*kc}(X^*)$. We claim that $F_m \xrightarrow{w^*} F$ in $\mathcal{L}^1_{w^*kc}(X^*)$.

Let $f \in S_F^1$. According to Lemma 1.3 of Hiai and Umegaki [**18**, p. 153], given $\varepsilon > 0$ we can find $\{A_k\}_{k\geq 1}^N$ a finite Σ-partition of Ω such that $\|f - \sum_{k=1}^N \chi_{A_k} f_k\|_1 \leq \varepsilon/2$. So, for $x \in \overline{B}_1 = \{z \in X : \|z\| \leq 1\}$ and $A \in \Sigma$, we have

$$\left| \int_A \left(f(\omega) - \sum_{k=1}^N \chi_{A_k}(\omega) f_k(\omega), x \right) d\mu(\omega) \right| \leq \frac{\varepsilon}{2}.$$

Since

$$\int_A \left(\sum_{k=1}^N \chi_{A_k}(\omega) f_m^k(\omega), x \right) d\mu(\omega) \longrightarrow \int_A \left(\sum_{k=1}^N \chi_{A_k}(\omega) f_k(\omega), x \right) d\mu(\omega)$$

as $m \to \infty$, we can find $m_0 = m_0(\varepsilon) \geq 1$ such that, for $m \geq m_0$, we have

$$\left| \int_A \left(f(\omega) - \sum_{k=1}^N \chi_{A_k}(\omega) f_m^k(\omega), x \right) d\mu(\omega) \right| \leq \varepsilon.$$

Observe that $\sum_{k=1}^N \chi_{A_k}(\cdot) f_m^k(\cdot) \in S_{F_m}^1$. So we see that, given $f \in S_F^1$ and $\varepsilon > 0$, we can find $m_0 \geq 1$ such that, for $m \geq m_0$ there exists $f_m \in S_{F_m}^1$ for which we have $|\int_A ((f(\omega) - f_m(\omega)), x)\, d\mu(\omega)| \leq \varepsilon$; hence

$$\int_A (f(\omega), x)\, d\mu(\omega) - \varepsilon \leq \int_A (f_m(\omega), x)\, d\mu(\omega).$$

Now choose $f \in S_F^1$ such that

$$\sigma\left(x, \int_A F(\omega)\, d\mu(\omega) \right) - \varepsilon \leq \int_A (f(\omega), x)\, d\mu(\omega).$$

So we have, for $m \geq m_0$,

$$\sigma(x, \int_A F(\omega)\, d\mu(\omega)) - \varepsilon \leq \int_A (f(\omega), x)\, d\mu(\omega) \leq \int_A (f_m(\omega), x)\, d\mu(\omega) + \varepsilon,$$

and so $\sigma(x^*, \int_A F(\omega)\, d\mu(\omega)) - 2\varepsilon \leq \sigma(x^*, \int_A F_m(\omega)\, d\mu(\omega))$. Since $\varepsilon > 0$ was arbitrary, we deduce that, for every $x \in X$ and $A \in \Sigma$, we have

$$\sigma\left(x, \int_A F(\omega)\, d\mu(\omega) \right) \leq \varliminf_{m\to\infty} \sigma\left(x, \int_A F_m(\omega)\, d\mu(\omega) \right). \tag{1}$$

On the other hand, given $\varepsilon > 0$, $x \in \overline{B}_1$ and $A \in \Sigma$, for every $m \geq 1$ find $f_m \in S^1_{F_m}$ such that $\sigma(x, \int_A F_m(\omega)\, d\mu(\omega)) - \varepsilon \leq \int_A (f_m(\omega)\, d\mu(\omega), x)$. As above, via Lemma 1.3 of [**18**] we can find $\{A_k\}_{k\geq 1}^N$ a Σ-partition of Ω such that

$$\sigma\bigg(x, \int_A F_m(\omega)\, d\mu(\omega)\bigg) - \varepsilon \leq \bigg(\int_A \sum_{k=1}^N \chi_{A_k}(\omega) f_m^k(\omega)\, d\mu(\omega), x\bigg) + \varepsilon$$

and so

$$\begin{aligned}\overline{\lim_{m\to\infty}}\, \sigma(x, \int_A F_m(\omega)\, d\mu(\omega)) - 2\varepsilon &\leq \bigg(\int_A \sum_{k=1}^N \chi_{A_k}(\omega) f_m(\omega)\, d\mu(\omega), x\bigg) \\ &\leq \sigma\bigg(\int_A F(\omega)\, d\mu(\omega)\bigg).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce for every $x \in X$ and every $A \in \Sigma$ we have

$$\overline{\lim_{m\to\infty}}\, \sigma\bigg(x, \int_A F_m(\omega)\, d\mu(\omega)\bigg) \leq \sigma\bigg(x, \int_A F(\omega)\, d\mu(\omega)\bigg). \tag{2}$$

From (1) and (2) above, we have that

$$\begin{aligned}\lim_{m\to\infty} \sigma\bigg(x, \int_A F_m(\omega)\, d\mu(\omega)\bigg) &= \lim_{m\to\infty} \int_A \sigma(x, F_m(\omega))\, d\mu(\omega) \\ &= \int_A \sigma(x, F(\omega))\, d\mu(\omega) \\ &= \sigma\bigg(x, \int_A F(\omega)\, d\mu(\omega)\bigg).\end{aligned}$$

Since $x \in X$ and $A \in \Sigma$ are arbitrary and $\sigma(\cdot, F_m(\omega)), \sigma(\cdot, F(\omega))$, $m \geq 1$, $\omega \in \Omega$ are continuous on X, by a standard density argument involving Corollary 3 of Diestel and Uhl [**12**, p. 42], we can finally conclude that $\int_\Omega \sigma(u(\omega), F_m(\omega))\, d\mu(\omega) \to \int_\Omega \sigma(u(\omega), F(\omega))\, d\mu(\omega)$ as $m \to \infty$ for every $u \in L^\infty(\Omega, X)$ and so $F_n \xrightarrow{w^*} F$ in $\mathcal{L}^1_{w^*kc}(X^*)$. $\square$

Using this theorem we can now have the following extension of Theorem 3 of Neveu [**26**, p. 3] who considers multi-valued martingales.

Another extension to multi-valued w^*-amarts can be found in Bagchi [**3**], Theorems 2.2 and 2.4.

Proposition 3.2. *If $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued submartingale in $\mathcal{L}^1_{w^*kc}(X^*)$ and $\sup_{n\geq 1} |||F_n|||_1 < \infty$, then there exists $F \in \mathcal{L}^1_{w^*kc}(\Sigma, X^*)$ such that $F_n(\omega) \xrightarrow{w^*} F(\omega)$ μ-almost everywhere, i.e., there exists a μ-null set N such that, for $\omega \in \Omega \backslash N$ and for every $x \in X$ we have $\sigma(x, F_n(\omega)) \to \sigma(x, F(\omega))$ as $n \to \infty$.*

Proof. First assume that $\omega \to \sup_{n\geq 1} |F_n(\omega)| = \vartheta(\omega) \in L^1(\Omega)$. From Theorem 3.1 we know that there exists a subsequence $\{F_{n_m} = F_m\}_{m\geq 1}$ of $\{F_n\}_{n\geq 1}$ and $F \in \mathcal{L}^1_{w^*kc}(X^*)$ such that $F_m \xrightarrow{w^*} F$ in $\mathcal{L}^1_{w^*kc}(X^*)$. On the other hand, if D is a countable dense subset of X, let $\hat{D} = \mathrm{span}_Q D$, the "rational span" of D. Evidently, $\hat{D}$ is still countable and of course dense in X. From Doob's submartingale convergence theorem, we know that, for all $\omega \in \Omega \backslash N$, $\mu(N) = 0$ and all $x \in \hat{D}$ we have $\sigma(x, F_n(\omega)) \to \varphi(\omega, x)$.

By setting $\varphi(\omega, x) = 0$ for $\omega \in N$ we see that $\omega \to \varphi(\omega, x)$ is measurable from Ω into R. Also for every $\omega \in \Omega$ and every $x, y \in \hat{D}$ we have $|\varphi(\omega, x) - \varphi(\omega, y)| \leq \lim |\sigma(x - y, F_n(\omega))| \leq \|x - y\| \vartheta(\omega)$.

Assuming without loss of generality that, for every $\omega \in \Omega$, $\vartheta(\omega)$ is finite, we conclude that $\varphi(\omega, \cdot)|_{\hat{D}}$, $\omega \in \Omega$ is uniformly continuous and so it has a unique continuous extension on all of X. Note that, for every $x \in X$ and every $A \in \Sigma$, we have

$$\int_A \sigma(x, F(\omega))\, d\mu(\omega) = \int_A \varphi(x, \omega)\, d\mu(\omega)$$

and so $\sigma(x, F(\omega)) = \varphi(x, \omega)$ for every $\omega \in \Omega \backslash N_1$, $\mu(N_1) = 0$ and every $x \in \hat{D}$ and then, by continuity for $\omega \in \Omega \backslash N_1$ and every $x \in X$. Therefore, $F_n(\omega) \xrightarrow{w^*} F(\omega)$, μ almost everywhere.

Now we pass to the general case according to which $\sup_{n\geq 1} |||F_n|||_1 < \infty$.

Using a standard stopping time technique we will reduce it to the case $\omega \to \sup_{n\geq 1} |F_n(\omega)| = \vartheta(\omega)$ belongs in $L^1(\Omega)$, considered above. Fix $\lambda > 0$ and $A_\lambda = \{\omega \in \Omega : \vartheta(\omega) \leq \lambda\}$.

Define $\tau \in T^*$ by

$$\tau(\omega) = \begin{cases} +\infty & \text{if } \omega \in A_\lambda \\ \min[n \geq 1 : |F_n(\omega)| > \lambda] & \text{if } \omega \in A_\lambda^c. \end{cases}$$

We claim that $\sup_{n\geq 1} |F_{n\wedge\tau}(\cdot)| \in L^1(\Omega)$. Indeed, note that, on $\{\tau < \infty\}$ we have $\lim_{n\to\infty} |F_{n\wedge\tau}| = |F_\tau|$.

So, via Fatou's lemma, we have

$$\begin{aligned}
\int_{\{\tau<\infty\}} |F_\tau(\omega)|\, d\mu(\omega) &\leq \varliminf_{n\to\infty} \int_{\{\tau<\infty\}} |F_{n\wedge\tau}(\omega)|\, d\mu(\omega) \\
&\leq \varliminf \int_\Omega |F_{n\wedge\tau}(\omega)|\, d\mu(\omega) \\
&\leq \varliminf_{n\to\infty} \int_\Omega |F_n(\omega)|\, d\mu(\omega) \\
&\leq \sup_{n\geq 1} \| |F_n| \|_1 < \infty.
\end{aligned}$$

The third inequality is a consequence of the optional sampling theorem for submartingales, since $\{|F_n|, \Sigma_n\}_{n\geq 1}$ is an R_+-valued submartingale.

Thus, we have

$$\begin{aligned}
\int_\Omega \sup_{n\geq 1} |F_{n\wedge\tau}(\omega)|\, d\mu(\omega) &= \int_{\{\tau=\infty\}} \sup_{n\geq 1} |F_{n\wedge\tau}(\omega)|\, d\mu(\omega) \\
&\quad + \int_{\{\tau<\infty\}} \sup_{n\geq 1} |F_{n\wedge\tau}(\omega)|\, d\mu(\omega) \\
&\leq \lambda + \sup_{n\geq 1} \| |F_n| \|_1 < \infty.
\end{aligned}$$

Since $\{|F_n|, \Sigma_n\}_{n\geq 1}$ is a submartingale, from the maximal inequality we have $\mu\{\omega \in \Omega : \sup_{n\geq 1} |F_n(\omega)| \geq \lambda\} \leq (1/\lambda) \sup_{n\geq 1} \| |F_n| \|_1$. So $\{F_{n\wedge\tau}\}_{n\geq 1}$ and $\{F_n\}_{n\geq 1}$ coincide outside a set whose μ-measure can become arbitrarily small as $\lambda \uparrow \infty$.

Therefore, without any loss of generality, we can assume that $\sup_{n\geq 1} |F_n(\cdot)| = \vartheta(\cdot) \in L^1(\Omega)$. □

Remark 3.3. Multi-valued supermartingales are considered in de Korvin and Kleyle [**20**], Papageorgiou [**28**], Hess [**17**] and in the recent interesting work of Wang and Xue [**38**].

4. Uniform amarts. We start this section with a weak convergence theorem for vector-valued random variables which is actually of independent interest and can be a useful tool in obtaining convergence theorems for random processes without assuming that the range space has the RNP. Our result extends an earlier one due to Brunel and Sucheston [**7**] and in addition our proof is simpler.

Recall that since, by hypothesis, X is separable, X^* furnished with the Mackey topology $m(X^*, X)$ is separable too. Let D_1^* be a countable m-dense subset of $\overline{B}_1^* = \{x^* \in X^* : \|x^*\| \leq 1\}$.

Proposition 4.1. *If* $\{f_n\}_{n\geq 1} \subseteq L^1(\Omega, X)$, $\underline{\lim}\, \|f_n\|_1 < \infty$, $\overline{\{f_n(\omega)\}}^w_{n\geq 1} \in P_{wk}(X)$, *$\mu$ almost everywhere, and for every* $x^* \in D_1^*$, $\lim(x^*, f_n(\omega))$ *exists for μ almost all $\omega \in \Omega$, then there exists an* $f \in L^1(\Omega, X)$ *such that* $f_n(\omega) \xrightarrow{w} f(\omega)$, *$\mu$ almost everywhere in X as $n \to \infty$.*

Proof. Since D_1^* is countable, we can find $N \in \Sigma$, $\mu(N) = 0$ such that, for every $\omega \in \Omega \backslash N$ and every $x^* \in D_1^*$, $\lim_{n\to\infty}(x^*, f_n(\omega))$ exists $\overline{\{f_n(\omega)\}}^w_{n\geq 1} \in P_{wk}(X)$. We claim that this is true for every $x^* \in \overline{B}_1^*$.

Indeed, fix $\omega \in \Omega\backslash N$ and let $z^* \in \overline{B}_1^*$. Then, for $x^* \in D_1^*$ and $n, m \geq 1$, we have

$$\begin{aligned} |(z^*, f_n(\omega) - f_m(\omega))| \leq |(z^* &- x^*, f_n(\omega))| \\ &+ |(x^*, f_n(\omega) - f_m(\omega))| \\ &+ |(x^* - z^*, f_m(\omega))|. \end{aligned}$$

By hypothesis, $\lim(x^*, f_n(\omega) - f_m(\omega)) = 0$. So, given $\varepsilon > 0$ we can find $n_0(\varepsilon) \geq 1$ such that, for $n, m \geq n_0(\varepsilon)$ we have $|(x^*, f_n(\omega) - f_m(\omega))| \leq (\varepsilon/3)$. Also, since D_1^* is m-dense in $\overline{B}_1^*$ and $\overline{\{f_n(\omega)\}}^w_{n\geq 1} \in P_{wk}(X)$ from the definition of the Mackey topology $m(X^*, X)$, we can find $x^* \in D_1^*$ such that $|(z^* - x^*, f_n(\omega))| \leq \varepsilon/3$ and

$$|(x^* - z^*, f_m(\omega))| \leq \varepsilon/3 \quad \text{for every } n, m \geq 1.$$

Hence, for $n, m \geq n_0(\varepsilon)$, we have $|(z^*, f_n(\omega) - f_m(\omega))| \leq \varepsilon$ which shows that $\{(z^*, f_n(\omega))\}_{n\geq 1}$ is Cauchy in R and so $\lim_{n\to\infty}(z^*, f_n(\omega))$ exists for every $(\omega, z^*) \in \Omega\backslash N \times X^*$.

Applying the Banach-Steinhaus theorem we get $f : \Omega \to X$ weakly measurable such that $f_n(\omega) \xrightarrow{w} f(\omega)$. Because X is separable from the Pettis measurability theorem, we get that $f(\cdot)$ is strongly measurable. Moreover, using Fatou's lemma and the fact that the norm of X is weakly lower semi-continuous, we have $\|f\|_1 \leq \int_\Omega \underline{\lim} \|f_n(\omega)\| \, d\mu(\omega) \leq \underline{\lim} \|f\|_1 < \infty$. Thus, $f \in L^1(\Omega, X)$ and the proof is complete. $\square$

This result leads us to the following convergence theorem for a broad class of vector-valued random processes, which includes uniform amarts, hence martingales too.

Definition 4.2. An adapted sequence $\{f_n, \Sigma_n\}_{n \geq 1}$ in $L^1(\Omega, X)$ is said to be a *pramart* (short for "amart in probability"), if for every given $\varepsilon > 0$ there is a $\sigma_0 \in T$ such that if $\tau, \sigma \in T(\sigma_0)$ with $\tau \geq \sigma$ we have $\mu\{\omega \in \Omega : \|E^{\Sigma_\sigma} f_\tau(\omega) - f_\sigma(\omega)\| > \varepsilon\} \leq \varepsilon$, i.e., $\|E^{\Sigma_\sigma} f_\tau - f_\sigma\|$ goes to zero in probability for $\sigma \in T$, uniformly in $\tau \in T(\sigma)$.

Theorem 4.3. *If $\{f_n, \Sigma_n\}_{n \geq 1}$ is pramart such that $\underline{\lim} \|f_n\|_1 < \infty$ and $\overline{\cup_{n \geq 1} f_n(\omega)}^w \in P_{wk}(X)$, μ almost everywhere, then there exists $f \in L^1(\Omega, X)$ such that $f_n(\omega) \to f(\omega)$, μ almost everywhere.*

Proof. Evidently, for every $x^* \in X^*$, $\{(x^*, f_n), \Sigma_n\}_{n \geq 1}$ is an R-valued pramart and $\underline{\lim} \|(x^*, f_n)\|_1 < \infty$. So we can apply the convergence theorem of Millet and Sucheston [**25**] and deduce that, for every $x^* \in X^*$, $\lim(x^*, f_n(\omega))$ exists, μ almost everywhere.

Apply Proposition 4.1 to generate an $f \in L^1(\Omega, X)$ such that $f_n(\omega) \xrightarrow{w} f(\omega)$, μ almost everywhere in X as $n \to \infty$. Since X is separable we can apply the classical Kadec-Klee renorming theorem and get an equivalent norm $|\cdot|$ on X which has the Kadec-Klee property, cf. Section 2. Let $\{x_k^*\}_{k \geq 1}$ be a sequence which is dense in $\overline{B}_1^*$ (it exists because X is separable). Now observe that

$$\begin{aligned} &\lim_{\sigma \in T} \sup_{\tau \in T(\sigma)} \mu\{\omega \in \Omega : \sup_{k \geq 1}[(x_k^*, f_\sigma(\omega)) - (x_k^*, E^{\Sigma_\sigma} f_\tau(\omega))] > \varepsilon\} \\ &\qquad \leq \lim_{\sigma \in T} \sup_{\tau \in T(\sigma)} \mu\{\omega \in \Omega : |f_\sigma(\omega) - E^{\Sigma_\sigma} f_\tau(\omega)| > \varepsilon\} = 0, \end{aligned}$$

the last limit being zero since $\{f_n, \Sigma_n\}_{n \geq 1}$ is a pramart. This then allows us to apply Lemma 2.2 of Wang and Xue [**38**] and get that

$$\begin{aligned}\lim_{n\to\infty}(\sup_{k\geq 1}(x_k^*, f_n(\omega))) &= \lim_{n\to\infty}|f_n(\omega)| \\ &= \sup_{k\geq 1}(x_k^*, f(\omega)) = |f(\omega)|, \quad \mu - \text{a.e.}\end{aligned}$$

Recalling that $|\cdot|$ has the Kadec-Klee property, we conclude that $f_n(\omega) \to f(\omega)$ on X as $n \to \infty$. $\square$

Since a uniform amart, in particular a martingale, is a pramart, we can state the following extension of a well-known martingale convergence theorem due to Chatterji [**8**].

Corollary 4.4. *If $\{f_n, \Sigma_n\}_{n\geq 1}$ is a uniform amart in $L^1(\Omega, X)$ such that $\underline{\lim}\,\|f_n\|_1 < \infty$ and $\overline{\cup_{n\geq 1} f_n(\omega)}^w \in P_{wk}(X)$, μ almost everywhere, then there exists an $f \in L^1(\Omega, X)$ such that $f_n(\omega) \to f(\omega)$, μ almost everywhere, in X as $n \to \infty$.*

We will now extend this corollary to multi-valued uniform amarts. It should be pointed out that the first to consider multi-valued amarts was Bagchi [**3**]. Bagchi was able to extend the result of Neveu to w^*-amarts.

Theorem 4.5. (a) *If $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued uniform amart in $\mathcal{L}^1_{wkc}(X)$ such that $\sup_{n\geq 1}|||F_n|||_1 < \infty$ and $\overline{\cup_{n\geq 1} F_n(\omega)}^w \in P_{wk}(X)$, μ almost everywhere, then there exists $F \in \mathcal{L}^1_{wkc}(\Sigma, X)$ such that $F_n(\omega) \underset{w}{\overset{M}{\to}} F(\omega)$, μ almost everywhere.*

(b) *If $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued uniform amart in $\mathcal{L}^1_{wkc}(X)$ such that $\sup_{\tau\in T}|||F_\tau|||_1 < \infty$, i.e., is of class* (B), *and $\overline{\cup_{n\geq 1} F_n(\omega)}^w \in P_{wk}(X)$, μ almost everywhere, then there exists $F \in \mathcal{L}^1_{wkc}(\Sigma, X)$ such that $F_n(\omega) \underset{W}{\overset{M,w}{\to}} F(\omega)$, μ almost everywhere, and $|F_n(\omega)| \to |F(\omega)|$, μ almost everywhere.*

Proof. (a) From Theorem 2.1 of Luu, [**23**, p. 64], we know that, for every $n \geq 1$, $F_n(\omega) = \overline{\{f_n^k(\omega)\}}_{k\geq 1}$, μ almost everywhere, with $\{f_n^k, \Sigma_n\}_{n\geq 1}$, $k \geq 1$ being uniform amart in $L^1(\Omega, X)$ such that $f_n^k(\omega) \in$

$F_n(\omega)$, μ almost everywhere for every $n, k \geq 1$, i.e., $\{f_n^k, \Sigma_n\}_{n\geq 1}$, $k \geq 1$ is a sequence of uniform amart selectors of $\{F_n\}_{n\geq 1}$, denoted by $\{f_n^k\}_{n\geq 1} \in UAS(F_n)$ for every $k \geq 1$.

From Corollary 4.3 we know that, for every $k \geq 1$, there exists $f_k \in L^1(\Omega, X)$ such that $f_n^k(\omega) \to f_k(\omega)$, μ almost everywhere in X as $n \to \infty$. Set $F(\omega) = \overline{\text{conv}}\,\{f_k(\omega)\}_{k\geq 1} \in P_{wkc}(X)$, μ almost everywhere. For each $k \geq 1$, let $f_n^k = u_n^k + p_n^k$ be the Riesz decomposition of f_n^k with $\{u_n^k, \Sigma_n\}_{n\geq 1}$ being a vector-valued martingale and $\{p_n^k, \Sigma_n\}_{n\geq 1}$ is a uniform potential such that $\|p_n^k(\omega)\| \leq s_n(\omega)$, μ almost everywhere, for every $k \geq 1$ with $s_n(\omega) \to 0$, μ almost everywhere as $n \to \infty$, see Luu [**23**]. Hence, $u_n^k(\omega) \to f_k(\omega)$, μ almost everywhere, for every $k \geq 1$.

Now let D^* be a countable set dense in X^* for the $m(X^*, X)$-topology. Then, from Lemma 4 of Neveu [**26**, p. 4], we know that for every $(\omega, x^*) \in (\Omega \backslash N) \times D^*$, $\mu(N) = 0$, we have

$$\sup_{k\geq 1}(x^*, u_n^k(\omega)) \longrightarrow \sup_{k\geq 1}(x^*, f_k(\omega)) \quad \text{as } n \to \infty.$$

Note that

$$\begin{aligned}
\sigma(x^*, F_n(\omega)) &= \sup_{k\geq 1}(x^*, f_n^k(\omega)) \\
&\leq \sup_{k\geq 1}(x^*, u_n^k(\omega)) + \sup_{k\geq 1}(x^*, p_n^k(\omega)) \\
&\leq \sup_{k\geq 1}(x^*, u_n^k(\omega)) + \|x^*\| s_n(\omega) \\
&\longrightarrow \sup_{k\geq 1}(x^*, f_k(\omega)) \\
&= \sigma(x^*, F(\omega)).
\end{aligned}$$

So, for every $(\omega, x^*) \in (\Omega \backslash N) \times D^*$, $\mu(N) = 0$, we have

$$\overline{\lim_{n\to\infty}}\, \sigma(x^*, F_n(\omega)) \leq \sigma(x^*, F(\omega)).$$

Let $z^* \in X^*$ be arbitrary. Then, if

$$W(\omega) = \overline{\text{conv}}\left[\left(\bigcup_{n\geq 1} F_n(\omega)\right) \cup \left(-\bigcup_{n\geq 1} F_n(\omega)\right)\right] \in P_{wkc}(X),$$

for $(\omega, x^*) \in (\Omega \backslash N) \times D^*$, we have

$$\begin{aligned}\sigma(z^*, F_n(\omega)) - \sigma(z^*, F(\omega)) &= \sigma(z^*, F_n(\omega)) - \sigma(x^*, F_n(\omega)) \\ &\quad + \sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega)) \\ &\quad + \sigma(x^*, F(\omega)) - \sigma(z^*, F(\omega)) \\ &\leq 2\sigma(z^* - x^*, W(\omega)) \\ &\quad + \sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega)).\end{aligned}$$

Recalling that $\sigma(\cdot, W(\omega))$ is m-continuous, given $\varepsilon > 0$, we can find $x^* \in D^*$ such that $2\sigma(z^* - x^*, W(\omega)) \leq \varepsilon$. So we have

$$\overline{\lim}[\sigma(z, F_n(\omega)) - \sigma(z^*, F(\omega))] \leq \varepsilon + \overline{\lim}[\sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega))] \leq \varepsilon,$$

hence $\overline{\lim}\,\sigma(z^*, F_n(\omega)) \leq \sigma(z^*, F(\omega))$ for every $(\omega, z^*) \in (\Omega \backslash N) \times X^*$. Invoking Proposition 4.1 of [**29**], we get that

$$w - \overline{\lim}\, F_n(\omega) \subseteq F(\omega), \quad \mu\text{-a.e.} \tag{3}$$

On the other hand, note that, for every $k \geq 1$, we have that $f_k(\omega) \in \underline{\lim}\, F_n(\omega)$, μ almost everywhere.

Since the latter set is closed and convex, we get

$$F(\omega) \subset \underline{\lim}\, F_n(\omega), \quad \mu\text{-a.e.} \tag{4}$$

From (3) and (4) we get that $F_n(\omega) \xrightarrow{M} F(\omega)$, μ almost everywhere. Note that, for every $(\omega, z^*) \in (\Omega \backslash N_1) \times X^*$, $\mu(N_1) = 0$, we have

$$(z^*, f_k(\omega)) = \lim_{n\to\infty} (z^*, f_n^k(\omega)) \leq \underline{\lim}_{n\to\infty}\, \sigma(z^*, F_n(\omega)), \tag{5}$$

hence $\sigma(z^*, F(\omega)) \leq \underline{\lim}\,\sigma(z^*, F_n(\omega))$. Combining (5) with the fact that $\overline{\lim}\,\sigma(z^*, F_n(\omega)) \leq \sigma(z^*, F(\omega))$ for every $(\omega, z^*) \in (\Omega \backslash N) \times X^*$, $\mu(N) = 0$, we finally have that $F_n(\omega) \xrightarrow{w} F(\omega)$, μ almost everywhere.

(b) We already know from part (a) that there exists an $F \in \mathcal{L}^1_{wkc}(\Sigma, X)$ such that $F_n(\omega) \underset{w}{\xrightarrow{M}} F(\omega)$, μ almost everywhere.

Let $D_1^* = \{x_k^*\}_{k\geq 1}$ be a sequence in $\overline{B}_1^*$ dense for the Mackey topology $m(X^*, X)$. From the duality formula for the distance function, see Holmes [**19**, p. 62], for every $x \in X$ we have $d(x, F_n(\omega)) = \sup[(x_k^*, x) - \sigma(x_k^*, F_n(\omega))]$. Observe that

$$\lim_{\sigma \in T} \sup_{\tau \in T(\sigma)} \mu\{\omega \in \Omega : \sup_{k \geq 1}(\sigma(x_k^*, E^{\Sigma_\sigma} F_\tau(\omega)) - \sigma(x_k^*, F_\sigma(\omega))) > \varepsilon\}$$
$$\leq \lim_{\sigma \in T} \sup_{\tau \in T(\sigma)} \mu\{\omega \in \Omega : h(F_\sigma(\omega), E^{\Sigma_\sigma} F_\tau(\omega)) > \varepsilon\} = 0,$$

since $\{F_n, \Sigma_n\}_{n \geq 1}$ is a multi-valued amart. Thus, we can apply Lemma 2.2 of Wang and Xue [**38**] and get that, if D is a countable dense subset of X, for every $x \in D$ we have $d(x, F_n(\omega)) \to d(x, F(\omega))$, μ almost everywhere.

Let $y \in X$ be arbitrary. We have

$$\begin{aligned} |d(y, F_n(\omega)) - d(y, F(\omega))| &\leq |d(y, F_n(\omega)) - d(x, F_n(\omega))| \\ &\quad + |d(x, F_n(\omega)) - d(x, F(\omega))| \\ &\quad + |d(x, F(\omega)) - d(y, F(\omega))| \\ &\leq \|x - y\| \sup_{n \geq 1} |F_n(\omega)| \\ &\quad + |d(x, F_n(\omega)) - d(x, F(\omega))| \\ &\quad + \|x - y\| |F(\omega)|. \end{aligned} \tag{6}$$

Since by hypothesis $\{F_n\}_{n \geq 1}$ is of class (B), from the maximal inequality, see, for example Egghe [**15**, p. 23]; for every $\lambda > 0$, we have

$$\mu\{\omega \in \Omega : \sup_{n \geq 1} |F_n(\omega)| \geq \lambda\} \leq \frac{1}{\lambda} \sup_{\tau \in T} \int_\Omega |F_\tau(\omega)| \, d\mu(\omega) < \infty.$$

Thus $\mu\{\omega \in \Omega : \sup_{n \geq 1} |F_n(\omega)| = \infty\} = 0$. Hence, using (6) above, we see that, for all $(\omega, y) \in (\Omega \backslash N_1) \times X$, $\mu(N_1) = 0$ we have that $d(y, F_n(\omega)) \to d(y, F(\omega))$ and so $F_n(\omega) \xrightarrow{W} F(\omega)$, μ almost everywhere.

Finally note that, for every $n \geq 1$, $|F_n(\omega)| = \sup_{k \geq 1} \|f_n^k(\omega)\|$. Also, from Lemma 4 of Neveu [**26**, p. 4] we have $\sup_{k \geq 1} \|u_n^k(\omega)\| \to \sup_{k \geq 1} \|f^k(\omega)\|$, μ almost everywhere as $n \to \infty$. But note that

$$|F_n(\omega)| \leq \sup_{k \geq 1} \|u_n^k(\omega)\| + s_n(\omega) \to \sup_{k \geq 1} \|f_k(\omega)\|,$$

μ almost everywhere; hence

$$\overline{\lim}\,|F_n(\omega)| \le |F(\omega)|, \quad \mu\text{-a.e. .} \tag{7}$$

On the other hand, for every $k \geq 1$, we have

$$\|f_k(\omega)\| = \lim_{n\to\infty} \|f_n^k(\omega)\| \le \underline{\lim}\,|F_n(\omega)|, \quad \mu\text{-a.e.,}$$

and so

$$|F(\omega)| \le \underline{\lim}_{n\to\infty} |F_n(\omega)|, \quad \mu\text{-a.e. .}$$

Therefore, from (7) and (8) we conclude that $|F_n(\omega)| \to |F(\omega)|$, μ almost everywhere. $\square$

5. Weak sequential amarts. In this section we consider multi-valued weak sequential amarts. First we prove a weak convergence theorem for them and then establish a weak decomposition theorem. First we need to introduce a special class of sets that we will need in the sequel.

We give first the following definition.

Definition 5.1. By $\wp$ we denote the set

$$\begin{aligned}\wp = \{C \in 2^X \backslash \{\varnothing\} : \, & C \text{ is } w\text{-closed and} \\ & \text{for every } r > 0,\ C \cap \overline{B}_r \text{ is weakly compact}\}.\end{aligned}$$

Here $\overline{B}_r = \{x \in X : \|x\| \le r\}$. Also, let $\wp_c = \{C \in \wp : C \text{ is convex}\}$.

Remark 5.2. The family $\wp$ is closed under finite unions, arbitrary intersections and of course contains the weakly closed weakly locally compact subsets of X. If X is reflexive, then $\wp$ consists of all the nonempty and weakly closed subsets of X and $\wp_c = P_{fc}(X)$.

Theorem 5.3. *If X has the Radon-Nikodym property* (RNP)*, X^* is separable and $\{F_n, \Sigma_n\}_{n\ge1}$ is a multi-valued WS-amart in $\mathcal{L}^1_{fc}(X)$ which is of class* (B)*, i.e.,* $\sup_{\tau\in T}\int_\Omega |F_\tau(\omega)|\,d\mu(\omega) < \infty$*, and for* *every* $A \in \Sigma$, $\Gamma(A) = \overline{\cup_{n\geq 1}\int_A F_n(\omega)\,d\mu(\omega)}^w \in \wp$, *then there exists* $F \in \mathcal{L}^1_{fc}(\Sigma, X)$ *such that* $F_n(\omega) \xrightarrow{w} F(\omega)$, μ *almost everywhere.*

Proof. We claim that, for every $m \geq 1$, and every $A \in \Sigma_m$ and for every increasing sequence $\{\tau_n\}_{n\geq 1} \subseteq T$, the sequence

$$\left\{\mathrm{cl}\int_A F_{\tau_n}(\omega)\,d\mu(\omega)\right\}_{n\geq 1}$$

converges weakly in $P_{fc}(X)$. Indeed, define $\sigma_n = \chi_A\tau_n + \chi_{A^c}m$, $n \geq 1$. Then $\{\sigma_n\}_{n\geq 1}$ is an increasing sequence of stopping times in T. Since $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued WS-amart, the sequence $\{\mathrm{cl}\int_A F_{\sigma_n}(\omega)\,d\mu(\omega)\}_{n\geq 1}$ converges weakly in $P_{fc}(X)$. Since

$$\mathrm{cl}\int_\Omega F_{\sigma_n}(\omega)\,d\mu(\omega) = \mathrm{cl}\left[\int_A F_{\tau_n}(\omega)\,d\mu(\omega) + \int_{A^c} F_m(\omega)\,d\mu(\omega)\right],$$

we see at once that the sequence $\{\mathrm{cl}\int_A F_{\tau_n}(\omega)\,d\mu(\omega)\}_{n\geq 1}$ converges weakly in $P_{fc}(X)$. So, for every $A \in \cup_{n\geq 1}\Sigma_n$ and every $x^* \in X^*$, we have that $\sigma(x^*, \int_A F_{\tau_n}(\omega)\,d\mu(\omega)) \to \varphi(x^*, A)$.

First assume that $\omega \to \vartheta(\omega) = \sup_{n\geq 1}|F_n(\omega)|$ belongs in $L^1(\Omega)$. Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ be such that if $A \in \Sigma$ and $\mu(A) \leq \delta$, then $\int_A \vartheta(\omega)\,d\mu(\omega) \leq \varepsilon$.

Now let $A \in \Sigma$ and find $A' \in \cup_{n\geq 1}\Sigma_n$ such that $\mu(A\Delta A') \leq \delta$. Then we have

$$\begin{aligned}
&\left|\sigma\left(x^*, \int_A F_{\tau_n}(\omega)\,d\mu(\omega)\right) - \sigma\left(x^*, \int_{A'} F_{\tau_n}(\omega)\,d\mu(\omega)\right)\right| \\
&\qquad = \left|\int_A \sigma\left(x^*, F_{\tau_n}(\omega)\right)d\mu(\omega) - \int_{A'} \sigma(x^*, F_{\tau_n}(\omega)\right)d\mu(\omega)\right| \\
&\qquad \leq \int_{A\Delta A'} |\sigma(x^*, F_{\tau_n}(\omega))|\,d\mu(\omega) \\
&\qquad \leq \int_{A\Delta A'} \vartheta(\omega)\,d\mu(\omega) \leq \varepsilon, \quad n \geq 1.
\end{aligned}$$

Hence, $\lim \sigma(x^*, \int_A F_{\tau_n}(\omega)\,d\mu(\omega))$ exists for every $(A, x^*) \in \Sigma \times X^*$. Note that $x^* \to \varphi(x^*, A)$ is sublinear, while from the Vitali-Hahn-Saks theorem we have that $A \to \varphi(x^*, A)$ is a signed measure.

Also set $\Gamma_0(A) = \overline{\mathrm{conv}}\,[(\Gamma(A) \cap \|\vartheta\|_1 \overline{B}_1) \cup (-(\Gamma(A) \cap \|\vartheta\|_1 \overline{B}_1))] \in P_{wkc}(X)$, cf. Definition 5.1. Then $|\varphi(x^*, A)| \le \sigma(x^*, \Gamma_0(A))$ for every $(A, x^*) \in \Sigma \times X^*$ and so since $\sigma(\cdot, \Gamma_0(A))$ is m-continuous, we deduce that $\varphi(\cdot, A)$ is m-continuous. Hence, there exists $M(A) \in P_{wkc}(X)$ such that $\varphi(x^*, M(A)) = \sigma(x^*, M(A))$. Since $M : \Sigma \to P_{wkc}(X)$ from Proposition 3 of Godet-Thobie [**16**, p. 113], we have that $M(\cdot)$ is a multi-measure (set-valued measure) in the sense of Coste [**9**].

Evidently, $M \ll \mu$ and $M(\cdot)$ is of bounded variation, i.e., $|M(\Omega)| = \sup_\pi \sum_{k=1}^N |M(A_k)| < \infty$ where the supremum is taken over all finite Σ-partitions π of Ω. So we can apply Theorem 2 of Coste [**9**, p. 1517] and get $F \in \mathcal{L}^1_{fc}(\Sigma, X)$ such that for every $(A, x^*) \in \Sigma \times X^*$ we have $\sigma(x^*, M(A)) = \int_A \sigma(x^*, F(\omega))\, d\mu(\omega)$.

Now note that, for every $x^* \in X^*$, $\{\sigma(x^*, F_n), F_n\}_{n \ge 1}$ is an R-valued amart and $|\sigma(x^*, F_n(\omega))| \le \|x^*\| \vartheta(\omega)$, μ almost everywhere. So, from the convergence theorem for R-valued amarts, see, for example, Egghe [**15**, p. 137], we have that for every $(\omega, x^*) \in (\Omega \backslash N) \times D^*$, $\mu(N) = 0$,

$$\lim \sigma(x^*, F_n(\omega)) = \psi(\omega, x^*);$$

here D^* is a countable subset of X^* dense for the norm topology.

Through the dominated convergence theorem for every $(A, x^*) \in \Sigma \times D^*$ we have $\int_A \sigma(x^*, F(\omega))\, d\mu(\omega) = \int_A \psi(\omega, x^*)\, d\mu(\omega)$, hence $\sigma(x^*, F(\omega)) = \psi(\omega, x^*)$ for every $(\omega, x^*) \in (\Omega \backslash N_1) \times D^*$, $\mu(N_1) = 0$. So, for $(\omega, x^*, y^*) \in (\Omega \backslash N) \times D^* \times D^*$, we have

$$|\psi(\omega, x^*) - \psi(\omega, y^*)| = |\sigma(x^*, F(\omega)) - \sigma(y^*, F(\omega))|$$

and, since $\sigma(\cdot, F(\omega))$ is strongly continuous, we can apply Theorem 5.3 of Degundji [**13**, p. 216] and get a unique continuous extension of $\psi(\cdot, \cdot)$ on all $(\Omega \backslash N_1) \times X^*$.

Evidently, $\psi(\omega, x^*) = \sigma(x^*, F(\omega))$ for all $(\omega, x^*) \in (\Omega \backslash N_1) \times X^*$. Therefore, for all such pairs (ω, x^*) we have $\sigma(x^*, F_n(\omega)) \to \sigma(x^*, F(\omega))$ and so we conclude that $F_n(\omega) \xrightarrow{w} F(\omega)$, μ almost everywhere.

Now we remove the extra hypothesis that $\vartheta(\cdot) = \sup_{n \ge 1} |F_n(\cdot)| \in L^1(\Omega)$. This is achieved by employing the same stopping time technique used in the proof of Proposition 3.2. So for $\lambda > 0$ define $A_\lambda = \{\omega \in \Omega : \vartheta(\omega) \le \lambda\}$. Then let $\tau \in T^*$ be defined by

$$\tau(\omega) = \begin{cases} +\infty & \text{if } \omega \in A_\lambda, \\ \min[n \ge 1 : |F_n(\omega)| > \lambda] & \text{if } \omega \in A_\lambda^c. \end{cases}$$

Again we claim that $\sup_{n\geq 1}|F_{n\wedge\tau}(\cdot)| \in L^1(\Omega)$. Note that, on $\{\tau<\infty\}$, we have $\lim_{n\to\infty}|F_{n\wedge\tau}| = |F_\tau|$. So

$$\begin{aligned}\int_{\{\tau<\infty\}}|F_\tau(\omega)|\,d\mu(\omega) &\leq \underline{\lim}\int_{\{\tau<\infty\}}|F_{n\wedge\tau}(\omega)|\,d\mu(\omega)\\ &\leq \underline{\lim}\int_\Omega |F_{n\wedge\tau}(\omega)|\,d\mu(\omega)\\ &\leq \sup_{\sigma\in T}\int_\Omega |F_\sigma(\omega)|\,d\mu(\omega)<\infty.\end{aligned}$$

Since on $\{\tau<\infty\}$, $|F_{n\wedge\tau}(\omega)|\leq|F_\tau(\omega)|$, we have

$$\begin{aligned}\int_\Omega \sup_{n\geq 1}|F_{n\wedge\tau}(\omega)|\,d\mu(\omega) &= \int_{\{\tau=\infty\}}\sup_{n\geq 1}|F_{n\wedge\tau}(\omega)|\,d\mu(\omega)\\ &\quad+\int_{\{\tau<\infty\}}\sup_{n\geq 1}|F_{n\wedge\tau}(\omega)|\,d\mu(\omega)\\ &\leq \lambda+\int_{\{\tau<\infty\}}|F_\tau(\omega)|\,d\mu(\omega)\\ &\leq \lambda+\sup_{\sigma\in T}\int_\Omega|F_\sigma(\omega)|\,d\mu(\omega)<\infty.\end{aligned}$$

Then, using the maximal inequality, see Lemma II.1.5 of Egghe [**15**, p. 23], we get that

$$\mu\{\omega\in\Omega:\sup_{n\geq 1}|F_n(\omega)|>\lambda\}\leq\frac{1}{\lambda}\sup_{\sigma\in T}\int_\Omega|F_{\sigma(\omega)}|\,d\mu(\omega),$$

and the righthand side in the above inequality can get arbitrarily small as $\lambda\to\infty$.

So $|F_n|$ and $F|_{n\wedge\tau}|$ are in fact equal outside a set whose μ-measure can be made arbitrarily small. Therefore, we conclude that there is no loss of generality in assuming that $\sup_{n\geq 1}|F_n(\cdot)|=\vartheta(\cdot)\in L^1(\Omega)$. $\square$

For multi-valued WS-amarts we can have the following weak Riesz decomposition type theorem. A Riesz decomposition theorem for w^*-amarts can be found in Bagchi [**4**, Theorem 4.7].

Theorem 5.4. *If X has the* RNP, *X^* is separable and $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued* WS-*amart in $\mathcal{L}^1_{fc}(X)$ which is of class* (B), *i.e., $\sup_{\tau\in T}\int_\Omega |F_\tau(\omega)|\,d\mu(\omega) < \infty$, and for every $A \in \Sigma$,*

$$\Gamma(A) = \overline{\bigcup_{n\geq 1}\int_A F_n(\omega)\,d\mu(\omega)}^w \in \wp,$$

then there exists $\{G_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued martingale in $\mathcal{L}^1_{fc}(X)$ such that $G_n(\omega) \xrightarrow{w} F(\omega)$, μ almost everywhere, where $F(\cdot) \in \mathcal{L}^1_{fc}(X)$ is the limit multi-function in the conclusion of Theorem 5.3.

Proof. From the proof of Theorem 5.3 we know that, for every $n \geq 1$, $M_n = M|_{\Sigma_n}$ is a multi-measure of bounded variation. So, once again, via Theorem 2 of Coste [**9**, p. 1517], we can get $G_n \in \mathcal{L}^1_{fc}(\Sigma_n, X)$ such that $M_n(A) = \mathrm{cl}\int_A^{(\Sigma_n)} G_n(\omega)\,d\mu(\omega)$ for every $A \in \Sigma_n$, $n \geq 1$. Then, for $n \geq m \geq 1$ and $A \in \Sigma_m$ we have $\mathrm{cl}\int_A^{(\Sigma_m)} G_m(\omega)\,d\mu(\omega) = M_m(A) = M_n(A) = \mathrm{cl}\int_A^{(\Sigma_n)} G_n(\omega)\,d\mu(\omega)$ and this by virtue of Theorem 5.4 of Hiai and Umegaki [**18**, p. 173] implies that $E^{\Sigma_m}G_n(\omega) = G_m(\omega)$, μ almost everywhere, hence, $\{G_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued martingale in $\mathcal{L}^1_{fc}(X)$. Observe that

$$\sup_{n\geq 1}\int_\Omega |G_n(\omega)|\,d\mu(\omega) = \sup_{n\geq 1}|M_n|(\Omega) \leq \sup_{\tau\in T}\int_\Omega |F_\tau(\omega)|\,d\mu(\omega) < \infty.$$

From Lemma 3.2 of Luu [**24**, p. 8], see also Proposition 1.4 of Luu [**23**, p. 64], we know that, for every $n \geq 1$, $G_n(\omega) = \overline{\{g_n^k(\omega)\}}_{k\geq 1}$, μ almost everywhere with $\{g_n^k, \Sigma_n\}_{n\geq 1}$, $k \geq 1$ being a martingale selection of $\{F_n\}_{n\geq 1}$, i.e., for every $k \geq 1$, $\{g_n^k\}_{n\geq 1} \in MS(F_n)$. Since X has the RNP and for every $k \geq 1$ $\{g_n^k\}_{n\geq 1}$ is an L^1-bounded martingale, we know that there exists $g_k \in L^1(\Omega, X)$, $k \geq 1$, such that $g_n^k(\omega) \to g_k(\omega)$, μ-almost everywhere in X as $n \to \infty$, see, for example, Egghe [**15**, p. 44]. Set $G(\omega) = \overline{\mathrm{conv}}\{g_k(\omega)\}_{k\geq 1}$. Because of Lemma 4 of Neveu [**26**, p. 4], if D^* is a countable strongly dense subset of X^*, for every $(\omega, x^*) \in (\Omega\backslash N) \times D^*$, $\mu(N) = 0$, we have $\sigma(x^*, G_n(\omega)) = \sup_{k\geq 1}(x^*, g_n^k(\omega)) \to \sup_{k\geq 1}(x^*, g_k(\omega)) = \sigma(x^*, G(\omega))$.

Note that $\{|G_n|, \Sigma_n\}_{n\geq 1}$ is an R_+-valued submartingale. So, from the maximal inequality for submartingales for every $\lambda > 0$, we have

$$\mu\{\omega \in \Omega : \sup_{n\geq 1} |F_n(\omega)| > \lambda\} \leq \frac{1}{\lambda} \sup_{\sigma\in T} \int_\Omega |G_n(\omega)|\, d\mu(\omega) < \infty,$$

hence $\mu\{\omega \in \Omega : \sup_{n\geq 1} |F_n(\omega)| = \infty\} = 0$. Using this fact and a density argument as in previous proofs, we get that

$$\sigma(x^*, G_n(\omega)) \longrightarrow \sigma(x^*, G(\omega)) \quad \text{for all} \quad (\omega, x^*) \in (\Omega \backslash N_1) \times X^*, \\ \mu(N_1) = 0.$$

Then, from Theorem 2.1 of Bellow and Egghe [**6**, p. 346], we know that there exist two increasing sequences $\{\sigma_n\}_{n\geq 1}$ and $\{\tau_n\}_{n\geq 1}$ in T such that $n \leq \sigma_n \leq \tau_n$ and, for every $(\omega, x^*) \in (\Omega\backslash N_2)\times X^*$, $\mu(N_2) = 0$, we have

$$\text{(9)} \quad \overline{\lim_{m\to\infty}} |\sigma(x^*, F_m(\omega)) - \sigma(x^*, G_m(\omega))| \\ \leq \underline{\lim} |\sigma(x^*, E^{\Sigma_{\sigma_n}} F_{\tau_n}(\omega)) - \sigma(x^*, F_{\sigma_n}(\omega))|.$$

Since $\{\sigma(x^*, F_n), \Sigma_n\}_{n\geq 1}$ is an R-valued amart, thus a uniform amart, the righthand side in (9) equals 0. So, finally, we get that, for all $(\omega, x^*) \in (\Omega\backslash N_2) \times X^*$, $\mu(N_2) = 0$,

$$\sigma(x^*, F(\omega)) = \sigma(x^*, F(\omega));$$

hence, $F(\omega) = G(\omega)$, μ almost everywhere. Therefore, $G_n(\omega) \xrightarrow{w} F(\omega)$, μ almost everywhere.

6. Amarts of infinite order. In this section we look at amarts of infinite order and establish some Riesz decomposition type results for them. Amarts of finite order were introduced by Luu [**22**]. Their multi-valued analogs were first introduced and studied by Bagchi [**4**].

We start by establishing a Riesz decomposition for vector-valued amarts of infinite order. Recall that the class of vector-valued amarts of finite order is denoted by AM^∞ and it includes the class of vector-valued amarts denoted by AM, i.e., $\mathrm{AM} \subseteq \mathrm{AM}^\infty$. For vector-valued amarts, the Riesz decomposition was established by Edgar and Sucheston [**14**].

More recently, a Riesz decomposition theorem for vector-valued amarts of infinite order, with values in a separable dual Banach space, was proved by Bagchi [**4**, Theorem 3.4]. Here we extend the result of Bagchi to processes with values in a Banach space which is not necessarily dual.

Proposition 6.1. *If* X *has the* RNP, $\{f_n, \Sigma_n\}_{n\geq 1} \subseteq \mathrm{AM}^\infty$ *and* $\underline{\lim}\, \|f_n\|_1 < \infty$, *then* $f_n = u_n + p_n$, $n \geq 1$, *with* $\{u_n, \Sigma_n\}_{n\geq 1}$ *an* $L^1(\Omega, X)$*-bounded martingale and* $\{p_n, \Sigma_n\}_{n\geq 1} \subseteq AM^\infty$ *such that* $\lim_{\sigma\in T^i} \sup_{\|x^*\|<1} \int_\Omega |(x^*, p_\sigma(\omega))|\, d\mu(\omega) = 0$.

Proof. Fix $l \in N$, $\sigma \in T$ and $\varepsilon > 0$. We claim that there exists an $N \geq \sigma$ such that if $\tau, \tau' \in T^l(N)$, we have $\sup_{A\in\Sigma_\sigma} \| \int_A f_\tau(\omega)\, d\mu(\omega) - \int_A f_{\tau'}(\omega)\, d\mu(\omega)\| \leq \varepsilon$.

Indeed, pick $N \geq 1$ large enough so that if $\tau_1, \tau_1' \in T^{l+1}(N)$, then $\| \int_\Omega f_{\tau_1}(\omega)\, d\mu(\omega) - \int_\Omega f_{\tau_1'}(\omega)\, d\mu(\omega)\| \leq \varepsilon$.

Now let $\tau, \tau' \in T^l(N)$ and fix $A \in \Sigma_\sigma$, $N_1 \geq \max[\tau, \tau']$. Define $\tau_1 = \chi_A\tau + \chi_{A^c}N_1$, and $\tau_1' = \chi_A\tau' + \chi_{A^c}N_1$.

Evidently, $\tau_1, \tau_1' \in T^{l+1}(N)$, and we have

$$\left\| \int_\Omega f_{\tau_1'}(\omega)\, d\mu(\omega) - \int_\Omega f_{\tau_1}(\omega)\, d\mu(\omega) \right\| \leq \varepsilon;$$

hence,

$$\begin{aligned}\bigg\| \int_A f_\tau(\omega)\, d\mu(\omega) &+ \int_{A^c} f_{N_1}(\omega)\, d\mu(\omega) \\ &- \int_A f_{\tau'}(\omega)\, d\mu(\omega) - \int_{A^c} f_{N_1}(\omega)\, d\mu(\omega) \bigg\| \\ &\qquad = \left\| \int_A f_\tau(\omega)\, d\mu(\omega) - \int_A f_{\tau'}(\omega)\, d\mu(\omega) \right\| \leq \varepsilon.\end{aligned}$$

Since $A \in \Sigma_\sigma$ was arbitrary, we finally have

$$\sup_{A\in\Sigma_\sigma} \left\| \int_A f_\tau(\omega)\, d\mu(\omega) - \int_A f_{\tau'}(\omega)\, d\mu(\omega) \right\| \leq \varepsilon$$

as claimed. Therefore, there exists $m(A) \in X$ such that

$$\lim_{\sigma\in T^l} \sup_{A\in\Sigma_\sigma} \| \int_A f_\tau(\omega)\, d\mu(\omega) - m(A)\| = 0.$$

From the Vitali-Hahn-Saks theorem, we know that $m|_{\Sigma_\sigma}$ is a vector measure which is clearly absolutely continuous with respect to μ. Also, we claim that it is of bounded variation. Indeed, given $\varepsilon > 0$, let $\{A_k\}_{k\geq 1}^M$ be a Σ_σ-partition of Ω. Then, for $n \geq 1$ large, we have

$$\left\| \int_{A_{n_k}} f_n(\omega)\, d\mu(\omega) - m(A_k) \right\| \leq \frac{\varepsilon}{M} \quad \text{for} \quad k \in \{1, 2, \dots, M\},$$

hence

$$\sum_{k=1}^{M} m(A_k) \leq \int_\Omega \|f_n(\omega)\|\, d\mu(\omega) + \varepsilon$$

and so

$$|m|(\Omega) \leq \underline{\lim} \int_\Omega \|f_n(\omega)\|\, d\mu(\omega) < \infty.$$

Since, by hypothesis, X has the RNP, we can find $u_n \in L^1(\Sigma_n, X)$, $n \geq 1$, such that $m_n(A) = m|_{\Sigma_n}(A) = \int_A u_n(\omega)\, d\mu(\omega)$ for every $A \in \Sigma_n$, $n \geq 1$.

Then $\int_A u_{n+1}(\omega)\, d\mu(\omega) = m(A) = \int_A u_n(\omega)\, d\mu(\omega)$ for every $A \in \Sigma_N$, $n \geq 1$, which shows that $\{u_n, \Sigma_n\}_{n\geq 1}$ is a vector-valued martingale. Moreover, for every $n \geq 1$, $\int_\Omega \|u_n(\omega)\|\, d\mu(\omega) \leq \underline{\lim}\, \|f_n\|_1 < \infty$, which shows that $\{u_n\}_{n\geq 1}$ is $L^1(\Omega, X)$-bounded.

For fixed $n \geq 1$, we have $\lim_{\tau\in T^i} \sup_{A\in\Sigma_n} \| \int_A f_\tau(\omega)\, d\mu(\omega) - m_n(A)\|$ for every $l \geq 1$. Choose $N \geq 1$ large so that if $\tau \geq \sigma \geq N$ we have

$$\sup_{A\in\Sigma_\sigma} \left\| \int_A f_\tau(\omega)\, d\mu(\omega) - \int_A f_\sigma(\omega)\, d\mu(\omega) \right\| \leq \varepsilon.$$

Fix $\sigma \in T^l$ such that $\sigma \geq N$, and let $k \geq \sigma$. Choose $\tau \geq k$, $\tau \in T^l$, such that

$$\begin{aligned} &\sup_{A\in\Sigma_k} \left\| \int_A f_\tau(\omega)\, d\mu(\omega) - m_k(A) \right\| \\ &\qquad\qquad = \sup_{A\in\Sigma_k} \left\| \int_A f_\tau(\omega)\, d\mu(\omega) - \int_A u_k(\omega)\, d\mu(\omega) \right\| \leq \varepsilon. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sup_{A\in\Sigma_\sigma} \left\| \int_A f_\sigma(\omega)\,d\mu(\omega) - \int_A u_\sigma(\omega)\,d\mu(\omega) \right\| \\
&\le \sup_{A\in\Sigma_\sigma} \left\| \int_A f_\sigma(\omega)\,d\mu(\omega) - \int_A f_\tau(\omega)\,d\mu(\omega) \right\| \\
&\quad + \sup_{A\in\Sigma_\sigma} \left\| \int_A f_\tau(\omega)\,d\mu(\omega) - \int_A u_k(\omega)\,d\mu(\omega) \right\| \\
&\le \sup_{A\in\Sigma_\sigma} \left\| \int_A f_\sigma(\omega)\,d\mu(\omega) - \int_A f_\tau(\omega)\,d\mu(\omega) \right\| \\
&\quad + \sup_{A\in\Sigma_k} \left\| \int_A f_\tau(\omega)\,d\mu(\omega) - \int_A u_k(\omega)\,d\mu(\omega) \right\| \\
&\le 2\varepsilon.
\end{aligned}$$

(The penultimate inequality is a consequence of the fact that $\Sigma_\sigma \subseteq \Sigma_k$.) So we have

$$\sup_{A\in\Sigma_\sigma} \left\| \int_A f_\sigma(\omega)\,d\mu(\omega) - \int_A u_\sigma(\omega)\,d\mu(\omega) \right\| \le 2\varepsilon.$$

Thus, if we set $p_n = f_n - u_n$, we have

$$\lim_{\sigma\in T^i} \sup_{A\in\Sigma_\sigma} \left\| \int_A p_\sigma(\omega)\,d\mu(\omega) \right\| = 0.$$

But recalling that

$$\sup_{A\in\Sigma_\sigma} \left\| \int_A p_\sigma(\omega)\,d\mu(\omega) \right\|$$

is a norm equivalent to the Pettis norm

$$\|p_\sigma\|_w = \sup_{\|x^*\|\le 1} \int_\Omega |(x^*, p_\sigma(\omega))|\,d\mu(\omega),$$

see Egghe [**15**, p. 5], we get that

$$\lim_{\sigma\in T^l} \|p_\sigma\|_w = \lim_{\sigma\in T^l} \sup_{\|x^*\|\le 1} \int_\Omega |(x^*, p_\sigma(\omega))|\,d\mu(\omega) = 0. \qquad \square$$

Remark 6.2. We can easily check that the above decomposition is unique.

An immediate consequence of Proposition 6.1 is the following convergence result.

Corollary 6.3. *If* $\dim X < \infty$ *and* $\{f_n, \Sigma_n\}_{n\geq 1}$ *is an amart of infinite order such that* $\underline{\lim}\,\|f_n\|_1 < \infty$, *then there exists an* $f \in L^1(\Omega, X)$ *such that* $f_n \xrightarrow{\mu} f$, *here* $\xrightarrow{\mu}$ *denotes convergence in probability* μ.

We can have a corresponding decomposition for multi-valued amarts of infinite order. A similar result for multi-valued amarts of infinite order, with values in a dual Banach space, was proved earlier by Bagchi [**40**, Theorem 4.2]. In fact, the result of Bagchi characterizes such multi-valued random processes.

Theorem 6.4. *If* X *has the* RNP, X^* *is separable and* $\{F_n, \Sigma_n\}_{n\geq 1}$ *is a multi-valued amart of infinite order in* $\mathcal{L}^1_{fc}(X)$ *such that*

$$\sup_{n\geq 1} |||F_n|||_1 < \infty,$$

and for every $A \in \Sigma$,

$$\Gamma(A) = \overline{\bigcup_{n\geq 1} \int_A F_n(\omega)\, d\mu(\omega)}^{\,w} \in P_{wk}(X),$$

then there exists a multi-valued martingale $\{G_n, \Sigma_n\}_{n\geq 1}$ *in* $\mathcal{L}^1_{wkc}(X)$ *such that* $\sup_{n\geq 1} |||G_n|||_1 < \infty$, *and for every* $l \geq 1$ *and every* $\sigma \in T$ *we have*

$$\lim_{\tau\in T^l} \sup_{A\in\Sigma_\sigma} h\bigg(\int_A F_\tau(\omega)\, d\mu(\omega), \int_A G_\tau(\omega)\, d\mu(\omega)\bigg) = 0.$$

Proof. Let $x^* \in X^*$ and consider the R-valued amart of the infinite order $\{\sigma(x^*, F_n(\cdot)), \Sigma_n\}_{n\geq 1}$. By Proposition 6.1, $\sigma(x^*, F_n(\cdot)) =$

$u_n(x^*)(\cdot) + p_n(x^*)(\cdot)$, $n \geq 1$, with $\{u_n(x^*)(\cdot), \Sigma_n\}_{n\geq 1}$ being a martingale and $\{p_n(x^*)(\cdot), \Sigma_n\}_{n\geq 1} \in \mathrm{AM}^\infty$ with $\lim_{\sigma\in T^l} \|p_\sigma(x^*)(\cdot)\|_1 = 0$. From the proof of Proposition 6.1 we know that

$$\lim_{\sigma\in T^l} \sup_{A\in\Sigma_\tau} \left| \int_A (\sigma(x^*, F_\tau(\omega)) - u_\tau(x^*)(\omega))\, d\mu(\omega))\right| = 0,$$

hence

$$\lim_{n\to\infty} \sup_{A\in\Sigma_m} \left| \int_A (\sigma(x^*, F_n(\omega)) - u_n(x^*)(\omega))\, d\mu(\omega))\right| = 0$$

for fixed $m \geq 1$, and so

$$\lim_{n\to\infty} \sup_{A\in\Sigma_m} \left| \int_A \sigma(x^*, E^{\Sigma_m} F_n(\omega))\, d\mu(\omega) - \int_A u_m(x^*)(\omega)\, d\mu(\omega))\right| = 0,$$

which implies that $\sigma(x^*, E^{\Sigma_m} F_n(\cdot)) \to u_m(x^*)(\cdot)$ in $L^1(\Sigma_m)$ as $n \to \infty$. Then $x^* \to u_m(x^*)(\cdot)$ is sublinear and

$$|\beta_m(x^*)(A)| = \left| \int_A u_m(x^*)(\omega)\, d\mu(\omega))\right| \leq \sigma(x^*, \Gamma_0(A))$$

where $\Gamma_0(A) = \overline{\mathrm{conv}}[\Gamma(A) \cup (-\Gamma(A))] \in P_{wkc}(X)$.

So $x^* \to \beta_m(x^*)$ is $m(X^*, X)$-continuous and this implies that there exists $M_m(A) \in P_{wkc}(X)$ such that $\beta_m(x^*)(A) = \sigma(x^*, M_m(A))$. As before, via Theorem 3 of Coste [**9**, p. 1517], we can get $G_m \in \mathcal{L}^1_{wkc}(\Sigma_m, X)$ such that $M_m(A) = \int_A G_m(\omega)\, d\mu(\omega))$ for every $A \in \Sigma_m$. Clearly $\{G_m, \Sigma_m\}_{m\geq 1}$ is a multi-valued martingale which is L^1-bounded, i.e., $\sup_{m\geq 1} \||G_m(\cdot)|\|_1 < \infty$, and $\sigma(x^*, G_m(\cdot)) = u_m(x^*)(\cdot)$ for all $(\omega, x^*) \in (\Omega\backslash N) \times X^*$, $\mu(N) = 0$.

Let $\sigma \in T$ and $\varepsilon > 0$, and choose $N \geq \sigma$ such that, if $\tau, \tau' \in T^l(N)$, $N \leq \tau \leq \tau'$, we have

$$\sup_{A\in\Sigma_\sigma} h\biggl(\int_A F_\tau(\omega)\, d\mu(\omega), \int_A F_{\tau'}(\omega)\, d\mu(\omega))\biggr) \leq \varepsilon.$$

Thus,

$$\lim_{\tau'\in T^l} \sup_{A\in\Sigma_\sigma} h\biggl(\int_A F_{\tau'}(\omega)\, d\mu(\omega), M(A)\biggr) = 0.$$

Because $\{G_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued martingale in $\mathcal{L}^1_{wkc}(X)$, we have $\int_A G_\sigma(\omega)\,d\mu(\omega)) = \int_A G_\tau(\omega)\,d\mu(\omega))$ for $A \in \Sigma_\sigma$ and $\tau \geq \sigma$. So

$$\lim_{\tau\in T^l} \sup_{A\in\Sigma_\sigma} \left|\sigma\bigg(x^*, \int_A F_\tau(\omega)\,d\mu(\omega)\bigg) - \sigma\bigg(x^*, \int_A G_\sigma(\omega)\,d\mu(\omega)\bigg)\right| = 0,$$

hence

$$M(A) = \int_A G_\sigma(\omega)\,d\mu(\omega) = \int_A G_\tau(\omega)\,d\mu(\omega) \quad \text{for} \quad A \in \Sigma_\sigma, \quad \tau \geq \sigma.$$

Therefore,

$$\lim_{\tau\in T^l} \sup_{A\in\Sigma_\sigma} h\bigg(\int_A F_\tau(\omega)\,d\mu(\omega), \int_A G_\sigma(\omega)\,d\mu(\omega)\bigg) = 0. \qquad \square$$

We conclude this section with a final useful observation concerning multi-valued amarts of infinite order. First a definition, see Luu [**24**].

Definition 6.5. If $F, G \in \mathcal{L}^1_{fc}(X)$, the *Pettis distance* $\Delta_w(F,G)$ between them is defined by $\Delta_w(F,G) = \sup_{\|x^*\|\leq 1} \int_\Omega |\sigma(x^*, F(\omega)) - \sigma(x^*, G(\omega))|\,d\mu(\omega)$.

Remark 6.6. In a manner similar to the vector-valued case, we can show that $\Delta_w(F,G) \leq \Delta(F,G)$. For Σ_0 a sub σ-field of Σ, we have

$$\Delta_w(E^{\Sigma_0}F, E^{\Sigma_0}G) \leq \Delta_w(F,G),$$

and

$$\begin{aligned}\sup_{A\in\Sigma} h\bigg(\int_A F(\omega)\,d\mu(\omega), \int_A G(\omega)\,d\mu(\omega)\bigg) \\ \leq \Delta_w(F,G) \leq 4 \sup_{A\in\Sigma} h\bigg(\int_A F(\omega)\,d\mu(\omega), \int_A G(\omega)\,d\mu(\omega)\bigg).\end{aligned}$$

Proposition 6.7. *If $\{F_n, \Sigma_n\}_{n\geq 1}$ is an adapted sequence in $\mathcal{L}^1_{fc}(X)$, and if there exists $F \in \mathcal{L}^1_{fc}(X)$ such that $\Delta_w(F_n, F) \to 0$ as $n \to \infty$, then $\{F_n, \Sigma_n\}_{n\geq 1}$ is a multi-valued amart of finite order in $\mathcal{L}^1_{fc}(X)$.*

Proof. Let $l \geq 1$ and $\varepsilon > 0$. Then we can find $N \geq 1$ such that, for $k, n \geq N$, we have $\Delta w(F_k, F_n) \leq \varepsilon / l^2$. Let $\sigma, \tau \in T^l$ be such that $\sigma, \tau \geq N$. We have

$$
\begin{aligned}
h\bigg(\int_\Omega F_\sigma(\omega)\,d\mu(\omega), \int_\Omega F_\tau(\omega)\,d\mu(\omega)\bigg) \\
&\leq \sum_{k,n \geq N} h\bigg(\int_{\{\sigma=k\}\cap\{\tau=n\}} F_k(\omega)\,d\mu(\omega), \\
&\qquad\qquad \int_{\{\sigma=k\}\cap\{\tau=n\}} F_n(\omega)\,d\mu(\omega)\bigg) \\
&\leq l^2 \sup_{k,n \geq N} \Delta_w(F_k, F_n) \leq \varepsilon,
\end{aligned}
$$

which proves that $\{F_n, \Sigma_n\}_{n \geq 1}$ is a multi-valued amart of finite order $l \geq 1$. But $l \geq 1$ was arbitrary. So we conclude that $\{F_n, \Sigma_n\}_{n \geq 1}$ is a multi-valued amart of infinite order. $\square$

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