ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1999

ASYMPTOTIC THEORY FOR A GENERAL THIRD-ORDER DIFFERENTIAL EQUATION OF EULER TYPE

A.A. AL-HAMMADI

1. Introduction. In this paper we investigate the asymptotic form of three linearly independent solutions of the third-order differential equation

$$(1.1)\quad \{q(x)(q(x)y'(x))'\}' + \{(q_1(x)y(x))' + q_1(x)y'(x)\}/2 + (p_0(x)y'(x))' + p_1(x)y(x) = 0$$

as $x \to \infty$. The functions q, q_1, p_0 and p_1 are defined on the interval $[a, \infty)$ with q nowhere zero. We do not need to restrict ourselves to real-valued coefficients nor to powers of x. Our aims are to identify relations between q, q_1, p_0 and p_1 corresponding to an Euler case for (1.1) and to obtain the asymptotic forms of the solutions in these cases. The various conditions imposed on the coefficients will be introduced when they are required in the development of the method. Al-Hammadi [**2**] considers (1.1) in the case where the solutions all have a similar exponential factor. A third-order equation similar to (1.1) has been considered previously by Al-Hammadi [**1**], Unsworth [**7**] and Pfeiffer [**6**].

Eastham [**4**] considered an Euler case for a fourth-order differential equation and showed that this case represents a borderline between situations where all solutions have a certain exponential character as $x \to \infty$ and where only two solutions have this character. The Euler cases for (1.1) we referred to, are given by

Case A.

$$(1.2)\qquad \frac{q_1'}{q_1} \sim \text{const.} \times \frac{p_0}{q^2},$$

Received by the editors on January 15, 1997, and in revised form on June 18, 1997.

and

$$\frac{(q^2p_0^{-2})'}{q^2p_0^{-2}} \sim \text{const.} \times \frac{p_0}{q^2}, \tag{1.3}$$

as $x \to \infty$.

Case B.

$$\frac{q_1'}{q_1} \sim \text{const.} \times \frac{p_1}{q_1}, \tag{1.4}$$

and

$$\frac{(q^2p_0^{-2})'}{q^2p_0^{-2}} \sim \text{const.} \times \frac{p_1}{q_1}, \tag{1.5}$$

as $x \to \infty$.

These cases will appear in the method in Sections 4–6, where we use the recent asymptotic theorem of Eastham [**3**, Section 2] to obtain the solutions of (1.1). Two examples are considered at the end of the paper in Section 6 with general remarks.

2. The first transformation. This section is heavily based on [**2**]. We write (1.1) in a standard way as a first-order system

$$Y' = AY, \tag{2.1}$$

where

$$Y = (y, qy', (1/2)q_1y + p_0y' + q(qy')')^t$$

and

$$A = \begin{bmatrix} 0 & q^{-1} & 0 \\ -\frac{1}{2}q_1q^{-1} & -p_0q^{-2} & q^{-1} \\ -p_1 & -\frac{1}{2}q_1q^{-1} & 0. \end{bmatrix} \tag{2.2}$$

We also express A in its diagonal form

$$T^{-1}AT = \Lambda \tag{2.3}$$

using the eigenvalues λ_j and the eigenvectors ν_j, $1 \le j \le 3$, of A. Writing

$$q^2 = q_0, \tag{2.4}$$

the characteristic equation of A is given by

$$q_0\lambda^3 + p_0\lambda^2 + q_1\lambda + p_1 = 0. \tag{2.5}$$

An eigenvector ν_j of A corresponding to λ_j is

$$\nu_j = (1, q_0^{1/2}\lambda_j, (1/2)q_1 + p_0\lambda_j + q_0\lambda_j^2)^t, \tag{2.6}$$

where the superscript t denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (2.3) by

$$T = (\,\nu_1 \quad \nu_2 \quad \nu_3\,)\,. \tag{2.7}$$

Hence, by [**2**, Section 2],

$$T^{-1} = \begin{bmatrix} m_1^{-1}r_1 \\ m_2^{-1}r_2 \\ m_3^{-1}r_3 \end{bmatrix} \tag{2.8}$$

where

$$m_j = 3q_0\lambda_j^2 + 2p_0\lambda_j + q_1, \quad 1 \le j \le 3, \tag{2.9}$$

and

$$r_j = (E\nu_j)^t, \quad 1 \le j \le 3, \tag{2.10}$$

with

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.11}$$

By (2.3), the transformation

$$Y = TZ \tag{2.12}$$

takes (2.1) into

$$Z' = (\Lambda - T^{-1}T')Z, \tag{2.13}$$

where

$$\Lambda = dg(\lambda_1, \lambda_2, \lambda_3). \tag{2.14}$$

Again, by [**1**], the matrix $T^{-1}T' = (t_{jk})$ is given by

$$t_{jj} = \frac{1}{2}\frac{m_j'}{m_j}, \quad 1 \le j \le 3, \tag{2.15}$$

and, for $j \neq k$, $1 \le j$, $k \le 3$,

$$t_{jk} = (\lambda_j - \lambda_k)^{-1} m_j^{-1} \{(\lambda_j + \lambda_k)(q_0' \lambda_j \lambda_k + q_1')/2 + (p_0' \lambda_j \lambda_k + p_1')\}. \tag{2.16}$$

We now have to work out the last expression in some detail in terms of q_0, q_1, p_0 and p_1 in order to determine the form of (2.13) and then make progress towards (1.1).

3. The matrices Λ and $T^{-1}T'$. At this stage we require the following conditions in the coefficients q_0, q_1, p_0 and p_1 as $x \to \infty$.

Condition 1. q_0, p_0 and q_1, p_1 are nowhere zero in some interval $[a, \infty)$, and

$$q_0 q_1 = o(p_0^2), \quad x \to \infty, \tag{3.1}$$

$$p_0 p_1 = o(q_1^2), \quad x \to \infty, \tag{3.2}$$

and we write

$$\varepsilon_1 = \frac{q_0 q_1}{p_0^2} = o(1), \quad x \to \infty, \tag{3.3}$$

$$\varepsilon_2 = \frac{p_0 p_1}{q_1^2} = o(1), \quad x \to \infty, \tag{3.4}$$

and

$$\varepsilon_3 = \frac{q_0 p_1}{q_1 p_0} = o(1), \quad x \to \infty. \tag{3.5}$$

Condition 2.

$$\frac{q_0'}{q_0}\varepsilon_1, \frac{q_0'}{q_0}\varepsilon_2, \frac{q_1'}{q_1}\varepsilon_1, \frac{q_1'}{q_1}\varepsilon_2, \frac{p_0'}{p_0}\varepsilon_1, \frac{p_0'}{p_0}\varepsilon_2, \frac{p_1'}{p_1}\varepsilon_2, \frac{p_1'}{p_1}\varepsilon_3, \tag{3.6}$$

are all $L(a, \infty)$.

As in [**1, 2**], we can solve (2.5) subject to (3.1) and (3.2). Then (2.5) gives distinct eigenvalues λ_j, $1 \leq j \leq 3$, as $x \to \infty$, such that

$$\lambda_1 = -\frac{p_1}{q_1}(1 + \delta_1), \tag{3.7}$$

$$\lambda_2 = -\frac{q_1}{p_0}(1 + \delta_2), \tag{3.8}$$

and

$$\lambda_3 = -\frac{p_0}{q_0}(1 + \delta_3), \tag{3.9}$$

where

$$\delta_1 = O(\varepsilon_2) \tag{3.10}$$

$$\delta_2 = O(\varepsilon_1) + O(\varepsilon_2), \tag{3.11}$$

and

$$\delta_3 = O(\varepsilon_1). \tag{3.12}$$

Hence, by (3.1) and (3.2),

$$\lambda_j = o(\lambda_{j+1}), \quad x \to \infty, \quad 1 \leq j \leq 2. \tag{3.13}$$

In (2.15), we investigate the behavior of m_j and m_j' as $x \to \infty$. First, by (3.1)–(3.5) and (3.7)–(3.12),

$$m_1 = q_1\{1 + O(\varepsilon_2)\}, \tag{3.14}$$

$$m_2 = -q_1\{1 + O(\varepsilon_1) + O(\varepsilon_2)\}, \tag{3.15}$$

and

$$m_3 = \frac{p_0^2}{q_0}\{1 + O(\varepsilon_1)\}. \tag{3.16}$$

Also, by substituting (3.7)–(3.9) into (2.9) and differentiating, we obtain

$$m_1' = q_1'\{1 + O(\varepsilon_2)\} + q_1\{O(\varepsilon_2') + O(\varepsilon_2\delta_1') + O(\varepsilon_2\varepsilon_3')\}, \tag{3.17}$$

$$m_2' = -q_1'\{1 + O(\varepsilon_1) + O(\varepsilon_2)\} + q_1\{O(\varepsilon_1') + O(\delta_2')\}, \tag{3.18}$$

and

$$m_3' = \frac{p_0^2}{q_0}\left[2\frac{p_0'}{p_0} - \frac{q_0'}{q_0}\right]\{1 + O(\varepsilon_1)\} + \frac{p_0^2}{q_0}\{0(\delta_3') + O(\varepsilon_1')\}. \tag{3.19}$$

Further, by (3.3)–(3.5),

$$\varepsilon_1' = O\left(\frac{q_0'}{q_0}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right) + O\left(\frac{p_0'}{p_0}\varepsilon_1\right), \tag{3.20}$$

$$\varepsilon_2' = O\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_2\right) + O\left(\frac{q_1'}{q_1}\varepsilon_2\right), \tag{3.21}$$

and

$$\varepsilon_3' = O\left(\frac{p_0'}{p_0}\varepsilon_3\right) + O\left(\frac{p_1'}{p_1}\varepsilon_3\right) + O\left(\frac{q_1'}{q_1}\varepsilon_3\right) + O\left(\frac{p_0'}{p_0}\varepsilon_3\right). \tag{3.22}$$

Then, for reference shortly, we note that upon substituting (3.7)–(3.9) into (2.5) and differentiating, we obtain:

$$\delta_2' = O(\varepsilon_2') + O(\varepsilon_2\varepsilon_3'), \tag{3.23}$$

$$\delta_2' = O(\varepsilon_1') + O(\varepsilon_2'), \tag{3.24}$$

and

$$\delta_3' = O(\varepsilon_1') + O(\varepsilon_1\varepsilon_3'). \tag{3.25}$$

Hence, by (3.20)–(3.25), and (3.6),

$$\varepsilon_j' \quad \text{and} \quad \delta_j' \text{ are } L(a,\infty), \quad 1 \le j \le 3. \tag{3.26}$$

Hence, for the diagonal elements t_{jj}, $1 \le j \le 3$, we can now substitute the estimates (3.14)–(3.19) into (2.15). We obtain

(3.27)
$$t_{11} = \frac{1}{2}\frac{q_1'}{q_1} + O\left(\frac{q_1'}{q_1}\varepsilon_2\right) + O(\varepsilon_2') + O(\varepsilon_2\delta_1') + O(\varepsilon_2\varepsilon_3'),$$

(3.28)
$$t_{22} = \frac{1}{2}\frac{q_1'}{q_1} + O\left(\frac{q_1'}{q^1}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_2\right) + O(\delta_2') + O(\varepsilon_1'),$$

(3.29)
$$t_{33} = 2\frac{p_0'}{p_0} - \frac{q_0'}{q_0} + O\left(\frac{p_0'}{p_0}\varepsilon_1\right) + O\left(\frac{q_0'}{q_0}\varepsilon_1\right) + O(\delta_3') + O(\varepsilon_1').$$

Now, for the nondiagonal elements t_{jk}, $j \ne k$, $1 \le j,\, k \le 3$, we consider (2.16). Now by (3.7), (3.9), (3.10), (3.12) and (3.14),

(3.30)
$$(1/2)(\lambda_1 - \lambda_3)^{-1} m_1^{-1}(\lambda_1 + \lambda_3)(q_0'\lambda_1\lambda_3 + q_1') = -(1/2)\frac{q_1'}{q_1} + O\left(\frac{q_0'}{q_0}\varepsilon_2\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_2\right) + O\left(\frac{q_1'}{q_1}\varepsilon_3\right),$$

and

(3.31)
$$(\lambda_1 - \lambda_3)^{-1} m_1^{-1}(p_0'\lambda_1\lambda_3 + p_1') = 0\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_3\right).$$

Thus, by (3.30) and (3.31), (2.16) gives for $j = 1$ and $k = 3$,

(3.32)
$$t_{13} = -(1/2)\frac{q_1'}{q_1} + O\left(\frac{q_0'}{q_0}\varepsilon_2\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_2\right) + O\left(\frac{q_1'}{q_1}\varepsilon_3\right) + O\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_3\right).$$

Again, by (3.7), (3.8), (3.10), (3.11) and (3.14),

(3.33)
$$(1/2)(\lambda_1 - \lambda_2)^{-1} m_1^{-1}(\lambda_1 + \lambda_2)(q_0'\lambda_1\lambda_2 + q_1') = -(1/2)\frac{q_0'}{q_0}\varepsilon_3\{1 + O(\varepsilon_1) + O(\varepsilon_2)\} - (1/2)\frac{q_1'}{q_1}\{1 + O(\varepsilon_1) + O(\varepsilon_2)\},$$

and

$$(3.34)\quad (\lambda_1-\lambda_2)^{-1}m_1^{-1}(p_0'\lambda_1\lambda_2+p_1') = \frac{p_0'}{p_0}\varepsilon_2\{1+O(\varepsilon_1)+O(\varepsilon_2)\} + \frac{p_1'}{p_1}\varepsilon_2\{1+O(\varepsilon_1)+O(\varepsilon_2)\}.$$

Hence, by (3.33) and (3.34), (2.16) gives, for $j=1$ and $k=2$,

$$(3.35)\quad t_{12} = -(1/2)\frac{q_1'}{q_1} + O\left(\frac{q_0'}{q_0}\varepsilon_3\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_2\right) + O\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_2\right).$$

Now, by (3.7), (3.8), (3.10), (3.11) and (3.15),

$$(3.36)\quad (1/2)(\lambda_2-\lambda_1)^{-1}m_2^{-1}(\lambda_2+\lambda_1)(q_0'\lambda_1\lambda_2+q_1') = O\left(\frac{q_0'}{q_0}\varepsilon_1\varepsilon_2\right) - (1/2)\frac{q_1'}{q_1}\{1+O(\varepsilon_1)+O(\varepsilon_2)\},$$

and

$$(3.37)\quad (\lambda_2-\lambda_1)^{-1}m_2^{-1}(p_0'\lambda_1\lambda_2+p_1') = O\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_2\right).$$

Hence, by (3.36) and (3.37), (2.16) gives, for $j=2$ and $k=1$,

$$(3.38)\quad t_{21} = -(1/2)\frac{q_1'}{q_1} + O\left(\varepsilon_1\frac{q_1'}{q_1}\right) + O\left(\varepsilon_2\frac{q_1'}{q_1}\right) + O\left(\frac{q_0'}{q_0}\varepsilon_1\varepsilon_2\right) + O\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_2\right).$$

Similar work can be done for the other elements t_{jk}, so we obtain

(3.39)

$$t_{23} = (1/2)\left(\frac{q_0'}{q_0}+\frac{q_1'}{q_1}\right) - \frac{p_0'}{p_0} + O\left(\frac{q_0'}{q_0}\varepsilon_1\right) + O\left(\frac{q_0'}{q_0}\varepsilon_2\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_2\right) + O\left(\frac{p_0'}{p_0}\varepsilon_1\right) + O\left(\frac{p_0'}{p_0}\varepsilon_2\right) + O\left(\frac{p_1'}{p_1}\varepsilon_3\right),$$

(3.40)

$$t_{31} = O\left(\frac{q_0'}{q_0}\varepsilon_3\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right) + O\left(\frac{p_0'}{p_0}\varepsilon_3\right) + O\left(\frac{p_1'}{q_1}\varepsilon_3\right),$$

and

(3.41)
$$t_{32} = \left(\frac{p_0'}{p_0}\varepsilon_1\right) + O\left(\frac{p_1'}{p_1}\varepsilon_3\right) + O\left(\frac{q_0'}{q_0}\varepsilon_1\right) + O\left(\frac{q_1'}{q_1}\varepsilon_1\right).$$

Now by (3.27)–(3.29), (3.32), (3.35), (3.38)–(3.41), (3.6) and (3.26), we can write the system (2.13) as

$$Z' = (\Lambda + R + S)Z, \tag{3.42}$$

where

$$R = \begin{pmatrix} -\eta & \eta & \eta \\ \eta & -\eta & -(\theta/2) - \eta \\ 0 & 0 & \theta \end{pmatrix} \tag{3.43}$$

with

$$\eta = (1/2)\frac{q_1'}{q_1}, \quad \theta = \frac{q_0'}{q_0} - 2\frac{p_0'}{p_0} = \frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}} \tag{3.44}$$

and S is $L(a,\infty)$.

4. Case A. Now we write (1.2) and (1.3) as

Condition 3.

$$\frac{q_1'}{q_1} = 2\sigma\frac{p_0}{q_0}(1+\phi), \tag{4.1}$$

$$\frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}} = \omega\frac{p_0}{q_0}(1+\psi), \tag{4.2}$$

where σ and ω are nonzero constants with $\omega(\neq 1, \neq 1-2\sigma, \neq 2)$, $\phi(x) \to 0$ and $\psi(x) \to 0$ as $x \to \infty$. The factor 2 is introduced only for convenience.

Also at this stage we let

Condition 4.

$$\phi'(x) \quad \text{and} \quad \psi'(x) \text{ are both } L(a,\infty). \tag{4.3}$$

We note that, by (3.7), (3.8), (3.44), (4.1) and (4.2), the condition of Eastham theorem [**3**, Section 2] is not satisfied. Indeed, the matrix Λ no longer dominates R. Therefore, we carry out a second diagonalization of the system (3.42).

First we write

$$\Lambda + R = \lambda_3\{S_1 + S_2\}, \tag{4.4}$$

with

$$S_1 = \begin{pmatrix} \sigma & -\sigma & -\sigma \\ -\sigma & \sigma & \omega/2+\sigma \\ 0 & 0 & 1-\omega \end{pmatrix} \tag{4.5}$$

and

$$S_2(x) = \begin{pmatrix} u_1 & u_2 & u_2 \\ u_2 & u_3 & u_4 \\ 0 & 0 & u_5 \end{pmatrix} \tag{4.6}$$

where

$$\begin{aligned} u_1 &= \frac{\lambda_1}{\lambda_3} - u_2, \\ u_2 &= -\sigma(\phi-\delta_3)(1+\delta_3)^{-1}, \\ u_3 &= \frac{\lambda_2}{\lambda_3} - u_2, \\ u_4 &= -\frac{1}{2}u_5 - u_2 \\ u_5 &= -\omega(\psi-\delta_3)(1+\delta_3)^{-1}. \end{aligned} \tag{4.7}$$

It is clear that $S_2(x) \to 0$ as $x \to \infty$.

Hence we diagonalize the constant matrix S_1 in (4.4). The distinct eigenvalues of the matrix S_1 are given by

$$\alpha_1 = 0, \quad \alpha_1 = 2\sigma, \quad \alpha_3 = 1-\omega, \tag{4.8}$$

using the transformation

$$Z = T_1 W, \tag{4.9}$$

where T_1 diagonalizes the constant matrix S_1. We can write (3.42) as

$$W' = (\Lambda_1 + M + T_1^{-1} S T_1) W \tag{4.10}$$

where

$$\Lambda_1 = \lambda_3 T_1^{-1} S_1 T_1 = \operatorname{diag}(\nu_1, \nu_2, \nu_3) = \lambda_3 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3), \tag{4.11}$$

$$M = \lambda_3 T_1^{-1} S_2 T_1, \tag{4.12}$$

and

$$T_1^{-1} S T_1 \in L(a, \infty). \tag{4.13}$$

Now we can apply the asymptotic theorem of Eastham in [**3**, Section 2] to (4.10) as in [**1, 2**], provided only that Λ_1 and M satisfy the conditions of [**3**, Section 2].

We first require that the ν_j, $1 \le j \le 3$, in (4.11) are distinct and this holds because the α_j, $1 \le j \le 3$ are distinct.

Second, we need to show that

$$\frac{M(x)}{\nu_i(x) - \nu_j(x)} \longrightarrow 0, \quad x \to \infty, \tag{4.14}$$

for $i \ne j$ and $1 \le i,\ j \le 3$. Now

$$\frac{M}{\nu_i - \nu_j} = (\alpha_i - \alpha_j)^{-1} T_1^{-1} S_2 T_1 \to 0, \quad x \to \infty.$$

Thus, (4.14) holds.

Third, we need to show that

$$\{(\nu_i - \nu_j)^{-1} M\}' \in L(a, \infty), \quad i \ne j,\ 1 \le i,\ j \le 3. \tag{4.15}$$

Hence, (4.15) holds if

$$S_2'(x) \in L(a, \infty). \tag{4.16}$$

Thus it suffices to show that

$$u_i'(x) \in L(a,\infty), \quad 1 \le i \le 5. \tag{4.17}$$

Now, by (3.7)–(3.9) and (4.7),

$$\begin{aligned} u_1' &= O(\varepsilon_3') + O(\varepsilon_3\delta_1') + O(\phi') + O(\delta_3'),\\ u_2' &= O(\phi') + O(\delta_3'),\\ u_3' &= O(\varepsilon_1') + O(\varepsilon_1\delta_2') + O(\phi') + O(\delta_3'),\\ u_4' &= O(\psi') + O(\delta_3') + O(\phi'),\\ u_5' &= O(\psi') + O(\delta_3'). \end{aligned} \tag{4.18}$$

Thus, by (3.26) and (4.3), (4.17) holds and consequently (4.15).

Now we state our main theorem for equation (1.1).

Theorem 4.1. *Let the coefficients q_0, q_1 and p_0 in (1.1) be $C^2[a,\infty)$, and let p_1 be $C^1[a,\infty)$. Let (3.1), (3.2), (3.6) and (4.1)–(4.3) hold. Let*

$$\operatorname{Re} I(x) \ \textit{be of one sign in } [a,\infty), \tag{4.19}$$

and

$$\operatorname{Re}\left\{\frac{\lambda_1+\lambda_2}{2} - \lambda_3 - (1/2)\frac{q_1'}{q_1'} - \frac{(q_0p_0^{-2})'}{q_0p_0^{-2}} \pm (1/2)I(x)\right\} \tag{4.20}$$

be of one sign in $[a,\infty)$, where

$$I(x) = \left[(\lambda_1-\lambda_2)^2 + \frac{q_1'^2}{q_1^2}\right]^{1/2}. \tag{4.21}$$

Then (1.1) has solutions

(4.22)
$$y_1 \sim q_1^{-1/2}\exp\left((1/2)\int_a^x\{\lambda_1+\lambda_2+I(t)\}\,dt\right),$$

(4.23)
$$y_2 = o\left[q_1^{-1/2}\exp\left((1/2)\int_a^x\{\lambda_1+\lambda_2-I(t)\}\,dt\right)\right]$$

and

$$y_3 \sim q_0 p_0^{-2} \exp\left(\int_a^x \lambda_3(t)\,dt\right). \tag{4.24}$$

Proof. Before applying the theorem in [**3**, Section 2], we show that the eigenvalues μ_k of $\Lambda_1 + M$ satisfy the dichotomy condition [**7**]. As in [**1, 2**], the dichotomy condition holds if

$$\operatorname{Re}(\mu_j - \mu_k) = f + g, \quad j \neq k, \quad 1 \leq k \leq 3, \tag{4.25}$$

where f has one sign in $[a, \infty]$ and g is $L(a, \infty)$ [**3**]. Now, since the eigenvalues of $\Lambda_1 + M$ are the same as the eigenvalues of $\Lambda + R$, hence by (3.43) and (2.3),

$$\mu_k(x) = \frac{\lambda_1 + \lambda_2}{2} - (1/2)\frac{q_1'}{q_1} + \frac{(-1)^{k+1}}{2} I(x), \quad k = 1, 2, \tag{4.26}$$

and

$$\mu_3(x) = \lambda_3 + \frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}}. \tag{4.27}$$

Thus, by (4.19) and (4.20), (4.25) holds. Since (4.10) satisfies all the conditions for the asymptotic result [**3**, Section 2], it follows that, as $x \to \infty$, (4.10) has three linearly independent solutions

$$W_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t)\,dt\right) \tag{4.28}$$

with e_k the coordinate vector with kth component unity and other components zero. Now we transform back to Y by means of (2.12) and (4.9) where T_1 in (4.9) is given by

$$T_1 = \begin{pmatrix} 1 & 1 & \sigma((\omega/2) - 1) \\ 1 & -1 & \sigma(-(3\omega/2) + 1) - (\omega/2)(\omega - 1) \\ 0 & 0 & (\omega - 1)(2\sigma + \omega - 1) \end{pmatrix}, \tag{4.29}$$

and using the fact that $\omega \neq 2$, we obtain the formula (4.24) and (4.22) after an adjustment of a constant multiple in y_k, $k = 1, 3$, while for y_2 we obtain (4.23).

5. Case B. Now we deal with Case B which is given by (1.4) and (1.5). We have the following theorem.

Theorem 5.1. *Let the coefficients q_0, q_1 and p_0 in* (1.1) *be $C^{(2)}[a,\infty)$, and let p_1 be $C^{(1)}[a,\infty)$. Let* (3.1), (3.2) *and* (3.6) *hold.*

Let

$$\frac{q_1'}{q_1} = \sigma_1 \frac{p_1}{q_1}(1+\phi_1), \tag{5.1}$$

and

$$\frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}} = \omega_1 \frac{p_1}{q_1}(1+\psi_1), \tag{5.2}$$

where σ_1 and ω_1 are nonzero constants with $\phi_1(x) \to 0$ and $\psi_1(x) \to 0$ as $x \to \infty$. Also let

$$\phi_1'(x) \quad \text{and} \quad \psi_1'(x) \text{ be } L(a,\infty). \tag{5.3}$$

Let (4.19), (4.20) *and* (4.21) *all hold. Then* (1.1) *has solutions*

$$y_k \sim q_1^{-1/2} \exp\left((1/2)\int_a^x \{\lambda_1 + \lambda_2 + (-1)^{k+1} I\}\, dt\right), \quad k = 1, 2, \tag{5.4}$$

$$y_3 \sim q_0 p_0^{-2} \exp\left(\int_a^x \lambda_3(t)\, dt\right). \tag{5.5}$$

Proof. As in [**2**], we apply Eastham theorem [**3**, Section 2] to the system (3.42) provided only that Λ and R satisfy the required condition. We shall use (3.43), (3.44), (5.1) and (5.2).

We first require that

$$\frac{q_1'}{q_1} = o\{(\lambda_i - \lambda_j)\}, \frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}} = o\{(\lambda_i - \lambda_j)\},$$

$i \neq j$, $1 \leq i, j \leq 3$, this being [**3**] for our system. By (5.1), (5.2), (3.1), (3.2), (3.7), (3.8) and (3.9), this requirement holds. We also require that

$$\left\{\frac{q_1'}{q_1}(\lambda_i - \lambda_j)^{-1}\right\}' \in L(a, \infty),$$

$$\left\{\frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}}(\lambda_i - \lambda_j)^{-1}\right\}' \in L(a, \infty),$$

for $i \neq j$, this begin [**3**] for our system. By (5.1), (5.2), (3.7), (3.8) and (3.9), this requirement is implied by (3.26) and (5.3). Finally, we require the eigenvalues of $\Lambda + R$ which are given by (4.26) and (4.27) to satisfy the dichotomy condition (4.25), and this is true by (4.19)–(4.21). Since (3.42) satisfies all the conditions for the result of [**3**, Section 2], it follows that, as $x \to \infty$, (3.42) has three linearly independent solutions $Z_k(x)$ such that

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t)\, dt\right). \tag{5.6}$$

We now transform back to Y by means of (2.12), (2.7) and (2.6). By taking the first component on each side of (2.12) and making use of (4.26) and (4.27), and carrying out the integration of $-(1/2)(q_1'/q_1)$ and $(q_0 p_0^{-2})'/q_0 p_0^{-2})$, we obtain (5.4) and (5.5) after an adjustment of a constant multiple in y_k, $1 \leq k \leq 3$.

6. Remarks and examples.

Remark 6.1. If (4.1) and (4.2) hold, and if (p_0, q_0) are both real or pure imaginary, then the dichotomy conditions (4.19) and (4.20) are satisfied. Moreover, the constants σ and ω are real.

Remark 6.2. If (5.1) and (5.2) hold, and if (p_0, q_0, q_1) are all real or pure imaginary, then (4.19) and (4.20) are satisfied.

Example 1.

$$q_0 = c_1 x^{\alpha_1}, \quad q_1 = c_2 x^{\alpha_2}, \quad p_0 = c_3 x^{\alpha_3}, \quad p_1 = c_4 x^{\alpha_4},$$

with α_i and c_j, $1 \le i \le 4$, are real constants with $c_i \neq 0$. Then (3.1), (3.2) and (3.6) hold under the condition

$$\begin{aligned} &\alpha_1 + \alpha_2 - 2\alpha_3 < 0, \\ &\alpha_3 + \alpha_4 - 2\alpha_2 < 0. \end{aligned} \tag{6.1}$$

Also, if we let $\alpha_2 \neq 0$, $\alpha_1 \neq 2\alpha_3$, then Euler case (4.1)–(4.2) is given by

$$\alpha_1 - \alpha_3 = 1, \tag{6.2}$$

and the nonzero numbers σ and ω are given by

$$\sigma = (1/2)\frac{c_1\alpha_2}{c_3}, \tag{6.3}$$

$$\omega = \frac{c_1}{c_3}(\alpha_1 - 2\alpha_3) = \frac{c_1}{c_3}(1 - \alpha_3), \tag{6.4}$$

and we require that

$$c_1(1-\alpha_3) \neq c_3, \quad c_1(1-\alpha_3) \neq 2c_3$$

and

$$c_3 \neq (1 + \alpha_2 - \alpha_3)c_1.$$

Also, $\phi = 0$ and $\psi = 0$ for this example. We note that (6.2) and (6.1) are equivalent to

$$\alpha_1 - \alpha_3 = 1, \quad \alpha_2 - \alpha_3 + 1 < 0, \quad \alpha_3 + \alpha_4 - 2\alpha_2 < 0. \tag{6.5}$$

Example 2. To give a quite different class of coefficients covered by our analysis, we consider

$$\begin{aligned} q_0(x) &= c_1 x^{\alpha_1} \exp x^b, \quad q_1(x) = c_2 x^{\alpha_2} \exp -x^b, \\ p_0(x) &= c_3 x^{\alpha_3} \exp x^b, \quad p_1(x) = c_4 x^{\alpha_4} \exp -4x^b, \end{aligned}$$

where α_i, c_j, $1 \leq i \leq 4$, and b are real constants with $c_i \neq 0$ and $b > 0$ satisfying $\alpha_1 + \alpha_2 - 2\alpha_3 < 0$. Then (3.1), (3.2) and (3.6) are all satisfied.

The Euler case (4.1)–(4.2) is given by

$$\alpha_3 - \alpha_1 = b - 1. \tag{6.6}$$

The values of σ and ω are given by

$$\sigma = -(1/2) b c_1 c_3^{-1}, \tag{6.7}$$

$$\omega = -b c_1 c_3^{-1} = 2\sigma, \tag{6.8}$$

and we require that $c_1 c_3^{-1} b \neq -1$, $c_1 c_3^{-1} b \neq -2$ and $c_1 c_3^{-1} b \neq -(1/2)$.

Now, in full, (4.1) and (4.2) are

$$-bx^{b-1} + \alpha_2 x^{-1} = -bx^{b-1}(1 + \phi),$$

and

$$-bx^{b-1} + (\alpha_1 - 2\alpha_3) x^{-1} = -bx^{b-1}(1 + \psi),$$

giving

$$\begin{aligned} \phi(x) &= -\alpha_2 b^{-1} x^{-b}, \\ \psi(x) &= -b^{-1}(\alpha_1 - 2\alpha_3) x^{-b}. \end{aligned} \tag{6.9}$$

Then $\phi(x)$ and $\psi(x)$ tend to zero as $x \to \infty$ and $\phi'(x)$ with $\psi'(x)$ are both $L(a, \infty)$. Similar examples can be given for Theorem 5.1.

Acknowledgment. I would like to thank Professor M.S.P. Eastham (London University) for his comments on this paper.

REFERENCES

1. A.S.A. Al-Hammadi, *Asymptotic theory for third-order differential equations*, Mathematika **35** (1988), 225–232.

2. ———, *Asymptotic theory for third-order differential equations with extension to higher odd-order equations*, Proc. Roy. Soc. Edinburgh Sect. A **117** (1991), 215–223.

3. M.S.P. Eastham, *The asymptotic solution of linear differential systems*, Mathematika **32** (1985), 131–138.

4. ———, *Asymptotic theory for a critical class of fourth-order differential equations*, Proc. Roy. Soc. London **383** (1982), 173–188.

5. N. Levinson, *The asymptotic nature of solutions of linear equations*, Duke Math. J. **15** (1948), 111–126.

6. G.W. Pfeiffer, *Asymptotic solutions of* $y''' + qy' + ry = 0$, J. Differential Equations **11** (1972), 145–155.

7. K. Unsworth, *Asymptotic expansions and deficiency indices associated with third-order self-adjoint differential operators*, Quart. Math. Oxford **24** (1973), 177–188.

Department of Mathematics, College of Science, University of Bahrain, P.O. Box 32038, Bahrain