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FINITE CODIMENSIONAL INVARIANT SUBSPACES OF BANACH SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let G be a bounded domain in the complex plane. Let $\mathcal{E}$ be a Banach space of functions analytic on G, such that for each $\lambda \in G$ the linear functional e_λ of evaluation at λ is bounded on $\mathcal{E}$. Assume further that $z\mathcal{E} \subset \mathcal{E}$ and, for every $\lambda \in G$, $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$. Here M_z is the operator of multiplication by z on $\mathcal{E}$ given by $f \mapsto zf$. In this article we characterize the finite codimensional subspaces of $\mathcal{E}$ which are invariant under M_z in some special cases.

1. Introduction. Let G be a bounded domain in the complex plane. Let $\mathcal{E}$ be a Banach space of functions analytic on G such that for each $\lambda \in G$ the linear functional e_λ of evaluation at λ is bounded on $\mathcal{E}$. Assume further that $z\mathcal{E} \subset \mathcal{E}$ and for every λ in G, $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$. A Banach space $\mathcal{E}$ with all the above properties is called a Banach space of analytic functions and is called a Banach space of functions if we only have $z\mathcal{E} \subset \mathcal{E}$. As a result we conclude that $M_z - \lambda$ is Fredholm for every $\lambda \in G$ and because $\dim \ker(M_z^* - \lambda) = 1$ we have $\operatorname{ind}(M_z - \lambda) = -1$ for $\lambda \in G$. A function $\varphi : G \to \mathbf{C}$ with the property $\varphi\mathcal{E} \subset \mathcal{E}$ is called a *multiplier* on $\mathcal{E}$, and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{E})$. If $\varphi \in \mathcal{M}(\mathcal{E})$, then the operator M_φ of multiplication by φ is bounded.

Richter [**11**] has shown that the commutant of the operator M_z is equal to $\{M_\varphi : \varphi \in \mathcal{M}(\mathcal{E})\}$. This makes $\mathcal{M}(\mathcal{E})$ into a Banach space by defining $\|\varphi\|_{\mathcal{M}(\mathcal{E})} = \|M_\varphi\|_{\mathcal{L}(\mathcal{E})}$. It is also true that $\mathcal{M}(\mathcal{E}) \subset H^\infty(G)$ and for each $\varphi \in \mathcal{M}(\mathcal{E}), \|\varphi\|_\infty \leq \|M_\varphi\|_{\mathcal{L}(\mathcal{E})} = \|\varphi\|_{\mathcal{M}(\mathcal{E})}$. Now suppose that $\mathcal{M}(\mathcal{E})$ contains a norm closed subalgebra $\mathcal{A}$ of $H^\infty(G)$. Then the above inequality shows that $\mathcal{A}$ is also closed in $\mathcal{M}(\mathcal{E})$ and the open

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mapping theorem applied to the map $i : (\mathcal{A}, \|.\|_{\mathcal{M}(\mathcal{E})}) \to (\mathcal{A}, \|.\|_\infty)$ yields $\|\varphi\|_{\mathcal{M}(\mathcal{E})} \le c\|\varphi\|_\infty$, for some positive constant c and all $\varphi \in \mathcal{A}$.

Let K be a compact subset of $\mathbf{C}$. We denote by $\mathrm{Rat}\,(K)$ the set of all rational functions with poles lying outside K. The closure of $\mathrm{Rat}\,(K)$ in the space $C(K)$ of all continuous complex valued functions on K is denoted by $R(K)$. A point $a \in K$ is said to be a peak point for $R(K)$ if there is a function $f \in R(K)$ such that $f(a) = 1$ and $|f(\zeta)| < 1$ for each $\zeta \ne a$.

Let F be a subset of the complex plane, and let $H(F)$ be the set of analytic functions f defined on $\mathbf{C}_\infty \backslash K$ for some compact subset K of F such that $f(\infty) = 0$ and $\|f\|_{\mathbf{C}_\infty \backslash K} \le 1$.

For a set $F \subset \mathbf{C}$, let $\gamma(F)$ denote the analytic capacity of F and define it by

$$\gamma(F) = \sup\{|f'(\infty)| : f \in H(F)\}.$$

The following results are useful and can be found in [**8**] and [**9**].

Proposition 1.1. (a) *If* $F_1 \subset F_2$*, then* $\gamma(F_1) \le \gamma(F_2)$*.*

(b) *If* K *is a compact subset of* $\mathbf{C}$*, then*

$$\gamma(K) = \gamma(\partial K) = \gamma(\hat{K}) = \gamma(\partial \hat{K}),$$

where $\hat{K}$ *is the union of* K *and bounded components of* K^c*.*

(c) *If* K *is a compact subset of* $\mathbf{C}$*, then*

$$\gamma(K) = \inf\{\gamma(U) : U \textit{ is an open set containing } K\}.$$

(d) *If* K *is compact and connected, then*

$$\gamma(K) \le \textit{diam}\, K \le 4\gamma(K).$$

Proposition 1.2. *If* K *is a compact set, there is a unique function* f *in* $H(\widehat{K})$*, which is called the* Ahlfors function*, such that* $f'(\infty) = \gamma(K)$*.*

Let $\mathcal{E}$ be a Banach space of analytic functions on G. A function $f \in \mathcal{E}$ is called a *cyclic vector* for the operator M_z if the polynomial multiples of f are dense in $\mathcal{E}$. Suppose $R(\overline{G}) \subset \mathcal{M}(\mathcal{E})$. We say that M_z is rationally cyclic if there is a function $g \in \mathcal{E}$ such that all rational multiples of g are dense in $\mathcal{E}$. That is, all elements of the form fg, $f \in \text{Rat}\,(\overline{G})$ are dense in $\mathcal{E}$.

Banach spaces of analytic functions exist in abundance, for example the Bergman spaces $L^p_a(G)$ for $1 \leq p < \infty$, the Banach algebra $H^\infty(G)$ of all bounded analytic functions on G, the Dirichlet spaces D_α, $-\infty < \alpha < \infty$, and many other examples which can be found in [**11**], [**8**] and [**2**].

Axler and Bourdon [**4**] characterized the finite codimensional invariant subspaces of $L^p_a(G)$ in the case when every connected component of ∂G contains more than one point, and Aleman [**3**] has done this for arbitrary bounded domains.

In this article we characterize the finite codimensional subspaces of $\mathcal{E}$ which are invariant under M_z in certain special cases. Our first assumption is that $\mathcal{E}$ contains a cyclic vector. In the second section we assume that $\mathcal{E}$ is a Banach algebra. We also characterize the finite codimensional subspaces of $\mathcal{E}$ which are invariant under M_z when the space contains a T-invariant subalgebra of $C(\overline{G})$ as a dense subset. We also consider reflexive Banach spaces $\mathcal{E}$. Finally we classify the boundary points of G.

For this characterization we need the next results which can be found in [**2**] and [**4**].

(a) If X is a normed linear space and $T : X \to X$ is a bounded operator such that $\dim X/[(T-\lambda)X]^- \leq 1$, for all $\lambda \in \sigma(T)$, then every finite codimensional invariant subspace E of T has the form $E = [p(T)X]^-$, where p is a polynomial whose degree equals the codimension of E and whose zeros lie in the residual spectrum of T.

(b) Let $\mathcal{E}$ be a Banach space of analytic functions on a plane domain G such that $\text{ran}\,(M_z - \lambda)$ is dense in $\mathcal{E}$ for every $\lambda \in \sigma(M_z)\backslash G$. Let $\mathcal{F}$ be a closed finite codimensional subspace of $\mathcal{E}$ that is invariant under M_z. Then $\mathcal{F} = p\mathcal{E}$ for some polynomial p whose roots lie in G.

Now if $\lambda \in G$, then $(z-\lambda)\mathcal{E}$ is closed and has codimension one, so that in order to apply (a) we investigate only the codimension of the subspace $[(z-\lambda)\mathcal{E}]^-$, $\lambda \in \sigma(M_z)\backslash G$.

We say that a Banach space of functions $\mathcal{E}$ has $*$-*property* if, for every

$\lambda \in \mathbf{C}$, $\dim[(z-\lambda)\mathcal{E}]^{\perp} \leq 1$. It is clear that, if $\mathcal{E}$ has *-property, then every finite codimensional subspace $\mathcal{M}$ of $\mathcal{E}$ which is invariant under M_z, has the form $\mathcal{M} = [p\mathcal{E}]^{-}$, where p is a polynomial whose degree equals the codimension of $\mathcal{M}$ in $\mathcal{E}$ and whose zeros lie in $\sigma(M_z)$ [**4**].

As a way of listing all the instances where a clear characterization of finite codimensional subspaces is obtained, we state the next theorem.

Theorem 1.3. (a) *Let $\mathcal{E}$ be a Banach space of analytic functions which contains a cyclic vector. Then $\mathcal{E}$ has $*$-property.*

(b) *If $\mathcal{E}$ is rationally cyclic, then $\mathcal{E}$ has $*$-property.*

(c) *Let $\mathcal{A}$ be a $T-$invariant subalgebra of $C(\overline{G})$. Then $\mathcal{A}$ has $*$-property.*

(d) *The finite codimensional subspaces of a reflexive Banach space of analytic functions $\mathcal{E}$ in either of the cases where $R(\overline{G}) \subset \mathcal{M}(\mathcal{E})$ and each point of ∂G is a peak point for $R(\overline{G})$ or $\mathcal{M}(\mathcal{E}) = H^{\infty}(G)$ and no connected component of ∂G is equal to a point can be completely characterized.*

Theorem 1.4. *Let $\mathcal{E}$ be a Banach space of functions. Furthermore, assume that each invariant subspace $\mathcal{M}$ of M_z with codimension one has the form $\mathcal{M} = [p\mathcal{E}]^{-}$, where p is a polynomial. Then $\mathcal{E}$ has $*$-property.*

Proof. Let $\lambda \in \mathbf{C}$ and x^* be a nonzero element of $[(z-\lambda)\mathcal{E}]^{\perp}$. Then $\mathcal{M} = \ker x^*$ is an invariant subspace of M_z of codimension one. Hence $\mathcal{M} = [p\mathcal{E}]^{-}$ for some polynomial p. Since, for each $f \in \mathcal{E}$, $\langle pf, x^* \rangle = p(\lambda)\langle f, x^* \rangle = 0$, it follows that $p(\lambda) = 0$. Therefore, $p = (z-\lambda)q$ for some polynomial q. We also note that $[(z-\lambda)q\mathcal{E}]^{-} \subset [(z-\lambda)\mathcal{E}]^{-} \subset \mathcal{M} \subset [p\mathcal{E}]^{-}$. Hence $\ker x^* = [(z-\lambda)\mathcal{E}]^{-}$ and the proof is complete. □

Theorem 1.5. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Banach spaces of functions, such that $\mathcal{E}_1 \subset \mathcal{E}_2$.*

(a) *Assume that $\mathcal{E}_1$ is closed in $\mathcal{E}_2$, $\dim \mathcal{E}_2/\mathcal{E}_1 = n < \infty$, and $\mathcal{E}_2$ has $*$-property. Then $\mathcal{E}_1$ has $*$-property.*

(b) *If $\mathcal{E}_1$ is dense in $\mathcal{E}_2$ and $\mathcal{E}_1$ has $*$-property, then $\mathcal{E}_2$ has $*$-property.*

Proof. Let $\mathcal{M}$ be an invariant subspace of $\mathcal{E}_1$ with codimension one in $\mathcal{E}_1$. Then $\dim \mathcal{E}_2/\mathcal{M} = n+1$. Therefore $\mathcal{M} = [q\mathcal{E}_2]^-$ and $\mathcal{E}_1 = [p\mathcal{E}_2]^-$, where p and q are polynomials whose degrees are equal to n and $n+1$, respectively. Because $\dim \mathcal{E}_1/\mathcal{M} = 1$, we have $[(z-\lambda)\mathcal{E}_1]^- \subset \mathcal{M}$ for some $\lambda \in \mathbf{C}$. But $[(z-\lambda)p\mathcal{E}_2]^- \subset [(z-\lambda)\mathcal{E}_1]^- \subset \mathcal{M} \subset \mathcal{E}_2$ and $n+1 = \dim \mathcal{E}_2/\mathcal{M} \le \dim \mathcal{E}_2/[(z-\lambda)p\mathcal{E}_2]^- \le n+1$. Therefore $\mathcal{M} = [(z-\lambda)\mathcal{E}_1]^-$ and the proof of (a) is complete by Theorem 1.4.

Let $\lambda \in \mathbf{C}$ and φ_1, φ_2 be two nonzero elements of $[(z-\lambda)\mathcal{E}_2]^\perp$. Then φ_1 and φ_2 are in $[(z-\lambda)\mathcal{E}_1]^\perp$. Hence $\varphi_1|_{\mathcal{E}_1} = \alpha\varphi_2|_{\mathcal{E}_1}$, for some constant α. Since $\mathcal{E}_1$ is dense in $\mathcal{E}_2$, it follows that $\varphi_1 = \alpha\varphi_2$. The proof of (b) is now complete. □

We now assume that $\mathcal{E}$ is a Banach space of analytic functions which contains a cyclic vector. We will give some examples of this type of Banach space at the end of this section.

Lemma 1.6. *Let $\mathcal{E}$ be a Banach space of analytic functions which contains a cyclic vector. If $\lambda \in \mathbf{C}$, then $\mathcal{E}$ has $*$-property.*

Proof. Let $\lambda \in \mathbf{C}$, $x^* \neq 0$ be an element of $[(z-\lambda)\mathcal{E}]^\perp$ and $g \in \mathcal{E}$ be a cyclic vector. Then, for each polynomial p, $\langle pg, x^*\rangle = p(\lambda)\langle g, x^*\rangle$. Hence $\langle g, x^*\rangle \neq 0$. Let $x_1^* \neq 0$ be another element of $[(z-\lambda)\mathcal{E}]^\perp$. Then $(\langle pg, x^*\rangle/\langle g, x^*\rangle) = (\langle pg, x_1^*\rangle/\langle g, x_1^*\rangle)$, for every polynomial p. The cyclicity of g in $\mathcal{E}$ shows that the above equality holds for each $f \in \mathcal{E}$; hence $\dim[(z-\lambda)\mathcal{E}]^\perp \le 1$. □

Example. Let $w \in C^2[0,1)$ be a positive integrable function. Denote by H_w the space of analytic functions f on the open unit disc $\mathbf{D}$ that satisfies

$$\|f\|_w^2 = |f(0)|^2 + \int_{\mathbf{D}} |f'(z)|^2 w(|z|)\,dA(z),$$

where dA is the area measure on $\mathbf{C}$. A simple computation shows that, if $f(z) = \sum_{n\ge 0} a_n z^n$ is analytic on $\mathbf{D}$, then $\|f\|_w^2 = \sum_{n\ge 0} |a_n|^2 w_n$, where $w_0 = 1$ and, for $n \ge 1$, $w_n = 2\pi n^2 \int_0^1 r^{2n-1} w(r)\,dr$. Hence

H_w is a separable Hilbert space of analytic functions in $\mathbf{D}$ and the polynomials are dense in H_w.

The *Dirichlet space* $\mathcal{D}$ is obtained when $w = 1$ and the *Hardy space* H^2 is obtained when $w(r) = 1 - r$, $r \in [0, 1)$. Therefore, if w is decreasing, concave, and satisfies $\lim_{r\to 0} w(r) = 0$, then $\mathcal{D} \subset H_w \subset H^2$.

The space H_w satisfies the conditions of Lemma 1.6 and, hence, the finite codimensional subspaces $\mathcal{M}$ of H_w which are invariant under M_z can be characterized accordingly.

Example. For a subarc I of $\partial\mathbf{D}$ and f in L^1, let $I(f) = 1/|I| \int_I f(t)\, dm(t)$, where dm is the arc measure. We say that f is of bounded mean oscillation and write $f \in$ BMO if $\|f\|_* = \sup_I(|f - I(f)|) < \infty$. BMO is a Banach space under the norm given by $\|f\| = |f(0)| + \|f\|_*$. Let VMO, the space of vanishing mean oscillations, be the closure of the continuous functions on $\partial\mathbf{D}$ in BMO. Let BMOA $=$ BMO $\cap H^1$ and VMOA $=$ VMO $\cap H^1$. One shows [**12**] that, if g is an outer function in VMOA, then g is a cyclic vector for M_z. Hence, the space VMOA satisfies the conditions of Lemma 1.6.

In [**6**] Bourdon has shown that if $G = \varphi(\mathbf{D})$ where φ is a weak-star generator of H^∞ the polynomials are dense in $L^2_a(G)$. Hence such spaces satisfy the above conditions. Another example can be found in [**8**].

2. Banach spaces of analytic functions. Let $\mathcal{A}$ be a Banach algebra of analytic functions on a bounded domain G which contains the constants and has $*$-property. It is clear that $\mathcal{M}(\mathcal{A}) = \mathcal{A} \subset H^\infty(G)$. We will show that $\mathcal{A}$ is a subalgebra of $C(\overline{G})$. At the end of this section we show that if a Banach space of analytic functions $\mathcal{E}$ contains a T-invariant subalgebra of $C(\overline{G})$ as a dense subset, then $\mathcal{E}$ has $*$-property.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra of analytic functions on a bounded domain G which contains the constants. If $\mathcal{A}$ has $*$-property, then $\mathcal{A} \subset C(\overline{G})$.*

Proof. Let $\lambda \in \overline{G}$ and $\{\beta_n\}$ be a sequence in G which converges to λ. Note that $\{e_{\beta_n}\}$ is a bounded sequence in the unit ball of $\mathcal{A}^*$. Hence there is a subsequence $\{e_{\beta_{n_i}}\}$ and $\varphi \in \mathcal{A}^*$ such that $e_{\beta_{n_i}} \to \varphi$ in the weak* topology. Since $1 \in \mathcal{A}$ and $e_{\beta_{n_i}}(1) = 1$, it follows that $\varphi(1) = 1$ and $\varphi \in [(z-\lambda)\mathcal{A}]^{\perp}$.

Let $\{\lambda_n\}$ be a sequence in G which converges to λ. Because $\{e_{\lambda_n}\}$ is a bounded sequence in the unit ball of $\mathcal{A}^*$, there exists a subsequence $\{e_{\lambda_{n_i}}\}$ and $\psi \in \mathcal{A}^*$ such that $e_{\lambda_{n_i}} \to \psi$ in the weak* topology. It is clear that ψ is an element of $[(z-\lambda)\mathcal{A}]^{\perp}$ and $\psi(1) = 1$. Hence $\psi = \varphi$ because $\mathcal{A}$ has $*$-property. Therefore $e_{\lambda_n} \to \varphi$ weak*, and it follows that f is continuous at λ. □

Remark. The above theorem shows that even $H^\infty(\mathbf{D})$ does not have $*$-property because $H^\infty(\mathbf{D}) \neq A(\overline{\mathbf{D}})$.

Let $g \in C_c^1$, and let f be a bounded Borel function on $\mathbf{C}$. Define $T_g f : \mathbf{C} \to \mathbf{C}$ by

$$T_g f(w) = \frac{1}{\pi} \int \frac{f(z) - f(w)}{z - w} \overline{\partial} g(z) dA(z).$$

The operator T_g is called the *Vitushkin localization operator*. If K is a compact subset of $\mathbf{C}$ and $\mathcal{A}$ is a closed subalgebra of $C(K)$, $\mathcal{A}$ is said to be *T-invariant* if $R(K) \subset \mathcal{A}$ and for each $f \in \mathcal{A}$ and $g \in C_c^1$, $T_g f \in \mathcal{A}$, where f is extended to $\mathbf{C}$ by letting it be identically zero off K.

Now let $\mathcal{E}$ be a Banach space of analytic functions on G, and let $\mathcal{A}$ be a T-invariant subalgebra of $C(\overline{G})$ such that $\mathcal{A} \subset \mathcal{E}$. Then the inclusion map $i : (\mathcal{A}, \|.\|_\infty) \longrightarrow (\mathcal{E}, \|.\|)$ is continuous, where $\|.\|_\infty$ is the supremum norm on $\mathcal{A}$. This follows from the continuity of point evaluations on $\mathcal{E}$. Therefore $\|f\| \leq c\|f\|_\infty$ for every $f \in \mathcal{A}$ and for some constant c.

The next two results are useful and can be found in Conway [**8**].

(c) Suppose $\mathcal{A}$ is a T-invariant subalgebra of $C(K)$ and $a \in K$. If $f \in \mathcal{A}$ and f has an analytic extension to a neighborhood of a, then $(f - f(a))/(z - a) \in \mathcal{A}$.

(d) If $\mathcal{A}$ is a T-invariant subalgebra of $C(K)$ and $a \in K$, then the subalgebra of $\mathcal{A}$ consisting of those functions in $\mathcal{A}$ that have analytic extension to a neighborhood of a is dense in $\mathcal{A}$.

Lemma 2.2. *Let $\lambda \in \partial G$ and $x^* \in [(z-\lambda)\mathcal{E}]^\perp$. Assume that $R(\overline{G}) \subset \mathcal{M}(\mathcal{E})$. Then, for each $f \in R(\overline{G})$ and $g \in \mathcal{E}$,*

$$\langle fg, x^* \rangle = f(\lambda)\langle g, x^* \rangle. \tag{1}$$

If $\mathcal{E}$ is rationally cyclic, then $\mathcal{E}$ has $*$-property.

Proof. Let $f \in R(\overline{G})$ have an analytic extension to a neighborhood of λ. Then $f - f(\lambda) = (z-\lambda)g_1$ for some $g_1 \in R(\overline{G})$. Hence (1) holds for such f.

By (d) every $f \in R(\overline{G})$ can be uniformly approximated by such functions, and for each $f \in R(\overline{G})$ and $g \in \mathcal{E}$,

$$\|fg\|_{\mathcal{E}} \leq \|f\|_{\mathcal{M}(\mathcal{E})}\|g\|_{\mathcal{E}} \leq c\|f\|_\infty \|g\|_{\mathcal{E}}.$$

Hence (1) holds for each $f \in R(\overline{G})$ and $g \in \mathcal{E}$. The proof of the second part follows the lines of the proof of Lemma 1.6. □

An example of spaces having the $*$-property can be constructed from the next theorem.

Theorem 2.3. *Let $\mathcal{A}$ be a T-invariant subalgebra of $C(\overline{G})$. Then $\mathcal{A}$ has $*$-property.*

Proof. Let $\lambda \in \partial G$, and let $x^* \neq 0$ be an element of $[(z-\lambda)\mathcal{A}]^\perp$. Assume that $f \in \mathcal{A}_\lambda$, the subalgebra of $\mathcal{A}$ consisting of those functions in $\mathcal{A}$ that have an analytic extension to a neighborhood of λ. Then, by (c), $f - f(\lambda) = (z-\lambda)g_1$ for some $g_1 \in \mathcal{A}$. Hence, $\langle f, x^* \rangle = f(\lambda)\langle 1, x^* \rangle$. By (d) it follows that this relation holds for each $f \in \mathcal{A}$. Hence, $\mathcal{A}$ has $*$-property. □

Aleman [**2**] has characterized the finite codimensional subspaces of Hilbert spaces of analytic functions when M_z is subnormal, $\sigma(M_z) = \overline{G}$ and $\mathcal{M}(\mathcal{E})$ contains $A(\overline{G})$, the space of continuous functions on $\overline{G}$ which are analytic on G. Here we do this for the case that $\mathcal{E}$ is a reflexive Banach space of analytic functions such that $R(\overline{G}) \subset \mathcal{M}(\mathcal{E})$ using the techniques of [**2**]. Note that the last condition implies that $\sigma(M_z) = \overline{G}$.

Theorem 2.4. *Let $\mathcal{E}$ be a reflexive Banach space of analytic functions such that $R(\overline{G}) \subset \mathcal{M}(\mathcal{E})$ and each point $\lambda \in \partial G$ is a peak point for $R(\overline{G})$. Let $\mathcal{F}$ be a closed finite codimensional invariant subspace of $\mathcal{E}$. Then $\mathcal{F} = p\mathcal{E}$ for some polynomial p whose roots lie in G.*

Proof. By (b) it is enough to show that $\operatorname{ran}(M_z - \lambda)$ is dense in $\mathcal{E}$ for each $\lambda \in \sigma(M_z)\backslash G$. Let $\lambda \in \partial G$ and $x^* \neq 0$ be an element of $[(z-\lambda)\mathcal{E}]^{\perp}$. There is an $f \in R(\overline{G})$ such that $f(\lambda) = 1$ and, for each $\zeta \neq \lambda$, $|f(\zeta)| < 1$, and there is a $g \in \mathcal{E}$ with $\langle g, x^* \rangle \neq 0$. For each $n \in \mathbf{N}$, $\langle f^n g, x^* \rangle = f^n(\lambda)\langle g, x^* \rangle = \langle g, x^* \rangle$ by Lemma 2.2, and $\|f^n g\|_{\mathcal{E}} \leq \|f^n\|_{\mathcal{M}(\mathcal{E})}\|g\|_{\mathcal{E}} \leq c\|f^n\|_{\infty}\|g\|_{\mathcal{E}} \leq c\|g\|_{\mathcal{E}}$. Hence, $\{f^n g\}$ is a bounded sequence in $\mathcal{E}$ and, for each $\zeta \neq \lambda$, $f^n(\zeta)g(\zeta) \to 0$ as $n \to \infty$. Since $\mathcal{E}$ is reflexive, there exists a subsequence $\{f^{n_i} g\}$ which converges to zero weakly, that is a contradiction. Therefore $\operatorname{ran}(M_z - \lambda)$ is dense in $\mathcal{E}$, and the proof is complete now. □

Remark. The conclusion of Theorem 2.4 does not hold without the assumption $R(\overline{G}) \subset \mathcal{M}(\mathcal{E})$. For example, let $\mathcal{E}$ be the weighted Dirichlet space D_α, $\alpha > 1$, and $\mathcal{M} = \{f \in D_\alpha : f(1) = 0\}$. It is clear that $\mathcal{M}$ is closed with codimension one which is invariant under M_z but cannot be written in the form pD_α for any polynomial p with zeros in the open unit disk. Note that in this case $\mathcal{M}(\mathcal{E}) = D_\alpha$. Let $f(z) = \sum(1/(n+1)^{\beta})z^n$, where $\beta = ((\alpha+1)/2)$. Then $f \in R(\overline{\mathbf{D}})$, and f is not in D_α.

The proof of the following theorem is mainly based on the proof of Curtis's peak point criterion which is cited in [**8**] and [**9**].

Theorem 2.5. *Let G be a bounded domain in the complex plane such that no connected component of ∂G is equal to a point. Assume further that $\mathcal{E}$ is a reflexive Banach space of analytic functions with $\mathcal{M}(\mathcal{E}) = H^\infty(G)$. Then, for each $\lambda \in \partial G$, $(z-\lambda)\mathcal{E}$ is dense in $\mathcal{E}$. Furthermore, every closed finite codimensional subspace $\mathcal{M}$ of $\mathcal{E}$ which is invariant under M_z can be written in the form $\mathcal{M} = p\mathcal{E}$ for some polynomial p whose roots lie in G.*

Proof. Let $\lambda \in \partial G$ and C_λ be the connected components of ∂G which contains λ. In [**4**] the authors showed that, for each $r < (\operatorname{diam} C_\lambda/2)$, the connected component K_r of $C_\lambda \cap B(\lambda, r)^-$ which contains λ meets $\partial B(\lambda, r)$. Therefore, for each $r < (\operatorname{diam} C_\lambda/2)$, there is a $\lambda_r \in \partial G$ so that $B(\lambda_r, r/4)^- \backslash G$ contains a connected subset F of K_r with $\operatorname{diam} F \geq (r/2)$ and λ is not in $B(\lambda_r, r/4)^- \backslash G$. Hence, we can choose a sequence $\{r_n\}$ of positive numbers and a sequence $\{\lambda_n\}$ in ∂G so that $r_n \to 0$, $B(\lambda_n, r_n/4)^- \backslash G \subset B(\lambda, r_n)^- \backslash G$, $\lambda \notin B(\lambda_n, r_n/4)^- \backslash G$ and

$$\gamma(B(\lambda_n, r_n/4)^- \backslash G) \geq r_n/8.$$

Hence, there is a sequence of functions $\{h_n\}$ such that, for each positive integer n, h_n is analytic on $\mathbf{C}_\infty \backslash L_n$, $\|h_n\|_{\mathbf{C}_\infty \backslash L_n} \leq 9$, $h_n(\lambda) = 1$. Also $h_n \to 0$ uniformly on compact subsets of G and $h_n(\lambda) = 1$, see Curtis's peak point criterion. Since $h_n \in H^\infty(\mathbf{C}_\infty \backslash L_n)$, there is an analytic function $k_n \in H^\infty(\mathbf{C}_\infty \backslash L_n)$ such that $h_n - h_n(\lambda) = (z - \lambda)k_n$.

For each $f \in \mathcal{E}$, the sequence $\{h_n f\}$ is a bounded sequence and hence $h_{n_j} f \to h$ weakly for some subsequence $\{h_{n_j} f\}$ and $h \in \mathcal{E}$. It is clear that $h = 0$. Now let $x^* \in [(z - \lambda)\mathcal{E}]^\perp$. Then, for each positive integer n,

$$\langle h_n f, x^* \rangle = h_n(\lambda)\langle f, x^* \rangle = \langle f, x^* \rangle.$$

Hence $x^* = 0$, and the proof is complete. □

Now assume $\mathcal{E}$ is a Banach space of analytic functions on a bounded domain G and $1 \in \mathcal{E}$. Denote the set of all $\lambda \in \partial G$ such that $\operatorname{ran}\,(M_z - \lambda)$ is closed by $\partial_r(\mathcal{E})$ and $\partial_e(\mathcal{E}) = \partial G \backslash \partial_r(\mathcal{E})$. Observe that if $\lambda \in \partial_r(\mathcal{E})$ then $\operatorname{ran}(M_z - \lambda) \neq \mathcal{E}$, otherwise $M_z - \lambda$ is invertible which contradicts the fact that $\overline{G} \subset \sigma(M_z)$. It will be seen that $\partial_r(\mathcal{E})$ is a relatively open subset of ∂G.

Next we study some properties of the boundary points of G. Part of the next lemma is similar to Subin's lemma; however, it is stated in our context.

Theorem 2.6. *Let $\lambda \in \partial_r(\mathcal{E})$. Then there is a neighborhood V of λ such that $V \cap \partial G \subset \partial_r(\mathcal{E})$, each $f \in \mathcal{E}$ has an analytic extension to V, and each point of V is an analytic bounded point evaluation. Moreover,* $\dim[(z - \lambda)\mathcal{E}]^\perp = 1$.

Proof. Because $\operatorname{ran}(M_z-\lambda)$ is closed, $M_z-\lambda$ is left invertible. Hence there is an operator B such that $B(M_z-\lambda)=I$. Let $V=\{\beta\in \mathbf{C}: |\beta-\lambda|<(1/\|B\|)\}$. Then $1-(\beta-\lambda)B$ is invertible for every $\beta\in V$ and $M_z-\beta=[1-(\beta-\lambda)B](M_z-\lambda)$. It follows that the $\operatorname{ran}(M_z-\beta)$ is closed for every $\beta\in V$. Hence $V\cap\partial G\subset\partial_r(\mathcal{E})$.

For the second part note that $\operatorname{ran}(M_z-\lambda)\neq\mathcal{E}$. Hence there exists $h_\lambda\in[(z-\lambda)\mathcal{E}]^\perp$ such that $\langle 1,h_\lambda\rangle\neq 0$. Replacing h_λ by a suitable multiple of itself, we may assume that $\langle 1,h_\lambda\rangle=1$. Let B and V be as before. Define $h:V\to\mathcal{E}^*$ by $h(\beta)=[1-(\beta-\lambda)B^*]^{-1}h_\lambda$. Clearly $h(\beta)\neq 0$ and $h(\beta)\in\ker(M_z-\beta)^*$. We also note that $h(\lambda)=h_\lambda$ and, because the function $\beta\mapsto\langle 1,h(\beta)\rangle$ is analytic on V by making V smaller, we may assume that $\langle 1,h(\beta)\rangle\neq 0$ for every $\beta\in V$. If $f\in\mathcal{E}$, then

$$\langle f,h(\beta)\rangle=\langle f,[1-(\beta-\lambda)B^*]^{-1}h_\lambda\rangle=\langle[1-(\beta-\lambda)B]^{-1}f,h_\lambda\rangle.$$

Hence the function $\beta\mapsto\langle f,h(\beta)\rangle$ is analytic on V. Let $\beta\in G\cap V$. Then there is a $k\in\mathcal{E}$ such that $f-f(\beta)=(z-\beta)k$. Therefore, $\langle f-f(\beta),h(\beta)\rangle=\langle(z-\beta)k,h(\beta)\rangle=0$. Hence $f(\beta)=(\langle f,h(\beta)\rangle/\langle 1,h(\beta)\rangle)$. For each $\beta\in V$, define $f_0(\beta)=(\langle f,h(\beta)\rangle/\langle 1,h(\beta)\rangle)$. It is clear that f_0 is an analytic extension of f on $G\cup V$. If we let $k(\beta)=(h(\beta)/\langle 1,h(\beta)\rangle)$, $\beta\in V$, then $k(\beta)$ is the reproducing kernel at β. Then $\dim[(z-\lambda)\mathcal{E}]^\perp=1$.

For the next part note that because $\operatorname{ran}(M_z-\beta)$ is closed for every $\beta\in V$, there is a $c>0$ such that $c\|f\|\leq\|(z-\lambda)f\|$, for each $f\in\mathcal{E}$. There is a sequence $\{\beta_n\}$ in G such that $\beta_n\to\lambda$. Hence there is $N>0$ so that $|\beta_n-\lambda|<c/2$ for each $n\geq N$. For such n and each $f\in\mathcal{E}$, $c\|f\|\leq\|(z-\lambda)f\|\leq\|(z-\beta_n)f\|+(c/2)\|f\|$. Therefore $(c/2)\|f\|\leq\|(z-\beta_n)f\|$. For each $n\geq N$ there is a $g_n\in\mathcal{E}$ so that $f-f(\beta_n)=(z-\beta_n)g_n$. It follows that $\{g_n\}$ is a bounded sequence in $\mathcal{E}$. Now let $\varphi\perp[(z-\lambda)\mathcal{E}]$. Therefore, $\varphi(f-f(\beta_n))=\varphi((z-\beta_n)g_n)=(\lambda-\beta_n)\varphi(g_n)$. Without loss of generality we may assume that $\varphi(g_n)\to\alpha$ for some $\alpha\in\mathbf{C}$. Therefore $\varphi(f)=f(\lambda)\varphi(1)$ and the proof is complete. □

Note that λ is a bounded point evaluation and $\operatorname{ran}(M_z-\lambda)\subset\ker e_\lambda$. Because both these spaces have codimension one, we easily conclude that $\operatorname{ran}(M_z-\lambda)=\ker e_\lambda$. An application of Cauchy's integral formula yields the following result.

Lemma 2.7. *Let G be a bounded domain in the complex plane, γ_0 a rectifiable simple closed curve in G and V the inside of γ_0. Assume further that $\mathcal{E}$ is a Banach space of analytic functions on G. Let $\mathcal{A}$ be a subset of $\mathcal{E}$ such that every $f \in \mathcal{A}$ is analytic on $G \cup V$. If $\mathcal{A}$ is dense in $\mathcal{E}$, then the following holds*

(i) *for each $\lambda \in G \cup V$ the point evaluation at λ is bounded.*

(ii) *each $f \in \mathcal{E}$ has an analytic extension to $G \cup V$.*

Proof. For each $f \in \mathcal{A}$ and $\lambda \in V \backslash G$ we have by the Cauchy's integral formula $f(\lambda) = (1/2\pi i) \int_{\gamma_0} (f(z)/(z-\lambda))\, dz$. Because γ_0 is a compact subset of G, there is a $c > 0$ such that, for each $f \in \mathcal{E}$ and $z \in \gamma_0$, $|f(z)| \leq c\|f\|$. Therefore, $|f(\lambda)| = |(1/(2\pi i)) \int_{\gamma_0} (f(z)/(z-\lambda))\, dz| \leq M_\lambda \|f\|$, for each $f \in \mathcal{A}$ and $\lambda \in V \backslash G$. Let $f \in \mathcal{E}$. Hence there is a sequence $\{f_n\}$ in $\mathcal{A}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$. Therefore, $\{f_n(\lambda)\}$ is a Cauchy sequence and hence $\lim_{n\to\infty} f_n(\lambda)$ exists. Define $f(\lambda) = \lim_{n\to\infty} f_n(\lambda)$. It is clear that $f(\lambda) = (1/(2\pi i)) \int_{\gamma_0} (f(z)/(z-\lambda))\, dz$, and $|f(\lambda)| = |(1/(2\pi i)) \int_{\gamma_0} (f(z)/(z-\lambda))\, dz| \leq M_\lambda \|f\|$, for each $f \in \mathcal{E}$ and $\lambda \in V \backslash G$. Therefore point evaluation at λ is bounded.

For any closed curve γ in $G \cup V$ and $f \in \mathcal{A}$, $\int_\gamma f(z)\, dz = 0$ and hence for each $f \in \mathcal{E}$, $\int_\gamma f(z)\, dz = 0$. Therefore, each $f \in \mathcal{E}$ is analytic on $G \cup V$. □

Example. Let $\mathbf{D}$ be the open unit disc in the complex plane. Delete from $\mathbf{D}$ a sequence of disjoint closed disc $\overline{B(x_n, r_n)}$, whose centers x_n lie on the positive real axis and decrease monotonically to zero. The region G obtained this way is called an L *region*.

Let $\mathcal{E}$ be a Banach space of analytic functions on G which contains the polynomials as a dense subset. Then by Lemma 2.7 the set of bounded point evaluations is $\mathbf{D}$, $\sigma(M_z) = \overline{\mathbf{D}}$ and each $f \in \mathcal{E}$ is analytic on $\mathbf{D}$.

These regions were first studied by L. Zalcman in [**13**], where he proved that for every $f \in H^\infty(G)$, $\lim_{z\to 0^-} f(z)$ exists if and only if $\sum_{n=1}^{\infty} (r_n/x_n) < \infty$. The equivalence condition for the existence of this limit for other Banach spaces of analytic functions can be found in [**10**].

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