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RANK 2 VECTOR BUNDLES IN A NEIGHBORHOOD OF AN EXCEPTIONAL CURVE OF A SMOOTH SURFACE

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ABSTRACT. Let $D \cong \mathbf{P}^1$ be an exceptional divisor on the smooth surface W and U the formal neighborhood of D in W. Let E be a rank 2 vector bundle on U. Here we associate to E an integer $t \geq 1$, a finite family E_i, $1 \leq i \leq t$, of rank 2 vector bundles on U and a finite sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$ of pairs of integers such that $E_i|D$ has splitting type (a_i, b_i), $E_1 = E$, $a_t = b_t$, $a_{i+1} + b_{i+1} = a_1 + b_1 + i$ and $b_i < b_{i+1} \leq a_{i+1} \leq a_i$ for $2 \leq i \leq t$. Vice versa, for any such sequence we prove the existence of at least one such bundle. We compute the second Chern class of E in terms of $\{(a_i, b_i)\}_{1 \leq i \leq t}$ and show that $\mathbf{O}_U(-a_1 D) \oplus \mathbf{O}_U(-b_1 D)$ is the unique bundle with splitting type (a_1, b_1) and maximal c_2.

0. Introduction. Let W be either a smooth connected quasi-projective surface defined over an algebraically closed field or a smooth connected two-dimensional manifold. We assume that W contains an exceptional divisor D, i.e., a smooth curve $D \cong \mathbf{P}^1$ with $\mathbf{O}_D(-1)$ as a normal bundle. Let U be either the formal completion of W along D or, if we work over $\mathbf{C}$, a small tubular neighborhood of D in W for the Euclidean topology. Let $\mathbf{I}$ be the ideal sheaf of D in U. Let E be a rank two vector bundle on U and (a, b) be the splitting type of $U|D$, i.e., let a, b be the integers with $a \geq b$ and such that $E|D \cong \mathbf{O}_D(a) \oplus \mathbf{O}_D(b)$. In the introduction of this paper we will associate to E an integer $t \geq 1$ and a finite sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$ of pairs of integers with $a_1 = a$, $b_1 = b$, $a_t = b_t = (a + b + t - 1)/2$ and a finite number of bundles E_i, $1 \leq i \leq t$, with $E_1 = E$, $E_i|D$ with splitting type (a_i, b_i).

Remark 0.1. It is well known and easy to check that if $E|D$ is trivial, then E is trivial. Furthermore, if W is quasi-projective, $E|D$ is trivial and $E \cong F|U$ with F algebraic vector bundle on W, then there exists a Zariski open neighborhood V of D with $F|V \cong \mathbf{O}_V^{\oplus 2}$ (use for

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instance the formal function theorem [**4**] for the blowing down of D). In particular we will have $E_t \cong \mathbf{O}_U(-a_tD)^{\oplus 2}$.

Fix a line bundle R on D and a surjection $\mathbf{r} : E \to R$ induced by a surjection $\rho : E|D \to R|D$. There exists such a surjection if and only if $\deg(R) \geq b$. If $\deg(R) = b < a$, then ρ is unique, up to a multiplicative constant. Hence if $\deg(R) = b < a$ the sheaf $\mathrm{Ker}\,(\mathbf{r})$ is uniquely determined, up to an isomorphism. Since D is a Cartier divisor, the sheaf $\mathrm{Ker}\,(\mathbf{r})$ is a vector bundle on U. We will say that $\mathrm{Ker}\,(\mathbf{r})$ is the bundle obtained from E making the negative elementary transformation induced by $\mathbf{r}$. Note that $\mathrm{Ker}\,(\rho)$ is a line bundle on D with $\deg(\mathrm{Ker}\,(\rho)) = a + b - \deg(R)$. Since $\deg(\mathbf{I}/\mathbf{I}^2) = 1$, it is easy to check that $\deg(\mathrm{Ker}\,(\mathbf{r})|D) = a + b + 1$ and that we have an exact sequence on D:

$$(1) \qquad \begin{aligned} 0 \longrightarrow \mathbf{O}_D(a + b + 1 - \deg(\mathrm{Ker}\,(\rho))) \\ \longrightarrow \mathrm{Ker}\,(\mathbf{r})|D \longrightarrow \mathrm{Ker}\,(\rho) \longrightarrow 0. \end{aligned}$$

Furthermore, using (1) we obtain a surjection $\mathbf{t} : \mathrm{Ker}\,(\mathbf{r}) \to \mathrm{Ker}\,(\rho)$ such that $\mathrm{Ker}\,(\mathbf{t}) \cong E(-D)$. In particular, $\mathrm{Ker}\,(\mathbf{t})|D \cong \mathbf{O}_D(a + 1) \oplus \mathbf{O}_D(b + 1)$. Thus, up to the twist by the line bundle $\mathbf{O}_U(-D)$, the negative elementary transformation induced by $\mathbf{r}$ has an inverse operation and we will say that E is obtained from $\mathrm{Ker}\,(\mathbf{r})$ making a positive elementary transformation supported by D. Note that if $\deg(R) = b$, then $\mathrm{Ker}\,(\mathbf{r})|D$ fits in an exact sequence

$$(2) \qquad 0 \longrightarrow \mathbf{O}_D(b + 1) \longrightarrow \mathrm{Ker}\,(\mathbf{r})|D \longrightarrow \mathbf{O}_D(a) \longrightarrow 0.$$

Hence if $b < a$ $\mathrm{Ker}\,(\mathbf{r})|D$ is more balanced than $E|D$. If $b \leq a - 3$, then (2) does not determine uniquely $\mathrm{Ker}\,(\mathbf{r})|D$. If $b \leq a - 2$ and $\mathrm{Ker}\,(\mathbf{r})|D$ is not balanced, we iterate the construction starting from $\mathrm{Ker}\,(\mathbf{r})$ and taking as R' the lowest degree factor of $\mathrm{Ker}\,(\mathbf{r})|D$ and the unique surjection (up to a multiplicative constant) $\rho' : \mathrm{Ker}\,(\mathbf{r}) \to R'$. In a finite number of steps, say $t - 1$ steps, we send E into a bundle which, up to a twist by $\mathbf{O}_U(-((a+b+t-1)/2)D)$, has trivial restriction to D and hence into a bundle isomorphic to $\mathbf{O}_U(-(a+b+t-1)/2)D)^{\oplus 2}$ by Remark 0.1. Set $E_1 := E$, $a_1 := a$ and $b_1 := b$. If $a_1 = b_1$, set $t := 1$ and stop. Assume $a_1 > b_1$. Hence we have defined the bundle $\mathrm{Ker}\,(\mathbf{r})$. Set $E_2 := \mathrm{Ker}\,(\mathbf{r})$. Let (a_2, b_2) be the splitting type of $\mathrm{Ker}\,(\mathbf{r})|D$. Note that $a_2+b_2 = a_1+b_1+1$ and $b_1 < b_2 \leq a_2 \leq a_1$. Hence $a_2-b_2 < a_1-b_1$.

If $a_2 = b_2$, set $t := 2$ and stop. If $a_2 > b_2$ iterate the construction. In a finite number of steps, say $t-1$ steps, we arrive at a bundle E_t with splitting type (a_t, b_t) with $a_t = b_t$. Call E_i, $2 \le i \le t$, the bundle we obtained after $i-1$ steps and (a_i, b_i) the splitting type of $E_i|D$. Note that the finite sequence $\{(a_i, b_i)\}_{1\le i\le t}$ of pairs obtained in this way has the following properties: $a_i \ge b_i$ for every $i > 0$, $a_i + b_i = a_1 + b_1 + i - 1$ for every $i > 1$, $a_i \ge a_{i+1} \ge b_{i+1} > b_i$ for every $i \ge 1$, $a_t = b_t$. We will call "admissible" any such finite sequence of pairs of integers. We will say that an admissible sequence $\{(a_i, b_i)\}_{1\le i\le t}$ is the admissible sequence associated to the bundle E if this sequence is created by the algorithm just described.

In Section 1 we will prove the following result which shows that every admissible sequence is associated to the process we described for some rank 2 vector bundle on W.

Theorem 0.2. *For every integer $t > 0$, every admissible sequence $\{(a_i, b_i)\}_{1\le i\le t}$ and every pair (W, D) there is a rank 2 vector bundle E on W with $\{(a_i, b_i)\}_{1\le i\le t}$ as an associated sequence.*

At the end of the unique section of this paper we will give two easy global existence theorems for vector bundles with a given associated sequence $\{(a_i, b_i)\}_{1\le i\le t}$, see Theorems 1.2 and 1.3, the second one concerning stable bundles.

Now assume W complete (or compact in the analytic case) and let $\pi : W \to Z$ be the blowing-down of D. Assume that $E \cong A|U$ with A rank 2 vector bundle on W. Set $d_2(A) := c_2(A) - c_2((\pi_*(A)^{**})$ and call the integer $d_2(A)$ the contribution of c_2 for A on D. Note that $d_2(A)$ depends only on $A|U$, i.e., on E. Note that $(\pi_*(A)^{**})$ is a vector bundle on Z because Z is smooth and two-dimensional. In Section 1 we will prove the following result which shows how to compute this integer in terms of the standard sequence $\{(a_i, b_i)\}_{1\le i\le t}$ associated to $A|U$.

Theorem 0.3. *Let E be a rank 2 vector bundle on W. Let $\{(a_i, b_i)\}_{1\le i\le t}$ be its admissible sequence with respect to D. We have $d_2(E) = \sum_{1\le i<t} a_i - a_t^2$.*

By Theorem 0.3 we may use the notation $d_2(\{(a_i, b_i)\}_{1\le i\le t})$ for the contribution, $d_2(E)$, of D to c_2 given by a vector bundle E with associated sequence $\{(a_i, b_i)\}_{1\le i\le t}$.

Remark 0.4. Note that if $b_1 = -a_1$ we have $a_1 \le d_2(A) \le a_1^2$, the lower bound being achieved if and only if $t = 3$ and $b_2 = 0$, and hence $a_2 = a_3 = b_3 = 1$, while the upper bound being achieved if and only if $a_i = a_1$ for every i or, equivalently, if and only if $t = 2a_1 + 1$ and $b_i = b_1 + i - 1$ for every i. These bounds are well-known, see, for instance, [**1**, Remark 5.4]. Note that if $b_1 = -a_1 - 1$ we have $a_1 \le d_2(A) \le a_1(a_1 + 1)$, the lower bound being achieved if and only if $t = 2$ and $b_2 = 0$, and hence $a_2 = 0$, while the upper bound being achieved if and only if $t = 2$ and $b_2 = 0$, and hence $a_2 = 0$, while the upper bound being achieved if and only if $a_i = a_1$ for every i or, equivalently, if and only if $t = 2a_1 + 2$ and $b_i = b_1 + i - 1$ for every i.

In Section 1 we will prove that if the upper bounds in Theorem 0.3 are achieved, then $E|U \cong \mathbf{O}_U(-a_1D) \oplus \mathbf{O}_U(-b_1D)$. Since $\mathrm{Pic}\,(U) \cong \mathbf{Z}$ and D is a generator of $\mathrm{Pic}\,(U)$, this is equivalent to the splitting of $E|U$ into the direct sum of two line bundles.

Theorem 0.5. *Let A be a rank 2 vector bundle on U with splitting type (a_1, b_1) on D. The following conditions are equivalent*:

(i) $A \cong \mathbf{O}_U(-a_1D) \oplus \mathbf{O}_U(-b_1D)$;

(ii) $d_2(A) = -a_1b_1$;

(iii) *the admissible associated sequence $\{(a_i, b_i)\}_{1\le i\le t}$ of A has $b_i = b_1 + i - 1$ for every i or, equivalently, the admissible associated sequence $\{(a_i, b_i)\}_{1\le i\le t}$ of A has $t = a_1 - b_1 + 1$.*

1. Proofs of 0.2, 0.3 and 0.5. In the unique section of this paper we give the proofs of Theorems 0.2, 0.3 and 0.5 and then the promised global existence theorems extending 0.2, see Theorems 1.2 and 1.3.

Proof of Theorem 0.2. We use induction on t. If $t = 1$ we take $E \cong \mathbf{O}_W(-a_1D)^{\oplus 2}$. Now we assume $t \ge 2$ and use the inductive assumption on t. Take a rank 2 vector bundle F on W with associated sequence $\{(a_i, b_i)\}_{2\le i\le t}$. Since $F|D \cong \mathbf{O}_D(a_2) \oplus \mathbf{O}_D(b_2)$ and $a_1 \ge a_2 \ge b_2$, there is a surjection $u : F \otimes \mathbf{O}_W(D)|D \to \mathbf{O}_D(a_1)$ and hence a surjection

$v : F \otimes \mathbf{O}_W(D) \to \mathbf{O}_D(a_1)$. Set $E := \mathrm{Ker}\,(v)$. By construction we have $\deg(E|D) = a_2 + b_2 - 1 = a_1 + b_1$. Since $\mathrm{Ker}\,(u)$ has degree b_1, $E|D$ is an extension of a line bundle of degree b_1 by a line bundle of degree a_1. Since $a_1 \geq b_1$, this extension splits, i.e., $E|D$ has splitting type (a_1, b_1). Let $\mathbf{r} : E \to \mathbf{O}_D(b_1)$ be the unique (up to a multiplicative constant) surjection. By the commutativity of elementary transformations, we have $F \cong \mathrm{Ker}\,(\mathbf{r})$. Hence F is the bundle E_2 associated to E in the first step of our reduction process. Thus E has associated sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$. □

Before proving Theorem 0.3 we give a brief discussion on Chern classes of coherent sheaves on U. We use the set-up and the notations of the statement of Theorem 0.3. We need to compute the total Chern class $c(F)$ of some coherent but nonlocally free sheaf on W. If W is not algebraic, the total Chern class $c(F)$ may be defined extending F to the associated real-analytic coherent sheaf, $F_{\mathbf{R}\text{-an}}$, taking a finite resolution $\mathbf{F} = \{F_i\}_{1 \leq i \leq s}$ of $F_{\mathbf{R}\text{-an}}$ by locally free coherent real analytic sheaves and setting $c(F) = c(\mathbf{F})$, where $c(\mathbf{F})$ is computed formally as $\prod_{1 \leq i \leq s} c(F_i)^{-i}$. We need it only in the following two cases. Let T be a torsion coherent sheaf on W. Using the multiplicative property of the total Chern character and the locally free resolution of the structural sheaf of complete intersection zero-dimensional subscheme of W, we obtain $c_1(T) = 0$ and $c_2(T) = \mathrm{length}\,(T) = h^0(W, T)$. Take $R \in \mathrm{Pic}^{\,t}(D)$ and see R as a torsion sheaf on W with resolution

$$0 \longrightarrow \mathbf{O}_W(-(t+1)D) \longrightarrow \mathbf{O}_W(-tD) \longrightarrow R \longrightarrow 0. \tag{3}$$

By the exact sequence (3) we have $c_1(R) = D$ and $c_2(R) = -t - 1$.

Proof of Theorem 0.3. If $t = 1$ we have $a_1 = b_1$ and hence we have $d_2(E) = a_1 c_1(E(a_1 D)) \cdot D = a_1^2$ by the triviality of $E(a_1 D)|U$, Remark 0.1. Now we assume $t > 1$ and use induction on t. We have an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow R \longrightarrow 0 \tag{4}$$

with F a rank 2 vector bundle on W with associated sequence $\{(a_i, b_i)\}_{2 \leq i \leq t}$ and $R \in \mathrm{Pic}^{\,b_1}(D)$. Hence by the inductive assumption we have $c_1(F) = c_1(E) - D = -(a_1 + b_1 + 1)D$ and $d_2(E) =$

$c_2(R)+d_2(F)+c_1(F)\cdot c_1(R) = -b_1-1+(\sum_{2\le i<t} a_i - a_t^2)+a_1+b_1+1 = \sum_{1\le i<t} a_i + a_t^2$, as wanted. □

Proof of Theorem 0.5. By Theorem 0.3 and Remark 0.4, conditions (ii) and (iii) are equivalent. Since $c_2(\mathbf{O}_U(-xD)\oplus \mathbf{O}_U(-yD)) = xyD^2 = -xy$ for every x, y, condition (i) implies condition (ii). Assume condition (iii). We use induction on t, the case $t = 1$ being trivial by Remark 0.1. Take the exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow \mathbf{O}_D(-b_1D) \longrightarrow 0 \tag{5}$$

giving the modification process for A. Hence B has admissible associated sequence $\{(a_i, b_i)\}_{2\le i\le t}$. By the inductive assumption we have $B \cong \mathbf{O}_U(-a_1D)\oplus\mathbf{O}_U(-b_2D)$. By the exact sequence (3) we have $\mathrm{Ext}^1(\mathbf{O}_D(xD), \mathbf{O}_U(yD)) \cong \mathbf{O}_D((y-x+1)D)$ for all integers x, y. Hence, by the local-to-global spectral sequence for the Ext-functors, we obtain $\mathrm{Ext}^1(U; \mathbf{O}_D(xD), \mathbf{O}_U(yD)) = 0$ if $x \ge y$ and $\dim(\mathrm{Ext}^1(U; \mathbf{O}_D(xD), \mathbf{O}_U((x+1)D))) = 1$. Since $B \cong \mathbf{O}_U(-a_1D)\oplus \mathbf{O}_U(-b_2D)$ and $b_2 = b_1 + 1$, there is a unique (up to a constant) nontrivial extension (5) and hence only one bundle may be the middle term of an exact sequence (5). Since $\mathbf{O}_U(-a_1D)\oplus\mathbf{O}_U(-b_1D)$ is one such middle term, A splits. □

Remark 1.1. We claim that in the algebraic case if $E|U$ splits as the direct sum $L \oplus L'$ of two line bundles, then there is a Zariski open neighborhood W' of D in W such that $\mathrm{Pic}(W') \cong \mathbf{Z}[D]$ and $E|W' \cong \mathbf{O}_{W'}(-a_1D)\oplus\mathbf{O}_{W'}(-b_1D)$. To check the claim just use that by Theorem A of Cartan-Serre for every affine neighborhood V of the point $\pi(D) \in Z$ any isomorphism between the stalk of $\pi_*(E)$ at $\pi_*(D)$ and the stalk of $\pi_*(L\oplus L')$ extends to V.

Using that a negative elementary transformation has an inverse (the positive elementary transformation) the proof of Theorem 0.2 gives immediately the following global result, both in the algebraic and in the analytic category.

Theorem 1.2. *Let $\pi : W \to Z$ be the blowing-down of D and B a rank two vector bundle on Z. Fix an admissible sequence $\{(a_i, b_i)\}_{1\le i\le t}$. Then there exists a rank two vector bundle F on W*

such that $F|U$ has associated admissible sequence $\{(a_i,b_i)\}_{1\le i\le t}$ and associated bundles E_i, $1\le i\le t$, with $E_i \cong F_i|U$, F_i vector bundle on W, $1\le i\le t$, and $F_t \cong \pi^*(B)\otimes \mathbf{O}_U(-a_tD)$.

We would like to have an improvement of Theorem 0.2, i.e., to have for every (or for some) admissible sequence the proof of the existence of a vector bundle with some nice properties and with $\{(a_i,b_i)\}_{1\le i\le t}$ as an associated sequence. In the case of projective surfaces the more interesting bundles are the stable ones (with respect to a polarization). For any smooth, connected projective surface X, every c_1, $H \in \mathrm{Pic}\,(X)$ with H ample and every integer d, let $M(X;2,c_1,d,H)$ be the moduli space of all H-stable rank 2 vector bundles, E, on X with $c_1(E)=c_1$ and $c_2(E)=d$. We have the following existence theorem.

Theorem 1.3. *Assume characteristic zero. Fix a smooth, connected, projective surface Z, $P\in Z$, L, $H\in \mathrm{Pic}\,(Z)$ with H ample and an admissible sequence $\{(a_i,b_i)\}_{1\le i\le t}$. Let $\pi: Y\to Z$ be the blowing-up of P and set $D:=\pi^1(P)$. Fix a small rational number $\varepsilon>0$ such that the Q-divisor $H_\varepsilon := \pi^*(H)-\varepsilon D$ is ample on Y. Then there exists an integer u, depending only on X, P, H, L, ε and a_1+b_1, such that for every integer $y\ge u$ and a general $B\in M(Z;2,L,y,H)$ there is an H_ε-stable bundle $E \in M(Y;2,\pi^*(L)\otimes \mathbf{O}_Y(D)^{\otimes(a_1+b_1)}$, $y+d_2(\{(a_i,b_i)\}_{1\le i\le t}),H_\varepsilon)$ with $\{(a_i,b_i)\}_{1\le i\le t}$ as associated admissible sequence, $E_t \cong \pi^*(B)\otimes \mathbf{O}_U(-a_tD)$ and such that every associated bundle E_i, $1\le i\le t$, is H_ε-stable.*

Here, since t is large, "general" means by [**5**, Theorem 2], "in a Zariski dense open subset of the equidimensional scheme $M(Z;2,L,y,H)$," but indeed by [**3**] we know that for large t the schemes $M(Z;2,L,y,H)$ and $M(Y;2,\pi^*(L)\otimes \mathbf{O}_Y(D)^{\otimes(a_1+b_1)}, y+d_2(\{(a_i,b_i)\}_{1\le i\le t}),H_\varepsilon)$ are irreducible. Theorem 1.3 follows from the following observation made independently by several peoples. The proof in [**5**, Theorem 2], gives an upper bound for the dimension of the "bad" set $\mathbf{B} := \{E \in M(Y;2,c_1,k,R) : h^0(Y,\mathrm{Hom}\,(E,E\otimes K_Y)) > h^0(K_Y)\}$ of a moduli space $M(Y;2,c_1,k,R)$, $R\in \mathrm{Pic}\,(Y)$, R ample, showing in this way that for large k, $M(Y;2,c_1,k,R)\backslash \mathbf{B}$ is Zariski dense in $M(Y;2,c_1,k,R)$. Fix an effective Cartier divisor Γ on Y and set $\mathbf{B}(\Gamma) := \{E \in M(Y;2,c_1,k,R) : h^0(Y,\mathrm{Hom}\,(E,E\otimes K_Y)) > h^0(K_Y\otimes \Gamma)\}$. The same computations give that for large k, $M(Y;2,c_1,k,R)\backslash \mathbf{B}(\Gamma)$ is Zariski dense in $M(Y;2,c_1,k,R)$. By Serre duality this means essentially that for large k we may find bundles $E \in M(Y;2,c_1,k,R)_{\mathrm{red}}$ with prescribed $E|\Gamma$. Take $X = Y$ and $\Gamma := (a_1 - b_1 + 1)D$. We obtain that for large y for the general $B \in M(Z;2,L,y,H)$ not only the bundle $\pi^*(B)$ is H_ε-stable, but the H_ε-stability is preserved by the positive elementary transformations needed to pass from $\pi^*(B)$ to a bundle with associated sequence and with $\pi^*(B)(-a_tD)$ as last associated bundle.

Up to formal (or complex analytic) isomorphisms any two formal, respectively tubular, neighborhoods of exceptional divisors of a quasi-projective, or complex, two-dimensional manifolds are equivalent. Hence, for every quadruple (W,D,U,E) as above and for every triple (W',D',U') as above, i.e., with D' exceptional divisor in W' and U' formal completion of D' or small, depending on U, tubular Euclidean neighborhood of D'in W', there is a unique bundle E' on U' with $(E',U') \cong (E,U)$ and in particular with $d_2(E') = d_2(E)$. Hence the explicit matrix calculations given in [**2**] on the blowing-up of $\mathbf{C}^2$ at the origin may be used in any abstract situation. Perhaps these computations may give some of the results proven in this paper, but we prefer our approach because it points out the role of the associated admissible sequences. We believe that the stratification of the set of all rank two vector bundles on U by their associated admissible sequence is a very natural set-theoretic stratification. We give a partial order on the set of all admissible sequences. Take two admissible sequences $\{(a_i,b_i)\}_{1\le i\le t}$ and $\{(c_i,d_i)\}_{1\le i\le t'}$. We will write $\{(a_i,b_i)\}_{1\le i\le t} \ge \{(c_i,d_i)\}_{1\le i\le t'}$ if $a_1 + b_1 = c_1 + d_1$ and for all integers i with $1 \le i \le \min\{t,t'\}$, we have $a_i \le c_i$. Note that if $\{(a_i,b_i)\}_{1\le i\le t} \ge \{(c_i,d_i)\}_{1\le i\le t'}$ we have $t \le t'$. This partial order is generated by a "proximity relation": we will say that $\{(a_i,b_i)\}_{1\le i\le t}$ is larger and proximal to $\{(c_i,d_i)\}_{1\le i\le t'}$ if $a_1 + b_1 = c_1 + d_1$ and either $t' = t$ and there is an integer $s < t$ with $a_i = c_i$ for $i \ne s$, $a_s = c_s - 1$, or $t' = t+2$, $a_i = c_i$ for $i < t$ and $c_t = d_t + 2$, i.e., $c_t = a_t + 1$. This partial order behaves well in flat families of rank 2 vector bundles on U in the following sense. Let $\{E_\lambda\}_{\lambda\in\Delta}$ be a flat family of rank 2 vector bundles parametrized by an integral variety Δ. Fix $P \in \Delta$. Call $\{a_i(\lambda),b_i(\lambda)\}_{1\le i\le t(\lambda)}$ the admissible sequence of E_λ. Notice that $a_1(\lambda)+b_1(\lambda) = a_1(P)+b_1(P)$ for every $\lambda \in \Delta$. By the semicontinuity of cohomology we have $a_1(\lambda) \le a_1(P)$ and $b_1(\lambda) \ge b_1(P)$ for a general $\lambda \in \Delta$. Assume $t(P) \ge 2$, $a_1(\lambda) = a_1(P)$ and $b_1(\lambda) = b_1(P)$

for a general $\lambda \in \Delta$. By the semicontinuity of cohomology we obtain $a_2(\lambda) \leq a_2(P)$ and $b_2(\lambda) \geq b_2(P)$ for a general $\lambda \in \Delta$. And so on. In particular, we have $t(\lambda) \leq t(P)$ for a general $\lambda \in \Delta$.

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