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CHAIN CATEGORIES OF MODULES AND SUBPROJECTIVE REPRESENTATIONS OF POSETS OVER UNISERIAL ALGEBRAS

DANIEL SIMSON

ABSTRACT. Filtered chain categories $\mathcal{C}(s, R)$ of modules over a commutative artinian uniserial ring R and their representation types are studied in the paper. A tame-wild dichotomy theorem is proved in case R is a finite dimensional K-algebra over an algebraically closed field K. The pairs (s, R) for which $\mathcal{C}(s, R)$ is of finite representation type are determined. In case $R = K[t]/(t^m)$ and K is algebraically closed, the pairs (s, m) for which $\mathcal{C}(s, R)$ is of tame representation type are listed. The problem is reduced to the study of categories of subprojective representations of posets over uniserial algebras and then to representations of posets over a field by applying a Galois covering functor technique.

1. Introduction. Let R be a unitary commutative artinian uniserial ring with the Jacobson radical $J(R)$. We recall that R is uniserial if the ideals of R form a finite chain. In this case $J(R)$ is the unique maximal ideal of R, and there is an integer $m \geq 1$ such that $J(R)^m = 0$, $J(R)^{m-1} \neq 0$ and any ideal of R appears in the chain

$$R \supset J(R) \supset J(R)^2 \supset \cdots \supset J(R)^{m-1} \supset J(R)^m = 0. \tag{1.1}$$

Examples of such rings R are the ring $\mathbf{Z}/p^m\mathbf{Z}$ of integers modulo p^m or the uniserial K-algebra $F_m = K[t]/(t^m)$, where $p \geq 2$ is a prime, $m \geq 1$ is an integer and K is a field.

Following Arnold [**1**] and [**2**], given an integer $s \geq 1$ we consider the filtered chain category $\mathcal{C}(s, R)$ whose objects are filtered s-chains

$$C = (C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{s-1} \subseteq C_s) \tag{1.2}$$

of finitely generated R-modules $C_1, \dots, C_s$, and a morphism from C to C' in $\mathcal{C}(s, R)$ is an R-module homomorphism $f : C_s \to C'_s$ such that

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$f(C_j) \subseteq C'_j$ for $j = 1, \ldots, s-1$. The direct sum of two objects C and C' in $\mathcal{C}(s, R)$ is defined to be the s-chain

$$C \oplus C' = (C_1 \oplus C'_1 \subseteq C_2 \oplus C'_2 \subseteq \cdots \subseteq C_{s-1} \oplus C'_{s-1} \subseteq C_s \oplus C'_s).$$

One can show that $\mathcal{C}(s, R)$ is an additive Krull-Schmidt category with enough relative projective objects and enough relative injective objects and that $\mathcal{C}(s, R)$ has almost split sequences. The category $\mathcal{C}(s, R)$ is said to be of *finite representation type* if the number of the isoclasses of indecomposable objects in $\mathcal{C}(s, R)$ is finite.

Following [**17**, Corollary 5.7], one shows that, for any R as above, relative projective objects in $\mathcal{C}(s, R)$ are relative injective, and relative injective objects in $\mathcal{C}(s, R)$ are relative projective. This means that the category $\mathcal{C}(s, R)$ is relatively quasi-Frobenius.

In [**2**], Arnold is interested in the problem when the category $\mathcal{C}(s, R)$ is of finite, tame or wild representation type, where the tame type is understood rather intuitively in the case R is not a finite dimensional algebra over an algebraically closed field. A tame-wild dichotomy result for $\mathcal{C}(s, R)$ is not established in [**2**]. The problem for $s = 2$ is known as Birkhoff's problem. It has been solved by Richman and Walker [**14**] in the representation-finite case.

In the case R is the uniserial K-algebra $F_m = K[t]/(t^m)$, $m \geq 1$, and the field K is algebraically closed, we define in Section 2 a tame type, a polynomial growth and a wild type for $\mathcal{C}(s, R)$ (see Definition 2.3) and we prove in Corollary 2.9 a tame-wild dichotomy for $\mathcal{C}(s, R)$. A complete solution of the above problem is a consequence of the following three theorems proved in (3.9).

Theorem 1.3. *Let R be a commutative artinian uniserial ring and $m \geq 1$ such that $J(R)^m = 0$ and $J(R)^{m-1} \neq 0$. The category $\mathcal{C}(s, R)$ is of finite representation type if and only if the pair (m, s) of integers satisfies any of the following conditions*:

(F1) $m = 1$ *or* $s = 1$,

(F2) $m \leq 5$ *and* $s = 2$,

(F3) $m \leq 3$ *and* $3 \leq s \leq 4$,

(F4) $m = 2$ *and* $s \geq 5$.

Theorem 1.4. *Let K be an algebraically closed field, $m \geq 1$ an integer and $F_m = K[t]/(t^m)$. The category $\mathcal{C}(s, F_m)$ is of wild representation type if and only if the pair (m, s) of integers satisfies any of the following conditions*:

(W1) $m \geq 7$ *and* $s \geq 2$,

(W2) $m \geq 5$ *and* $s \geq 3$,

(W3) $m \geq 4$ *and* $s \geq 5$,

(W4) $m \geq 3$ *and* $s \geq 6$.

Theorem 1.5. *Let K be an algebraically closed field, $m \geq 1$ an integer and $F_m = K[t]/(t^m)$. The following three conditions are equivalent*:

(a) *The category $\mathcal{C}(s, F_m)$ is of tame representation type.*

(b) *The category $\mathcal{C}(s, F_m)$ is tame of polynomial growth.*

(c) *The pair (m, s) of integers satisfies any of the conditions* (F1)–(F4) *of Theorem* 1.3*, or any of the following three conditions*:

(T1) $m = 6$ *and* $s = 2$,

(T2) $m = 4$ *and* $3 \leq s \leq 4$,

(T3) $m = 3$ *and* $s = 5$.

The proof of Theorems 1.3, 1.4 and 1.5 is given in (3.8) by applying the reduction functor $\mathbf{res} : \mathbf{fspr}\,(I, R) \to \mathcal{C}(s, R)$ (2.7), the reduction given in Proposition 2.8 and corresponding representation type results in categories $\mathbf{fspr}\,(I, R)$ of filtered subprojective R-representations of finite posets I presented in Theorems 3.4 and 3.6.

In case $s = 2$ and $m \leq 6$ the structure of the category $\mathcal{C}(s, \mathbf{Z}/p^m\mathbf{Z})$ has been described in [**15**]. Let us also recall that chain categories of modules and the geometric structure of the representation spaces has been investigated in [**4**].

Throughout this paper we denote by K a field and by $\mathrm{mod}\,(B)$ the category of finitely generated unitary right B-modules, where B is a ring with an identity element.

2. Filtered chain categories and a reduction to subprojective representations of finite posets. Let R be a commutative artinian uniserial ring. Consider the R-subalgebra

$$\mathbf{T}_s(R) = \begin{pmatrix} R & R & \cdots & R \\ 0 & R & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{pmatrix}$$

of the full matrix algebra $\mathbf{M}_s(R)$ consisting of all s by s upper triangular matrices $a = [a_{pq}]$ in $\mathbf{M}_s(R)$ with zeros below the main diagonal. Note that there is a natural functorial embedding

$$\mathcal{E} : \mathcal{C}(s, R) \longrightarrow \operatorname{mod} \mathbf{T}_s(R) \tag{2.0}$$

of categories defined by attaching to each object C of $\mathcal{C}(s, R)$ (see (1.2)) the group $\mathcal{E}(C) = C_1 \oplus \cdots \oplus C_s$ equipped with the right $\mathbf{T}_s(R)$-module structure defined by the formula $(c_1, \dots, c_s) \cdot [a_{pq}] = (c'_1, \dots, c'_s)$ where $c'_j = \sum_{t=j}^{s} c_t a_{tj}$. It is easy to see that $\mathcal{E}$ is a fully faithful exact functor and therefore $\mathcal{C}(s, R)$ may be viewed as a full subcategory of the module category $\operatorname{mod} \mathbf{T}_s(R)$. The indecomposable projective right $\mathbf{T}_s(R)$-modules $e_1\mathbf{T}_s(R), \dots, e_s\mathbf{T}_s(R)$ are in the image of $\mathcal{E}$, because it is easy to see that, for any $j \le s$, there is a $\mathbf{T}_s(R)$-module isomorphism $\mathcal{E}(P_j) \cong e_j\mathbf{T}_s(R)$, where

$$P_j = (0 \hookrightarrow \cdots \hookrightarrow 0 \hookrightarrow R \xrightarrow{\text{id}} R \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} R) \tag{2.1}$$

is the object of $\mathcal{C}(s, R)$ with the module R on the coordinates $j, j+1, \dots, s$ and zeros on the remaining coordinates. Here $\{e_1, \dots, e_s\}$ is the standard set of primitive matrix idempotents in $\mathbf{T}_s(R)$. It follows that $\{P_1, \dots, P_s\}$ is a complete set of indecomposable projective objects of $\mathcal{C}(s, R)$ up to isomorphism.

Following [**17**, Chapter 5] we define the contravariant functor $D^\bullet : \mathcal{C}(s, R) \to \mathcal{C}(s, R)$ by attaching to any s-chain C (1.2) the s-chain $D^\bullet(C) = (C_1^\bullet \subseteq C_2^\bullet \subseteq \cdots \subseteq C_{s-1}^\bullet \subseteq C_s^\bullet)$, where $C_s^\bullet = \operatorname{Hom}_R(C_s, R)$ and $C_j^\bullet$ is the kernel of the epimorphism $\operatorname{Hom}_R(C_s, R) \to \operatorname{Hom}_R(C_j, R)$ induced by the embedding $C_j \subseteq C_s$ for $j \le s - 1$. Since the ring R is self-injective, then the functor is a duality of categories. We call it a *reflection-duality*.

It is easy to see that $D^{\bullet}(P_j) \cong P_{s-j+1}$. Hence we easily conclude as in [**17**, Corollary 5.7] that the indecomposable projective objects of $\mathcal{C}(s,R)$ are relatively injective and, conversely, any indecomposable relative injective object of $\mathcal{C}(s,R)$ is projective. Then, by applying [**3**, Proposition 6.1], we get the following nice properties of $\mathcal{C}(s,R)$.

Proposition 2.2. *Let R be a commutative artinian uniserial ring.*

(a) *The filtered s-chain category $\mathcal{C}(s,R)$ is an additive Krull-Schmidt category with enough relative projective objects and enough relative injective objects.*

(b) *Any relative projective object of $\mathcal{C}(s,R)$ is relative injective, and any relative injective object of $\mathcal{C}(s,R)$ is relative projective.*

(c) *The category $\mathcal{C}(s,R)$ has almost split sequences.* □

Assume now that R is a finite dimensional uniserial K-algebra and K is algebraically closed. We view $\mathcal{C}(s,R)$ as a full exact subcategory of the module category $\operatorname{mod}\mathbf{T}_s(R)$ along the functor (2.0). Given an object C of $\mathcal{C}(s,R)$ (see (1.2)), we call the vector $\mathbf{dim}\, C = (\mathbf{dim}_K C_1, \dots, \mathbf{dim}_K C_s)$ the *dimension vector* of C. Following [**7**], [**17**, p. 368] and [**18**] we introduce tameness and wildness for the category $\mathcal{C}(s,R)$ as follows.

Definition 2.3. Assume that R is a finite dimensional uniserial K-algebra and K is an algebraically closed field.

(a) The category $\mathcal{C}(s,R)$ is of *wild representation type* if there exists a K-linear exact representation embedding $T : \operatorname{mod}\Gamma_3(K) \to \mathcal{C}(s,R)$ (see [**18**]), where

$$\Gamma_3(K) = \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix}$$

If, in addition, the functor T is fully faithful, we call $\mathcal{C}(s,R)$ of *fully wild representation type*, or strictly wild representation type (see [**7**], [**18**]).

(b) The category $\mathcal{C}(s,R)$ is of *tame representation type* if, for every dimension vector $v \in \mathbf{N}^s$, there exist $K[t]-\mathbf{T}_s(R)$-bimodules $L^{(1)}, \dots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules such that all but finitely many indecomposable objects C with $\mathbf{dim}\, C = v$ are of the form $C \cong K^1_\lambda \otimes L^{(j)}$ where $j \le r_v$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$. If there is a common bound for the numbers r_v of such $K[t] - \mathbf{T}_s(R)$-bimodules $L^{(1)}, \dots, L^{(r_v)}$ in each vector v, the tame category $\mathcal{C}(s,R)$ is called *domestic* (see [**24**, (2.1)], [**17**, Section 14.4]).

(c) Assume that the category $\mathcal{C}(s,R)$ is of tame representation type. We define the *growth function* $\boldsymbol{\mu}^1 : \mathbf{N}^3 \to \mathbf{N}$ as follows. Given a vector $v \in \mathbf{Z}^s$ we define $\boldsymbol{\mu}^1(v)$ to be the minimal number r_v of $K[t] - \mathbf{T}_s(R)$-bimodules $L^{(1)}, \dots, L^{(r_v)}$ satisfying the conditions in the definition of tame representation type. A tame category $\mathcal{C}(s,R)$ is said to be of *polynomial growth* if there exists an integer $g \ge 1$ such that $\boldsymbol{\mu}^1(v) \le \|v\|^g$ for all vectors $v \in \mathbf{Z}^s$ with $\|v\| = v_1 + \cdots + v_s \ge 2$.

Now we show how the study of the category $\mathcal{C}(s,R)$ is reduced to the study of the categories of filtered subprojective R-representations of finite posets studied in [**19**, Section 5] and [**20**]. Here we follow our notation introduced in [**19**, Section 5].

Assume that $I \equiv (I, \preceq)$ is a finite partially ordered set (abbr. poset) with a unique maximal element $\star$. Let F be a commutative ring. In [**19**] and [**20**] we have defined a *filtered subprojective F-representation* of I to be the system $X = (X_j)_{j\in I}$ of finitely generated F-modules X_j, $j \in I$, satisfying the following conditions:

(a) $X_\star$ is a projective F-module,

(b) X_j is a submodule of $X_\star$ for every $j \in I$ and $X_i \subseteq X_j$ if $i \preceq j$ in I.

By a morphism $f : X \to X'$ of filtered subprojective F-representations X and X' of the poset I we mean an F-module homomorphism $f : X_\star \to X'_\star$ such that $f(X_j) \subseteq X'_j$ for every $j \in I$.

We denote by $\mathbf{fspr}\,(I,F)$ the category of filtered subprojective F-representations of the poset I. Let FI be the incidence F-algebra of I (see [**17**], [**19**, Section 5]). Following [**19**, Section 5] and [**22**] we denote by $\mathrm{mod}_{\mathrm{pr}}(FI)$ the full subcategory of $\mathrm{mod}(FI)$ consisting of projectively adjusted FI-modules. By [**19**] there is a category equivalence

$$\rho : \mathbf{fspr}\,(I,F) \xrightarrow{\simeq} \mathrm{mod}_{\mathrm{pr}}(FI) \tag{2.4}$$

and therefore $\mathbf{fspr}\,(I,F)$ can be viewed as a full exact subcategory of the module category $\mathrm{mod}(FI)$. Consequently, if F is a finite dimensional K-algebra over an algebraically closed field K, then the tame representation type, the polynomial growth and the wild representation type of $\mathbf{fspr}\,(I,F)$ are well defined as above. By applying [**19**, Lemma 5.8] and [**22**, Theorems 6.5 and 6.10] to the category $\mathbf{fspr}\,(I,F) \cong \mathrm{mod}_{\mathrm{pr}}(FI)$ one gets the following important result.

Proposition 2.5. (a) *Let F be an artinian algebra. Then $\mathbf{fspr}\,(I,F)$ is an additive Krull-Schmidt category with enough relative projective objects and enough relative injective objects. The category $\mathbf{fspr}\,(I,F)$ has almost split sequences and every object of $\mathbf{fspr}\,(I,F)$ has a projective cover.*

(b) *If F is a finite-dimensional K-algebra over an algebraically closed field K, then $\mathbf{fspr}\,(I,F)$ is either of tame representation type or of wild representation type, and the types are mutually exclusive.*

Given an integer $s \geq 0$ we consider the totally ordered poset

$$\mathbf{A}_s^* : 1 \to 2 \to 3 \to \cdots \to s-1 \to s \to *. \tag{2.6}$$

For any commutative artinian uniserial ring R, we define the restriction functor

$$\mathbf{res}\, : \mathbf{fspr}\,(\mathbf{A}_s^*, R) \longrightarrow \mathcal{C}(s,R) \tag{2.7}$$

as follows. If $X = (X_1 \subseteq X_2 \subseteq \cdots \subseteq X_s \subseteq X_*)$ is an object of $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$, we set $\mathbf{res}\,(X) = (X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{s-1} \subseteq X_s)$. This can be viewed as the restriction of X to the subposet $1 \to 2 \to \cdots \to s$ of $\mathbf{A}_s^*$. If $f : X \to X'$ is a morphism in $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ we set $\mathbf{res}\,(f) = f|_{X_s}$, the restriction of f to X_s. The main properties of the functor $\mathbf{res}$ are collected in the following proposition.

Proposition 2.8. *Let R be a commutative artinian uniserial ring.*

(a) *The additive functor $\mathbf{res}\,(2.7)$ is full and dense.*

(b) *If $f : X \to X'$ is a morphism in $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$, then $\mathbf{res}\,(f) = 0$ if and only if f has a factorization through a direct sum of copies of the projective object $P_* : 0 \to 0 \to \cdots \to 0 \to R$. If the objects X and X' have no summands isomorphic with P_*, then f is an isomorphism if and only if $\mathbf{res}\,(f)$ is an isomorphism.*

(c) *The functor* $\mathbf{res}$ *defines a bijection between the isoclasses of the indecomposable objects* X *of* $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ *that are nonisomorphic with* P_* *and the isoclasses of indecomposable objects of* $\mathcal{C}(s, R)$.

(d) *If* R *is a finite dimensional* K*-algebra over an algebraically closed field* K, *then* $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ *is of tame representation type, respectively of polynomial growth or of wild representation type, if and only if the category* $\mathcal{C}(s, R)$ *is of tame representation type, respectively of polynomial growth or of wild representation type.*

Proof. (a) It follows from our assumption that the ring R is a self-injective. Hence it follows that the functor $\mathbf{res}$ is full because, for any X in $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ the R-module X_* is projective, and therefore it is injective. To see that $\mathbf{res}$ is dense, we associate with any object C (1.2) of $\mathcal{C}(s, R)$ the object $\mathcal{I}(C) = (C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{s-1} \subseteq C_s \subseteq C_*)$ of $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ viewed as a representation of $\mathbf{A}_s^\star$ by taking for $C_\star$ an injective envelope of C_s. It is clear that $C \cong \mathbf{res}\,(\mathcal{I}(C))$.

By applying (a) and the definition of $\mathbf{res}$ we easily prove the statement (b) and (c). We leave it to the reader.

(d) Assume that $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is of wild representation type. Then there exists an exact representation embedding K-linear functor $T : \operatorname{fin}(K[t_1, t_2]) \to \mathbf{fspr}\,(\mathbf{A}_s^*, R)$, where $\operatorname{fin}(K[t_1, t_2])$ is the category of finite dimensional modules over $K[t_1, t_2]$ (see [**17**, Chapter 14] and [**18**]). Without loss of generality we can suppose that the objects isomorphic with P_* are not in the image of T because otherwise we can replace the polynomial algebra $K[t_1, t_2]$ by a localization $K[t_1, t_2]_h$ at a polynomial $h \neq 0$ (see [**18**] and [**22**, Section 6]). It follows that the functor $\mathbf{res} \circ T : \operatorname{fin}(K[t_1, t_2]) \to \mathcal{C}(s, R)$ is a representation embedding and therefore $\mathcal{C}(s, R)$ is of wild representation type.

Assume that $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is of tame representation type. We shall show that $\mathcal{C}(s, R)$ is of tame representation type. Fix a vector $v = (v_1, \dots, v_s) \in \mathbf{N}^s$. Note that the number of R-modules U such that $\mathbf{dim}_K(U) = v_s$ is finite, up to isomorphism. Let $U_1, \dots, U_{t_s}$ be a set of representatives of the isoclasses of such R-modules U. Let $v^{(j)} = (v_1, \dots, v_s, v_*^{(j)}) \in \mathbf{N}^{s+1}$ for $j \leq t_s$ where $v_*^{(j)} = \dim_K E(U_j)$ and $E(U_j)$ is the injective envelope of U_j.

By our assumption, there exist $K[t]$-RI-bimodules $L^{(1)}, \dots, L^{(r_v)}$ which are finitely generated free $K[t]$-modules such that all but finitely many indecomposable objects X of $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ with $\mathbf{dim}\,X \in \{v^{(1)}, \dots, v^{(t_s)}\}$ are of the form $X \cong K_\lambda^1 \otimes L^{(j)}$ where $j \leq r_v$, $K_\lambda^1 = K[t]/(t-\lambda)$ and $\lambda \in K$.

Consider the $K[t] - \mathbf{T}_s(R)$-bimodules $\overline{L}^{(1)}, \dots, \overline{L}^{(r_v)}$, where $\overline{L}^{(i)} = \mathbf{res}\,(L^{(i)})$. Let C be an indecomposable object of $\mathcal{C}(s, R)$ such that $\mathbf{dim}\,C = v$. By (a), the object $X = \mathcal{I}(C)$ of $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ defined in the proof of (a) is indecomposable and its $*$-coordinate R-module is isomorphic to any of the modules $E(U_1), \dots, E(U_{t_s})$. It follows that $\mathbf{dim}\,X$ belongs to the set $\{v^{(1)}, \dots, v^{(t_s)}\}$ and therefore all but finitely many such objects X are of the form $X \cong K_\lambda^1 \otimes L^{(j)}$, where $j \leq r_v$ and $\lambda \in K$. Hence we get $C \cong \mathbf{res}\,(\mathcal{I}(C)) \cong \mathbf{res}\,(X) \cong K_\lambda^1 \otimes \overline{L}^{(j)}$. This shows that $\mathcal{C}(s, R)$ is of tame representation type. The polynomial growth implication is proved in a similar way.

Conversely, assume that $\mathcal{C}(s, R)$ is of wild representation type. To prove that $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is of wild representation type, suppose to the contrary that $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is not. By Proposition 2.5 (b), the category $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is of tame representation type and, by the implication proved above, the representation-wild category $\mathcal{C}(s, R)$ is of tame representation type. By applying to $\mathcal{C}(s, R) \subseteq \operatorname{mod} \mathbf{T}_s(R)$ the algebraic geometry arguments used in the proof of [**17**, Theorem 14.34] (with R and $\mathbf{T}_s(R)$ interchanged), we get a contradiction $1 \geq 2$ in counting corresponding algebraic variety dimensions. This proves that $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is of wild representation type.

In a similar way we show that $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is of tame representation type, if $\mathcal{C}(s, R)$ is of tame representation type. This finishes the proof. □

As a consequence of Propositions 2.5 and 2.8, we get the following tame-wild dichotomy result for the categories $\mathcal{C}(s, R)$.

Corollary 2.9. *If R is a commutative uniserial finite dimensional K-algebra over an algebraically closed field K, then $\mathcal{C}(s, R)$ is either of tame representation type or of wild representation type, and the types are mutually exclusive.*

Proof. The corollary reduces to a corresponding tame-wild dichotomy for bocses proved by Drozd in [**7**]. To see that we consider the following diagram

$$\operatorname{rep}(\mathcal{B}_{\mathbf{T}_{s+1}(R)}, K) \xrightarrow{\simeq} \operatorname{prin}(\mathbf{A}_s^*, R) \xrightarrow{\boldsymbol{\Theta}} \mathbf{fspr}\,(\mathbf{A}_s^*, R) \xrightarrow{\mathbf{res}} \mathcal{C}(s, R),$$

where **res** is the restriction functor (2.7) and $\mathcal{B}_{\mathbf{T}_{s+1}(R)}$ is a free triangular bocs (in the sense of Drozd [**7**]), associated to the bipartite K-algebra $\mathbf{T}_{s+1}(R) = \left(\begin{smallmatrix} \mathbf{T}_s(R) & M \\ 0 & R \end{smallmatrix}\right)$ in [**22**, Proposition 4.9], with M a direct sum of s copies of R. Furthermore, $\operatorname{prin}(\mathbf{A}_s^*, R)$ is the category of prinjective $\mathbf{T}_{s+1}(R)$-modules in the sense of [**17**, Chapter 17], which in our case may be identified with the category of the representations $Y = (Y_1 \overset{\varphi_1}{\to} Y_2 \overset{\varphi_2}{\to} \cdots \to Y_s \overset{\varphi_s}{\to} Y_*)$ of the quiver $\mathbf{A}_s^*$ such that $Y_1, \dots, Y_s$ are finitely generated R-modules, Y_* is a finitely generated injective R-module, $f_1, \dots, f_s$ are R-module homomorphisms such that the restriction $(Y_1 \overset{\varphi_1}{\to} Y_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{s-1}}{\to} Y_s)$ of the representation Y to the subposet $\mathbf{A}_s$ of $\mathbf{A}_s^*$ is isomorphic to a direct sum of copies of the projective representations $P_1, \dots, P_s$ (2.1). The functor $\boldsymbol{\Theta}$ associates to Y the object $\boldsymbol{\Theta}(Y) = (Y_1' \hookrightarrow Y_2' \hookrightarrow \cdots \hookrightarrow Y_s' \hookrightarrow Y_*)$, where $Y_j' = \varphi_s \cdots \varphi_j(Y_j)$. It follows from [**22**] that the category $\operatorname{prin}(\mathbf{A}_s^*, R)$ is equivalent with the category $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathbf{T}_{s+1}(R))_R^{\mathbf{T}_s(R)}$ defined in [**22**], the category $\mathbf{fspr}\,(\mathbf{A}_s^*, R)$ is equivalent with the category $\operatorname{mod}_{\mathrm{pr}}(\mathbf{T}_{s+1}(R))_R^{\mathbf{T}_s(R)}$ and the functor $\boldsymbol{\Theta}$ is the adjustment functor $\boldsymbol{\Theta}^{\mathbf{T}_s(R)}$ in [**22**]. Then, by [**22**, Theorem 6.10], the functor $\boldsymbol{\Theta}$ preserves tame representation type and wild representation type. Furthermore, by [**22**, Proposition 4.9], there exists an equivalence $\operatorname{rep}(\mathcal{B}_{\mathbf{T}_{s+1}(R)}, K) \cong \operatorname{prin}(\mathbf{A}_s^*, R)$, preserving the tame and wild representation type. Since, according to Proposition 2.8, the functor **res** preserves the tame representation type and wild representation type, then the corollary is a consequence of the well-known tame-wild dichotomy theorem of Drozd [**7**]. □

3. The representation type of the category $\mathbf{fspr}\,(I, R)$. Let I be a finite poset with a unique maximal element $\star$, $m \geq 1$ be an integer and let $F_m = K[t]/(t^m)$ where K is an algebraically closed field. Our main aim of this section is to present complete lists of pairs (I, m) for which the category $\mathbf{fspr}\,(I, F_m)$ is of tame representation type, of finite representation type, of wild representation type, or $\mathbf{fspr}\,(I, F_m)$ is tame of nonpolynomial growth, respectively.

For this purpose we recall from [**19**] that there is a K-linear functor

$$\tilde{\mathbf{F}} : \mathbf{fspr}^{-}(\widehat{I_m}^{*}, K) \longrightarrow \mathbf{fspr}\,(I, F_m), \tag{3.1}$$

where $\widehat{I_m}$ is the infinite poset with a $\mathbf{Z}$-action associated to (m, I) in [**19**, (5.9)], $\widehat{I_m}^{*} = \widehat{I_m} \cup \{*\}$ is the enlargement of $\widehat{I_m}$ by a unique maximal element $*$ and $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K)$ is the full subcategory of $\mathbf{fspr}\,(\widehat{I_m}^{*}, K)$ consisting of objects $X = (X_\beta; X_*)_{\beta \in \widehat{I_m}}$ such that $X_\beta = X_*$ for β sufficiently large. Note that $\mathbf{fspr}\,(\widehat{I_m}^{*}, K)$ is the category $\widehat{I_m}sp$ of $\widehat{I_m}$-spaces over K and $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K)$ is the full subcategory $\widehat{I_m}$-$\widetilde{sp}$ consisting of the $\widehat{I_m}$-spaces $\mathbf{M} = (M_\beta; M)_{\beta \in \widehat{I_m}}$ such that $M_\beta = M$ for β sufficiently large (see [**17**] and [**19**, Theorem 4.5]).

We recall from [**17**, Proposition 15.100] that the category $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K) = \widehat{I_m}$-$\widetilde{sp}$ is said to be *locally coordinate support finite* if there exists a finite subposet L of $\widehat{I_m}$ such that, for any indecomposable object $\mathbf{M}$ in $\widehat{I_m}$-$\widetilde{sp}$ the finite poset $\mathbf{csupp}\,(\mathbf{M}) = \{j \in \widehat{I_m}^{*};\ (\mathbf{cdn}\,\mathbf{M})_j \neq 0\}$ is contained in a $\mathbf{Z}$-shift of L, where $(\mathbf{cdn}\,\mathbf{M})_j = \dim_K(M_j / \sum_{t \prec j} M_t)$.

The following theorem collects the main properties of the functor (3.1).

Theorem 3.2. *Assume that $F_m = K[t]/(t^m)$, $m \geq 1$, I is a finite poset with a unique maximal element $\star$ and $\widehat{I_m}^{*} = \widehat{I_m} \cup \{*\}$ is the infinite poset associated to (m, I). Then the functor $\tilde{\mathbf{F}}$ (3.1) has the following properties.*

(a) *If X is an indecomposable object in $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K)$, then the object $\tilde{\mathbf{F}}(X)$ is indecomposable and $\tilde{\mathbf{F}}(X) \cong \tilde{\mathbf{F}}(\sigma X)$ where σX is the $\mathbf{Z}$-shift of X.*

(b) *If X and Y are indecomposable objects in $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K)$ and $\tilde{\mathbf{F}}(X) \cong \tilde{F}(Y)$, then $Y \cong \sigma^t Y$ for some $t \in \mathbf{Z}$.*

(c) *If the functor $\tilde{\mathbf{F}}$ is dense, then it induces a bijection between the set of $\mathbf{Z}$-orbits of isomorphism classes of indecomposable objects in $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K)$ and the set of isomorphism classes of indecomposable objects in $\mathbf{fspr}\,(I, F_m)$.*

(d) *The category $\mathbf{fspr}\,(I, F_m)$ is of finite representation type if and only if the poset $\widehat{I_m}$ does not contain the critical posets $\mathcal{K}_1 = (1, 1, 1, 1)$,*

$\mathcal{K}_2 = (2,2,2)$, $\mathcal{K}_3 = (1,3,3)$, $\mathcal{K}_4 = (N,4)$, $\mathcal{K}_5 = (1,2,5)$ of Kleiner [**11**] *listed in* [**17**]. *If* $\mathbf{fspr}\,(I, F_m)$ *is of finite representation type, then the functor* $\tilde{\mathbf{F}}$ *is dense.*

(e) *If* $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K)$ *is of wild representation type, then* $\mathbf{fspr}\,(I, F_m)$ *is of wild representation type.*

(f) *Assume that* $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K)$ *is locally coordinate support finite. Then the functor* $\tilde{\mathbf{F}}$ *is dense and* $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K)$ *is of tame representation type, respectively of polynomial growth, if and only if* $\mathbf{fspr}\,(I, F_m)$ *is of tame representation type, respectively of polynomial growth.*

Proof. Let $D = K[[t]]$ be the power series K-algebra in the indeterminate t. Obviously D is a complete discrete valuation domain with the unique maximal ideal $\mathfrak{p} = (t)$ and there are K-algebra isomorphisms $F_m/J(F_m) \cong D/\mathfrak{p} \cong K$ and $F_m \cong D/\mathfrak{p}^m$. Let

$$\Lambda = \Lambda(I, D/\mathfrak{p}^m) = \begin{pmatrix} D & \mathfrak{p}^{n_{12}} & \cdots & \mathfrak{p}^{n_{1s+1}} \\ \mathfrak{p}^m & D & \cdots & \mathfrak{p}^{n_{2s+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p}^m & \mathfrak{p}^m & \cdots & D \end{pmatrix} \subseteq \mathbf{M}_{s+1}(D)$$

be the classical D-suborder [**19**, (5.10)] of the hereditary D-order $\Gamma = \mathbf{M}_{s+1}(D)$ associated with (m, I) where $s+1 = |I|$, $\mathfrak{p}^{n_{ij}} = D$ for $i \preceq j$ in I and $\mathfrak{p}^{n_{ij}} = \mathfrak{p}^m$ for $i \not\preceq j$ in I. It follows from the definition [**19**, (5.14)] that the functor $\tilde{\mathbf{F}}$ is the composition

$$\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K) = \widehat{I_m} - \widetilde{sp} \xrightarrow{\mathbf{F}} \mathrm{latt}\,(\Lambda) \xrightarrow{\overline{\mathbf{G}_I}} \mathbf{fspr}\,(I, F_m), \tag{3.3}$$

where $\mathbf{F}$ is the Roggenkamp-Wiedemann [**16**] covering-type functor (viewed as the completion functor in [**17**, Chapter 13]) and the functor $\overline{\mathbf{G}_I}$ is constructed in [**19**, Section 5] as the composition of the functor $\mathbf{G}_I : \mathrm{latt}\,(\Lambda) \to \widehat{\mathbf{fspr}}\,(I^*, F_m)$ [**19**, (5.14)] with a category equivalence $\mathbf{res}_I : \widehat{\mathbf{fspr}}\,(I^*, F_m) \xrightarrow{\simeq} \mathbf{fspr}\,(I, F_m)$ (see [**19**, Lemma 5.2]). By [**19**, Lemma 5.15], the functor $\overline{\mathbf{G}_I}$ is a representation equivalence preserving the representation types.

It is easy to see that the poset $\widehat{I_m}$ is just the infinite poset $I(\Lambda)$ associated with Λ in [**25**] (see also [**16**], [**17**, Chapter 13] and [**19**, Section 4]).

It then follows that the statements (a), (b) and (c) are immediate consequences of the properties of the functor $\mathbf{F}$ established in [**16**] (see also [**17**, Chapter 13]).

(d) By the main result in [**25**], the D-order Λ is of finite lattice type if and only if the poset $\widehat{I_m} = I(\Lambda)$ does not contain the critical posets of Kleiner [**11**]. Hence the above observations yield the first statement of (d). To prove the second one, assume that $\mathbf{fspr}\,(I, F_m)$ is of finite representation type. By (c) the category $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K) = \widehat{I_m} - \widetilde{sp}$ has only finitely many isoclasses of indecomposable objects up to a $\mathbf{Z}$-shift. It follows from [**16**] and [**17**, Chapters 11 and 13] that $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K)$ coincides with its unique preprojective component. By [**16**], the functor $\mathbf{F}$ carries irreducible morphisms to irreducible ones. Furthermore, by [**19**, Lemma 5.15], the functor $\overline{\mathbf{G}_I}$ carries irreducible morphisms to irreducible ones. It follows that the composite functor $\tilde{\mathbf{F}} = \overline{\mathbf{G}_I}\mathbf{F}$ carries the preprojective component to a finite connected component $\mathbf{C}$ of the category $\mathbf{fspr}\,(I, F_m)$. By the representation-finite criterion of Auslander [**17**, Theorem 11.44], extended easily to our situation, the finite component $\mathbf{C}$ coincides with $\mathbf{fspr}\,(I, F_m)$ and consequently the functor $\tilde{\mathbf{F}}$ is dense.

(e) By the arguments given in [**17**, pp. 383–384], the functor $\mathbf{F}$ preserves wild representation type. Hence (e) follows because $\tilde{\mathbf{F}} = \overline{\mathbf{G}_I}\mathbf{F}$ and, according to [**19**, Lemma 5.15], the functor $\overline{\mathbf{G}_I}$ is a representation equivalence preserving the representation types.

(f) We only outline the proof. Assume that the category $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*}, K) = \widehat{I_m}\text{-}\widetilde{sp}$ is locally coordinate support finite. Since $\mathbf{F}$ is a covering-type functor with the group $\mathbf{Z}$, then the results of Dowbor and Skowroński [**5**, Theorem] and [**5**, Proposition 2.5] on Galois coverings on locally support finite locally bounded K-categories generalize to our locally coordinate support finite situation. Since according to [**19**, Lemma 5.15] the functor $\overline{\mathbf{G}_I}$ is a representation equivalence and preserves the representation types, then the composite functor $\tilde{\mathbf{F}} = \overline{\mathbf{G}_I}\mathbf{F}$ is dense, preserves and lifts tameness, wildness and the polynomial growth. $\square$

There is an alternative proof of a Theorem 3.2 (f) outlined in [**21**] (see also [**15**]) by viewing the category $\mathbf{fspr}\,(I, F_m)$ as a full subcategory of the category $\mathrm{rep}_K(Q, \Omega)$ of K-linear representations of a bounded quiver (Q, Ω) associated to (I, F_m) and applying the universal Galois covering functor $\mathrm{rep}_K(\tilde{Q}, \tilde{\Omega}) \to \mathrm{rep}_K(Q, \Omega)$ [**6**].

Theorem 3.4. *Assume that I is a finite poset with a unique maximal element $\star$, R is a commutative artinian uniserial ring and $m \geq 1$ an integer such that $J(R)^m = 0$ and $J(R)^{m-1} \neq 0$. The category* $\mathbf{fspr}\,(I, R)$ *is of finite representation type if and only if the pair (m, I) satisfies any of the following conditions*:

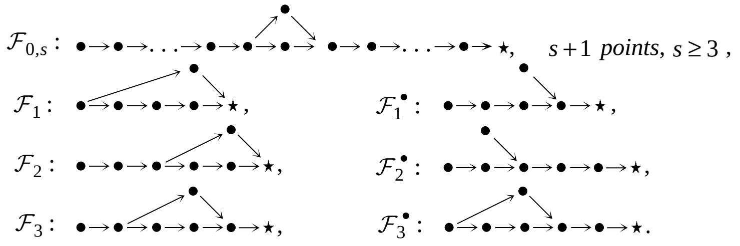
0° *$|I| = 1$ and m is arbitrary or $m = 1$ and I does not contain the critical posets $\mathcal{K}_1, \dots, \mathcal{K}_5$ of Kleiner* [**11**] *listed in* [**17**],

1° *I is linearly ordered, $|I| \geq 2$ and the pair $(m, |I| - 1)$ satisfies any of the conditions* (F1)–(F4) *of Theorem* 1.3, *or*

2° *I is not linearly ordered and the pair (m, I) satisfies any of the following two conditions*:

(F5) *$m = 3$ and I is the poset $\mathcal{F}_0 = (\bullet \to * \leftarrow \bullet)$, or*

(F6) *$m = 2$ and I is a one-peak subposet of any of the posets $\mathcal{F}_{0,s}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet}, \mathcal{F}_3^{\bullet}$ presented below*



Proof. It follows from the well-known Cohen structure theorem that there exists a complete discrete valuation domain D with the unique maximal ideal $\mathfrak{p}$ and ring isomorphisms $R/J(R) \cong D/\mathfrak{p}$ and $R \cong D/\mathfrak{p}^m$. Let $\Lambda = \Lambda(I, D/\mathfrak{p}^m)$ be the classical D-suborder [**19**] in the hereditary order $\Gamma = \mathbf{M}_{s+1}(D)$ associated with (m, I) where $s + 1 = |I|$. It follows from the definition that the poset $\widehat{I_m}$ is just the infinite poset $I(\Lambda)$ associated with Λ in [**25**] (see also [**17**, Chapter 13] and [**19**, Section 4]). According to the main result in [**25**], the order Λ is of finite lattice type if and only if the poset $\widehat{I_m}$ does not contain the critical posets $\mathcal{K}_1, \dots, \mathcal{K}_5$ of Kleiner. On the other hand, there is a representation equivalence functor $\overline{\mathbf{G}}_I : \operatorname{latt}(\Lambda) \to \mathbf{fspr}(I, R)$ constructed in [**19**, Section 5] as the composition of the functor $\overline{\mathbf{G}}_I : \operatorname{latt}(\Lambda) \to \widehat{\mathbf{fspr}}(I^*, R)$ [**19**, (5.14)] with an equivalence $\mathbf{res}_I : \widehat{\mathbf{fspr}}(I^*, R) \overset{\simeq}{\to} \mathbf{fspr}(I, R)$ (see [**19**, Lemma 5.2]). It then follows that $\mathbf{fspr}(I, R)$ is of finite representation type if and only if the poset $\widehat{I_m}$ does not contain the critical posets $\mathcal{K}_1, \dots, \mathcal{K}_5$ of Kleiner.

To prove the "only if" part we check by a case by case inspection that, if (I, m) is any of the pairs satisfying the conditions in the theorem, then the infinite poset $\widehat{I_m}$ does not contain the critical posets $\mathcal{K}_1, \dots, \mathcal{K}_5$ of Kleiner. It then follows that $\mathbf{fspr}(I, R)$ is of finite representation type. The reader is referred to the proof of Corollary 5.19 in [**19**] for an illustration of this technique in case the poset I is linearly ordered.

To prove the converse we show first that the category $\mathbf{fspr}(I, R)$ is of infinite representation type if I contains a critical poset of Kleiner or the pair (I, m) is of one of the following types:

(A) The poset I is linearly ordered and any of the following conditions is satisfied:

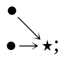
(A1) $m \geq 6$ and $|I| \geq 3$,

(A2) $m \geq 4$ and $|I| \geq 4$,

(A3) $m \geq 3$ and $|I| \geq 6$;

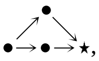
(B) The poset I is not linearly ordered and the pair (I, m) is of one of the following types:

(B1) $m \geq 4$ and I is the poset $\mathcal{I}_0$:

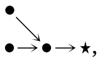


(B2) $m \geq 3$ and I is any of the following posets:

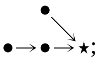
$\mathcal{I}_1$:



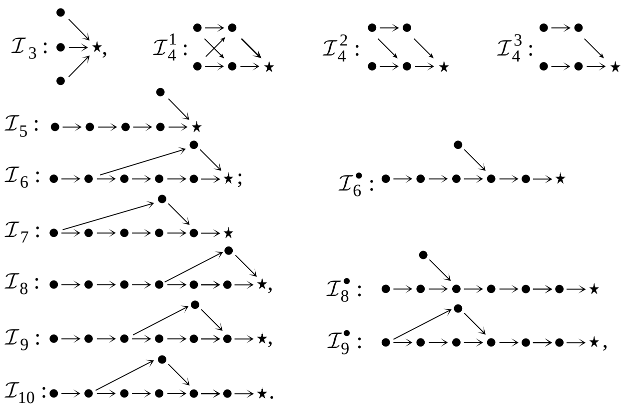
$\mathcal{I}_1^\bullet$:



$\mathcal{I}_2$:

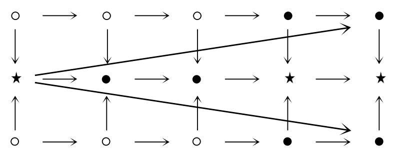


(B3) $m \geq 2$ and I is any of the following posets:



For this purpose we easily check that if (T, n) is of any of the types above, then the infinite poset $\widehat{T_n}$ contains one of the critical posets $\mathcal{K}_1, \ldots, \mathcal{K}_5$ of Kleiner. It then follows that the category $\mathbf{fspr}\,(T, R)$ is of infinite representation type, as required.

Let us illustrate it by an example. Take $I = \mathcal{I}_0$ and $m = 4$. Then the pair $(I, 4)$ is of type (B1) and the infinite poset $\hat{I}_4$ contains a finite subposet ${}_0\widehat{I_5}$ of the form



It contains a subposet isomorphic to the poset $\mathcal{K}_2 = (2, 2, 2)$ marked by the solid points.

Now assume that the category $\mathbf{fspr}\,(I, R)$ is of finite representation type. Then (I, m) does not contain any pair (T, n) listed above and simple combinatorial arguments show that the pair (I, m) satisfies any of the conditions listed in the theorem. This finishes the proof. □

Note that Theorem 3.4 can be also deduced from the main result of

Plahotnik [**13**] by passing from $\mathbf{fspr}\,(I,R)$ to matrix R-representations of I.

Remark 3.5. In case R is the K-algebra $F_m = K[t]/(t^m)$ and $\mathbf{fspr}\,(I,F_m)$ is representation-finite, there is a simple algorithm for determining the Auslander-Reiten quiver of the category $\mathbf{fspr}\,(I,F_m)$ by applying the functor (3.1), Theorem 3.2 and [**19**, Theorem 4.5]. For this, one determines the projective component of the category $\mathbf{fspr}^{\,-}(\widehat{I_m}^{\,*},K) = \widehat{I_m} - \widetilde{sp}$ as in Examples 13.18, 13.28 and 13.29 of [**17**], and then one glues it properly along $\mathbf{F}$ according to the $\mathbf{Z}$-action and then along the functor $\overline{\mathbf{G}_I}$.

In particular, one shows in this way that, for $m \le 5$, the number of isoclasses of indecomposable objects in $\mathbf{fspr}\,(\mathbf{A}_2^*,F_m)$ equals 3, 6, 11, 21 and 51 in case $m=1$, $m=2$, $m=3$, $m=4$ and $m=5$, respectively. Hence we conclude the fact proved in [**14**] that the number of the isoclasses of indecomposable objects in the chain category $\mathcal{C}(2,F_m)$ equals 2, 5, 10, 20 and 50, respectively, because of Proposition 2.8(c). By Theorem 3.6 below, the categories $\mathcal{C}(2,F_7)$ and $\mathbf{fspr}\,(\mathbf{A}_2^*,F_7)$ are of wild representation type, whereas $\mathcal{C}(2,F_6)$ and $\mathbf{fspr}\,(\mathbf{A}_2^*,F_6)$ are tame, representation-infinite of polynomial growth. The structure of $\mathbf{fspr}\,(\mathbf{A}_2^*,F_6)$ is described in [**15**]. It is proved here that $\mathbf{fspr}\,(\mathbf{A}_2^*,F_6)$ is tame of tubular type.

Theorem 3.6. *Assume that* $I = \mathbf{A}_s^*$, $m \ge 1$, $F_m = K[t]/(t^m)$ *and* K *is an algebraically closed field.*

(a) *The following three conditions are equivalent.*

(i) *The category* $\mathbf{fspr}\,(I,F_m)$ *is of tame representation type.*

(ii) *The category* $\mathbf{fspr}\,(I,F_m)$ *is tame of polynomial growth.*

(iii) *The pair* (m,I) *is of any of the types presented in Theorem* 3.4 *or* $(m,|I|)$ *is any of the following four pairs* $(6,3)$, $(4,4)$, $(4,5)$, (3.6).

(b) *The category* $\mathbf{fspr}\,(I,F_m)$ *is of wild representation type if and only if the pair* $(m,|I|)$ *of integers satisfies any of the following four conditions*:

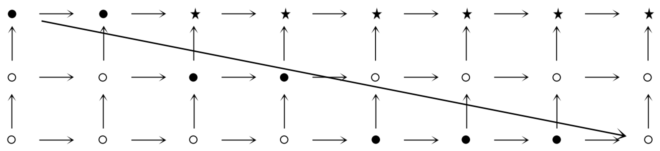
(W1$^+$) $m \ge 7$ *and* $|I| \ge 3$, (W3$^+$) $m \ge 4$ *and* $|I| \ge 6$,
(W2$^+$) $m \ge 5$ *and* $|I| \ge 4$, (W4$^+$) $m \ge 3$ *and* $|I| \ge 7$.

Proof. Consider the functor $\tilde{\mathbf{F}} : \mathbf{fspr}^-(\widehat{I_m}^*, K) \to \mathbf{fspr}\,(I, F_m)$ (3.1). It follows from [**17**, Theorem 15.99] that the category $\mathbf{fspr}^-(\widehat{I_m}^*, K) = \widehat{I_m}\text{-}\widetilde{sp}$ is of wild representation type if and only if the infinite poset $\widehat{I_m}$ contains any of the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$ of Nazarova [**12**] presented in [**17**, p. 309]. On the other hand, by a simple combinatorial checking we get the following two statements:

(A) The poset $\widehat{I_m}$ contains any of the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$ if and only if the pair $(m, |I|)$ satisfies any of the conditions (W1$^+$)–(W4$^+$).

(B) The poset $\widehat{I_m}$ does not contain the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$ if and only if the pair (m, I) satisfies any of the conditions stated in the statement (iii).

For example, if I is the poset $\mathbf{A}_2^* : \cdot \to \cdot \to *$ and $m = 7$, then the pair $(I, 7)$ is of type (W1$^+$) and the infinite poset $\hat{I}_7$ contains a finite subposet ${}_0\widehat{I_7}$ of the form



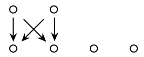
It contains a subposet isomorphic to the poset $\mathcal{N}_3 = (2, 2, 3)$ marked by the solid points.

Next we prove the following statement:

(C) If the pair (m, I) satisfies any of the conditions stated in (iii), then the category $\mathbf{fspr}\,(I, F_m)$ is tame of polynomial growth.

To prove (C) we consider two cases. First suppose that (m, I) is of any of the types presented in Theorem 3.4. It follows from Theorem 3.4 that $\mathbf{fspr}\,(I, F_m)$ is representation-finite and consequently it is tame of polynomial growth. Next suppose that $\mathbf{fspr}\,(I, F_m)$ is representation-infinite. Then $(m, |I|)$ is any of the following four pairs (6,3), (4,4), (4,5), (3,6). In each of the four cases (6,3), (4,4), (4,5), (3,6), a simple combinatorial analysis of the infinite poset $\widehat{I_m}$ shows that $\widehat{I_m}$ does not contain the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$ of Nazarova [**12**] and does not contain the poset

$\mathcal{NZ}$:



of Nazarova and Zavadskij. By [**17**, Theorems 15.89 and 15.99], the category $\widehat{I_m}$-$\widetilde{sp}$ is tame of polynomial growth. Moreover, it follows from [**17**, Theorem 15.100] that $\mathbf{fspr}^{-}(\widehat{I_m}^{*}, K) = \widehat{I_m}$-$\widetilde{sp}$ is locally coordinate support finite. Hence by applying Theorem 3.2 (f), we conclude that $\mathbf{fspr}\,(I, F_m)$ is tame of polynomial growth and our claim (C) follows.

(b) Assume that the pair $(m, |I|)$ satisfies any of the conditions (W1$^+$)–(W4$^+$). Since $\widehat{I_m} = I(\Lambda)$, it follows from (A) and [**17**, Theorems 15.3 and 15.99] that the category $\widehat{I_m}$-$\widehat{sp} = \mathbf{fspr}^{-}(\widehat{I_m}^{*}, K)$ is of wild representation type. Hence we conclude that $\mathbf{fspr}\,(I, F_m)$ is of wild representation type, because we know from Theorem 3.2 that the functor $\widetilde{\mathbf{F}}$ preserves the wild representation type.

Conversely, suppose that the category $\mathbf{fspr}\,(I, F_m)$ is of wild representation type. By (A) it is sufficient to prove that the poset $\widehat{I_m}$ contains any of the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$. Assume, to the contrary, that $\widehat{I_m}$ does not contain hypercritical posets. By (B) and (C), the category $\mathbf{fspr}\,(I, F_m)$ is tame of polynomial growth, and we get a contradiction with the tame-wild dichotomy of Proposition 2.5 (b).

(a) The implication (ii) $\Rightarrow$ (i) is trivial. The statement (C) yields the implication (iii) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (iii). Assume that $\mathbf{fspr}\,(I, F_m)$ is of tame representation type. By the tame-wild dichotomy of Proposition 2.5, $\mathbf{fspr}\,(I, F_m)$ is not of wild representation type. It follows from (b) and (A) that $\widehat{I_m}$ does not contain the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$. Then the statement (iii) is a consequence of (B). This finishes the proof. $\square$

Remark 3.7. For $m \leq 6$ and $p \geq 2$ a prime, a description of the Auslander-Reiten quiver of the category $\mathcal{C}(2, \mathbf{Z}/p^m\mathbf{Z})$ was presented by C.M. Ringel and M. Schmidmeier during the Fifth Budapest-Chemnitz-Praha-Toruń Conference in Algebra held in Budapest from 12–15

June 2001 (see [**15**]). In particular, a complete classification of the indecomposable objects of $\mathcal{C}(2, \mathbf{Z}/p^6\mathbf{Z})$ was given. This shows that the 2-chain category $\mathcal{C}(2, \mathbf{Z}/p^6\mathbf{Z})$ is tame of "tubular type."

Remark 3.8. In the present paper we prove Theorem 3.6 only in case the poset I is of the form $\mathbf{A}_s^*$. In [**20**] Theorem 3.6 is extended to the case I is an arbitrary poset with a unique maximal element.

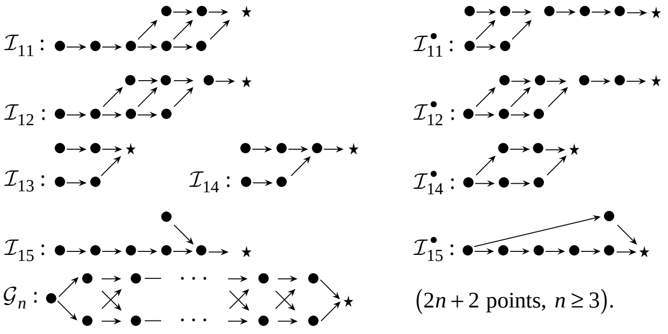
We show in [**20**] that if I is not a chain and $m \geq 2$, then:

(T1) The category $\mathbf{fspr}\,(I, F_m)$ is tame and representation infinite if and only if

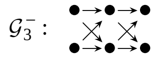
(1) $m = 4$ and $I = \mathcal{I}_0$, (see (B1)), or

(2) $m = 3$ and $I = \mathcal{I}_1$ or $I = \mathcal{I}_1^\bullet$, (see (B2)); or else

(3) $m = 2$, I is not a one-peak subposet or any of the posets presented in (F6), and I is a one-peak subposet of any of the posets $\mathcal{I}_3$, $\mathcal{I}_4^1$, $\mathcal{I}_4^2$, $\mathcal{I}_4^3$, $\mathcal{I}_6 \ldots, \mathcal{I}_{10}$, $\mathcal{I}_6^\bullet$, $\mathcal{I}_8^\bullet$, $\mathcal{I}_9^\bullet$ presented in (B3) or of any of the following nine posets



(T2) The category $\mathbf{fspr}\,(I, F_m)$ is tame of nonpolynomial growth if and only if $m = 2$, I is a one-peak subposet of the garland $\mathcal{G}_n$ with $n \geq 3$ and I contains the garland $\mathcal{G}_3^-$:



The characterizations (T1) and (T2) were presented in the International Conference on Representations of Algebras VIII in Geirenger, 4–10 August 1996 (see the abstract [**21**]).

(3.9) *Proof of Theorems* 1.3–1.5. Let **res** : $\mathbf{fspr}\,(\mathbf{A}_s^*, R) \to \mathcal{C}(s, R)$ be the functor (2.7). By Proposition 2.5 (c) and (d), the functor **res** preserves and respects the finite representation type, tame representation type, wild representation type and the polynomial growth. It follows that Theorem 1.3 is an immediate consequence of Theorem 3.4, whereas Theorems 1.4 and 1.5 follow easily from Theorem 3.6. □

4. Concluding remarks. Fix a prime integer $p \geq 2$ and $m \geq 1$. Let $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ be the finite field with p elements and denote by $K = \overline{\mathbf{Z}_p}$ the algebraic closure of $\mathbf{Z}_p$. The study of the category $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ of subprojective representations of a finite poset I has an important application to the study of the category $\mathrm{rep}\,(I, \hat{\mathbf{Z}}_{(p)}, m)$ defined in [**2**, Section 4.1] and related functorially with the category $B(T, m)_p$ of isomorphism at p of finite rank Butler groups (see [**2**, Section 4.3]), where $\hat{\mathbf{Z}}_{(p)}$ is the ring of p-adic integers. In particular, the results of this paper are strongly related with the open questions stated in [**2**, pp. 142, 164, 168] and related problems discussed by Dugas and Rangaswamy [**8**].

In the present paper we are interested in determining the representation type of the following three categories

$$(4.1) \quad \mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z}) \qquad \mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m)) \qquad \mathbf{fspr}\,(I, K[t]/(t^m))$$

and in a complete classification of their indecomposable objects. Unfortunately we have defined tame representation type and wild representation type only for the category $\mathbf{fspr}\,(I, K[t]/(t^m))$, because the field $K = \overline{\mathbf{Z}_p}$ is algebraically closed and Proposition 2.5 applies.

However, our results of this paper might help to define and determine a tame representation type and a wild representation type for the categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$. Without loss of generality we may suppose that $m \geq 2$, because in the case $m = 1$ the rings $\mathbf{Z}/p^m\mathbf{Z}$, $\mathbf{Z}_p[t]/(t^m)$ and $K[t]/(t^m)$ are fields and the results of Nazarova [**12**] presented in [**17**, Chapter 15] apply.

Assume that $m \geq 2$ and note that, according to Theorem 3.4, each of the categories in (4.1) is of infinite representation type if so is $\mathbf{fspr}\,(I, K[t]/(t^m))$. Moreover, the pairs (I, m) for which $\mathbf{fspr}\,(I, K[t]/$

$(t^m))$ is wild are determined by Theorem 3.6 and the statement (T1) of Remark 3.8. It should not be difficult to show that, for any such a pair (I, m) the categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$ are also "wild" in a reasonable sense, or at least are endo-wild in the sense of [**23**, Definition 5.1].

It then remains to show that the categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$ are of "tame representation type" in a reasonable sense, if (I, m) is any of the pairs (6,3), (4,4), (4,5), (3,6) in Theorem 3.6 or (I, m) is any of the pairs described by the conditions (1)–(3) in (T1) of Remark 3.8, by carrying out a classification of indecomposables from $\mathbf{fspr}\,(I, K[t]/(t^m))$ to the categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$. This idea was presented by Ringel in case $s = 2$ for the category $\mathcal{C}(2, \mathbf{Z}/p^6\mathbf{Z})$ in the Budapest Conference in Algebra in June 2001 (see [**15**]). In view of Proposition 2.8, this applies to the category $\mathbf{fspr}\,(I, \mathbf{Z}/p^6\mathbf{Z})$ with $I = (1 \to 2 \to *)$.

We hope that an analogous procedure and a classification of indecomposables in categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ might help us to find a proper definition of tame representation type in this case.

It seems to us that field extension arguments and a geometrical and a model theory technique developed for tame algebras by Kasjan in [**9**] and [**10**] might help to build up a "bridge" between $\mathbf{fspr}\,(I, K[t]/(t^m))$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$. To get a connection between the categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$ we note that the $\mathbf{Z}_p$-algebra $\mathbf{Z}_p[t]/(t^m)$ is the associated graded ring $\mathrm{gr}\,(\mathbf{Z}/p^m\mathbf{Z})$ of $\mathbf{Z}/p^m\mathbf{Z}$. This obvious observation might also help to show that the Auslander-Reiten quivers of the categories $\mathbf{fspr}\,(I, \mathbf{Z}/p^m\mathbf{Z})$ and $\mathbf{fspr}\,(I, \mathbf{Z}_p[t]/(t^m))$ are isomorphic (see [**19**, Problem 5.21(c)]).

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REFERENCES

1. D.M. Arnold, *Representations of partially ordered sets and abelian groups*, Contemp. Math. **87** (1989), 91–109.

2. ———, *Abelian groups and representations of finite partially ordered sets*, Canad. Math. Soc. Books in Math., Springer-Verlag, New York, 2000.

3. M. Auslander and S.O. Smalø, *Almost split sequences in subcategories*, J. Algebra **69** (1981), 426–454.

4. T. Brüstle, L. Hille, C.M. Ringel and G. Röhle, *Modules without self-extensions over the Auslander algebra of $K[T]/(T^n)$*, Algebras Represent. Theory **2** (1999), 295–312.

5. P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras*, Arch. Math. (Brno) **44** (1985), 522–529.

6. ———, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. **62** (1987), 311–337.

7. Ju.A. Drozd, *Tame and wild matrix problems*, in *Representations and quadratic forms*, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979, 39–74 (in Russian).

8. M. Dugas and K.M. Rangaswamy, *Completely decomposable abelian groups with a distinguished CD subgroup*, Rocky Mountain J. Math. **32** (2002), 1383–1395.

9. S. Kasjan, *Base field extensions and generic modules over finite dimensional algebras*, Arch. Math. (Basel) **77** (2001), 155–162.

10. ———, *On the problem of axiomatization of tame representation type*, Fund. Math. **171** (2002), 53–67.

11. M.M. Kleiner, *Partially ordered sets of finite type*, in Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28** (1972), 32–41 (in Russian).

12. L.A. Nazarova, *Partially ordered sets of infinite type*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 963–991 (in Russian).

13. V.V. Plahotnik, *Representations of partially ordered sets over commutative rings*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 527–543 (in Russian).

14. F. Richmann and E. Walker, *Subgroups of p^5 bounded groups*, in *Abelian groups and modules*, Birkhäuser, Boston, 1999, pp. 55–74.

15. C.M. Ringel and M. Schmidmeier, *Submodules of modules over uniserial algebras*, preprint, 2001.

16. K.W. Roggenkamp and A. Wiedemann, *Auslander-Reiten quivers of Schurian orders*, Comm. Algebra, **12** (1984), 2525–2578.

17. D. Simson, *Linear representations of partially ordered sets and vector space categories*, Algebra Logic Appl., Vol. 4, Gordon & Breach Sci. Publ., New York, 1992.

18. ———, *On representation types of module subcategories and orders*, Bull. Polish Acad. Sci. Math. **41** (1993), 77–93.

19. ———, *Socle projective representations of partially ordered sets and Tits quadratic forms with application to lattices over orders*, in *Proc. of Conf. on Abelian Groups and Modules* (Colorado Springs, 1995), Springer-Verlag, New York, 1996, pp. 73–111.

20. ———, *A covering functor and tame representation type for poset representations over artinian principal ideal algebras*, Faculty of Math. and Informatics, Nicholas Copernicus Univ., Toruń, 1996, preprint.

21. ———, *Tameness and a covering functor for poset representations over uniserial algebras*, Abstracts of Internat. Conf. on Representations of Algebras VIII, (Geirenger, Norway, 4–10 August), 1996.

22. ———, *Prinjective modules, propartite modules, representations of bocses and lattices over orders*, J. Math. Soc. Japan **49** (1997), 31–68.

23. ———, *An endomorphism algebra realisation problem and Kronecker embeddings for algebras of infinite representation type*, J. Pure Appl. Algebra, **172**, (2002), 293–303.

24. A. Skowroński, *Module categories over tame algebras*, in Workshop on Representations of Algebras (Mexico, 1994), Canad. Math. Soc. Conf. Proc., Vol. 19, Amer. Math. Soc., 1996, pp. 281–313.

25. A.G. Zavadskij and V.V. Kirichenko, *Semimaximal rings of finite type*, Mat. Sb. **103** (1977), 323–345 (in Russian).

Faculty of Mathematics and Informatics, Nicholas Copernicus University, 87-100 Toruń, ul. Chopina 12/18, Poland
Email address: simson@mat.uni.torun.pl