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# COMMUTATIVE ENDOMORPHISM ALGEBRAS OF TORSION-FREE, RANK-TWO KRONECKER MODULES WITH SINGULAR HEIGHT FUNCTIONS

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Dedicated to Professor J.D. Reid on the occasion of his retirement

ABSTRACT. As is the case with Abelian groups, rank-1, torsion-free Kronecker modules are characterized by height functions. A height function is called singular provided it never assumes the value infinity. The endomorphism algebras of singular, rank-1, torsion-free Kronecker modules are trivial. Here we consider the endomorphism algebras of torsion-free, rank-2 modules that are extensions of finite-dimensional modules by modules of rank-1. If $K$ is the ground field and $K(X)$ the field of rational functions, the endomorphism algebras of the rank-2, indecomposable modules are known to be commutative $K$-subalgebras of the matrix ring $M_2(K(X))$. When the height function is nonsingular the resulting endomorphism algebras can be varied, including, for example, coordinate rings of elliptic curves. This paper examines the possibilities for the singular case, which we show are more limited. Yet their endomorphism algebras offer examples from an important class of commutative rings, namely, zero-dimension, local rings.

**1. Introduction.** For an algebraically closed field $K$, a function $h : K \cup \{\infty\} \to \{\infty, 0, 1, \dots\}$ is called a *height function*. Height functions are just as pervasive in the theory of Kronecker modules as they are in the theory of abelian groups [**4, 7**]. Every $K[X]$-module may be considered a Kronecker module, see [**5**]. The concepts used in Abelian groups have seen fruitful extension to Kronecker modules. In turn, Kronecker modules have helped guide the development of the general representation theory of finite-dimensional algebras, posets and indirectly Abelian groups, see, e.g., [**1, 2, 8, 9, 14**].

The Kronecker modules that are torsion-free extensions of finite-dimensional, rank-1 modules by infinite-dimensional, rank-1 modules have no analogue in $K[X]$-module theory. These modules, which are

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the objects studied in this paper, can be explicitly described using three parameters: an integer $m > 0$, a height function $h$ and a functional $\alpha : K(X) \to K$. They will be denoted by $\mathbf{V}(m, h, \alpha)$. The scope for the endomorphism algebras of these $\mathbf{V}$s, unlike the rank-1 modules, is quite broad. The coordinate rings of elliptic curves all come up as End $\mathbf{V}$s, see [**11**]. These occur using a height function $h$ that takes the value $\infty$. A height function such as this, that assumes the value $\infty$ is called nonsingular. This nonsingular case seems to impinge on some problems about affine curves, for example, the detection of planarity of a given curve.

When $h$ is singular we show here that the possibilities for End $\mathbf{V}(m, h, \alpha)$ are limited to the following:

$K$, $K \oplus K$, the trivial extension $K \propto S$ of $K$ by a vector space $S$.

The third possibility, $K \propto S$, is a zero-dimensional, local ring. Such rings are also significant from a geometric point of view.

**2. Pole spaces, modules, endomorphisms.** In this section we lay down the notations for our Kronecker modules in a way that linear algebra can facilitate the study of their endomorphisms.

Throughout, $K$ will be an algebraically closed field, $X$ an indeterminate, and $K(X)$ the field of rational functions with coefficients in $K$. For each $\theta$ in $K$, the rational function $(X - \theta)^{-1}$ is ubiquitous in this paper. For brevity of notation we adopt the following convention

$$X_\theta = (X - \theta)^{-1}. \tag{1}$$

**Height functions and pole spaces.** Every nonzero rational function $t$ has a unique factorization

$$t = \lambda \prod_{\theta \in K} X_\theta^{j\theta},$$

where $\lambda \in K$, $j_\theta \in \mathbf{Z}$, and all but finitely many $j_\theta$ are 0. For each $\theta$ in $K$ the integer $j_\theta$ is denoted by $\mathrm{ord}_\theta(t)$ and the integer $-\sum_{\theta \in K} j\theta$ is denoted by $\mathrm{ord}_\infty(t)$.

Let us agree that $\mathrm{ord}_\theta(0) = -\infty$ for all $\theta$ in $K \cup \{\infty\}$. Subject to the usual arithmetic conventions with the symbol $\infty$, the $\mathrm{ord}_\theta$ functions come with the familiar properties:

$$\mathrm{ord}_\theta(st) = \mathrm{ord}_\theta(s) + \mathrm{ord}_\theta(t), \tag{2}$$

$$\mathrm{ord}_\theta(s+t) \leq \max\{\mathrm{ord}_\theta(s), \mathrm{ord}_\theta(t)\} \tag{3}$$

for all $\theta$ in $K \cup \{\infty\}$ and all $s, t$ in $K(X)$. Furthermore, if $\mathrm{ord}_\theta(s) < \mathrm{ord}_\theta(t)$, then

$$\mathrm{ord}_\theta(s+t) = \mathrm{ord}_\theta(t). \tag{4}$$

If $\theta \in K \cup \{\infty\}$ and $\mathrm{ord}_\theta(t) > 0$, the rational function $t$ is said to have a *pole* at $\theta$. In that case the positive integer $\mathrm{ord}_\theta(t)$ is called the *order of the pole* that $t$ has at $\theta$. For $\theta$ in $K$, a positive power of $X_\theta$ appears in the partial fraction expansion of $t$ if and only if $t$ has a pole at $\theta$. In that case the highest power of $X_\theta$ appearing is $X_\theta^{\mathrm{ord}_\theta(t)}$. Furthermore, a positive power of $X$ appears in the partial fraction expansion of $t$ if and only if $t$ has a pole at infinity. In that case the highest power of $X$ that appears is $X^{\mathrm{ord}_\infty(t)}$.

We say that a rational function $t$ *dominates* a rational function $s$ provided

$$\mathrm{ord}_\theta(s) \leq \mathrm{ord}_\theta(t) \quad \text{for all poles} \quad \theta \text{ in } K \cup \{\infty\} \text{ of } s. \tag{5}$$

In other words, $t$ dominates $s$ if and only if no higher power of $X_\theta$ or of $X$ appears in the partial fraction expansion of $s$ than the highest power which appears in the expansion of $t$.

**Definition.** A nonzero $K$-linear subspace $R$ of $K(X)$ will be called a *pole space* provided that $R$ contains $K$, and whenever $R$ contains a function $t$, then $R$ also contains all of the functions dominated by $t$.

There is a straightforward bijection between the family of all pole spaces and the family of all height functions as follows. For a height function $h$, the space

$$R_h = \{s \in K(X) : \mathrm{ord}_\theta(s) \leq h(\theta) \text{ for all } \theta \text{ in } K \cup \{\infty\}\}$$

is a pole space. Conversely, any pole space $R$ determines the height function $h_R : K \cup \{\infty\} \to \{\infty, 0, 1, 2, \dots\}$, given for each $\theta$ in $K \cup \{\infty\}$ by

$$h_R(\theta) = \sup\{\mathrm{ord}_\theta(s) : s \in R\}.$$

The correspondences $h \mapsto R_h$ and $R \mapsto h_R$ constitute our bijection.

For a given pole space $R_h$, the set of functions

$$\{X_\theta^n : 1 \le n \le h(\theta)\} \cup \{X^n : 0 \le n \le h(\infty)\} \tag{6}$$

is a basis of $R_h$ as vector space over $K$, which we call the *standard basis* of $R_h$. We shall move freely between the twin notions of pole space and its corresponding height function. Consequently, the family of all pole spaces forms a complete lattice of subspaces of $K(X)$.

**Rank-1 modules.** In [**4**, Section 3] and (subject to the Abelian group/$K[X]$-module dictionary) in [**7**, Section 85], height functions and their pole spaces are used to construct and classify torsion-free, rank-1 modules as well as their endomorphism algebras. In order to set the stage for our rank-2 Kronecker modules, we review the definition of a Kronecker module and describe those which are rank-1 and torsion-free.

A *Kronecker module* is a representation of the quiver $\bullet \;\substack{\longrightarrow \\ \longrightarrow}\; \bullet$.

Thus, it is an ordered pair $(a, b)$ of linear transformations between a pair of $K$-linear spaces $U, V$:

$$U \;\substack{\xrightarrow{a} \\ \xrightarrow{b}}\; V.$$

The pair $(a, b)$ can be used as the generic name for all such pairs of linear transformations and, when the actions of $a$ and $b$ are understood, we simply denote the module by the symbol $U \;\substack{\longrightarrow \\ \longrightarrow}\; V$.

Given two modules

$U \;\substack{\longrightarrow \\ \longrightarrow}\; V$ and $W \;\substack{\longrightarrow \\ \longrightarrow}\; Z$, a *homomorphism* between them is a pair of linear maps $U \xrightarrow{\psi} W$, $V \xrightarrow{\varphi} Z$ such that the following diagrams commute:

$$
\begin{array}{ccc} U & \xrightarrow{a} & V \\ \Big\downarrow \psi & & \Big\downarrow \varphi \\ W & \xrightarrow{a} & Z \end{array}
\qquad\qquad
\begin{array}{ccc} U & \xrightarrow{b} & V \\ \Big\downarrow \psi & & \Big\downarrow \varphi \\ W & \xrightarrow{b} & Z \end{array} \tag{7}
$$

For example, take any pole space $R_h$ and define the subspace

$$R_h^- = \{r \in R_h : \mathrm{ord}_\infty(r) < h(\infty)\}.$$

The space $R_h^-$ is the maximal subspace of $R_h$ such that $XR_h^- \subseteq R_h$. As we go over all height functions $h$, the modules $R_h^- \overset{\longrightarrow}{\longrightarrow} R_h$, given explicitly as

$$R_h^- \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} R_h \quad \text{where } a : r \mapsto r \text{ and } b : r \mapsto Xr, \tag{8}$$

represent all torsion-free modules of rank-1, see [**4**, Theorem 3.4].

**Functionals and derivers.** In addition to height functions $h : K \cup \{\infty\} \to \{\infty, 0, 1, 2, \dots\}$, we need to work with $K$-linear functionals $\beta : K(X) \to K$. If $\beta$ is such a functional and $r \in K(X)$, we denote by $\langle \beta, r\rangle$ the value in $K$ that $\beta$ takes at a rational function $r$. For a rational function $s$, let $\beta * s$ denote the functional given by

$$\langle \beta * s, r\rangle = \langle \beta, sr\rangle \quad \text{for each } r \in K(X). \tag{9}$$

For each functional $\beta$, we have the $K$-linear operator $\partial_\beta : K(X) \to K(X)$ as described in [**10**]. If $K(X)^\star$ is the space of functionals on $K(X)$, such $\partial_\beta$ arise from a $K$-bilinear map

$$\partial : K(X^\star) \times K(X) \longrightarrow K(X) \quad \text{where } (\beta, r) \longmapsto \partial_\beta(r),$$

that is uniquely determined by the requirements

$$\partial_\beta(rs) = r\partial_\beta(s) + \partial_{\beta * s}(r) \quad \text{and} \quad \partial_\beta(X) = \langle \beta, 1\rangle \tag{10}$$

for every $\beta$ in $K(X)^\star$ and every $r, s$ in $K(X)$.

Recalling the notation (1) for $X_\theta$ we have, more explicitly, that each $K$-linear map $\partial_\beta : K(X) \to K(X)$ is defined on the standard basis

$$\{X_\theta^n : \theta \in K \text{ and } n = 1, 2, 3, \dots\} \cup \{X^n : n = 0, 1, 2, \dots\}$$

of $K(X)$ by

$$\partial_\alpha(X_\theta^n) = -\langle \alpha, X_\theta^n\rangle X_\theta - \langle \alpha, X_\theta^{n-1}\rangle X_\theta^2 - \cdots - \langle \alpha, X_\theta\rangle X_\theta^n, \tag{11}$$

for $\theta$ in $K$ and $n \geq 1$, by

$$\partial_\beta(1) = 0, \tag{12}$$

and by

(13)

$$\partial_\alpha(X^n) = \langle \alpha, X^{n-1}\rangle + \langle \alpha, X^{n-1}\rangle X + \cdots + \langle \alpha, X\rangle X^{n-2} + \langle \alpha, 1\rangle X^{n-1},$$

for $n \geq 1$.

Each $\partial_\beta$ will be called a *deriver*. From (11), (12) and (13) and using partial fraction expansions, we can see some important properties of derivers.

**Proposition 2.1.** *If $\partial_\beta$ is a deriver and $s \in K(X)$, then*

$$\mathrm{ord}_\theta(s) \geq 0 \Longrightarrow \mathrm{ord}_\theta(\partial_\beta(s)) \leq \mathrm{ord}_\theta(s).$$

*The poles of $\partial_\beta(s)$ lie among the poles of $s$. Derivers leave pole spaces invariant. In fact, $\partial_\beta(R_h) \subseteq R_h^-$ for every pole space $R_h$.*

**Rank-2 modules.** We shall now describe the modules of rank-2, whose endomorphism algebras interest us.

For a positive integer $m$, we let $P_m$ denote the space of polynomials of degree strictly less than $m$ and let $P_0 = (0)$. Clearly $P_m$ is just the $m$-dimensional pole space that corresponds to the height function which is zero on $K$ and $m-1$ at infinity. The rank-1 modules $P_{m-1} \ \substack{\longrightarrow \\ \longrightarrow} \ P_m$ are precisely the finite-dimensional, torsion-free, rank-1 Kronecker modules, see, e.g., [**2**, Theorem 7.5].

We shall be working with $K$-linear subspaces of the space $K(X)^2$ of pairs of rational functions. Such pairs shall be written in column notation. Given a triplet $(m, h, \alpha)$ where $m$ is a positive integer, $h$ is a height function and $\alpha$ is a functional, put

(14)
$$V = V(m,h,\alpha) = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in K(X)^2 : r \in R_h \text{ and } \partial_\alpha(r) + s \in P_m \right\},$$

and put

(15)
$$V^- = V^-(m,h,\alpha) = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in V : r \in R_h^- \text{ and } \partial_\alpha(r) + s \in P_{m-1} \right\}.$$

Observe that $XV^- \subseteq V$. Indeed, if $\binom{r}{s} \in V^-$, then $r \in R_h^-$ and $\partial_\alpha(r) + s \in P_{m-1}$. Therefore, $Xr \in R_h$, and using (10) we get as well that

$$\partial_\alpha(Xr) + Xs = X\partial_\alpha(r) + \partial_{\alpha * r}(X) + Xs = X(\partial_\alpha(r) + s) + \langle \alpha, r \rangle \in P_m.$$

The Kronecker module $\mathbf{V}(m,h,\alpha)$ is defined to be the module

$$V^-(m,h,\alpha) \;\; \overset{a}{\underset{b}{\rightrightarrows}} \;\; V(m,h,\alpha)\,, \tag{16}$$

where

$$a : \begin{pmatrix} r \\ s \end{pmatrix} \longmapsto \begin{pmatrix} r \\ s \end{pmatrix} \quad \text{and} \quad b : \begin{pmatrix} r \\ s \end{pmatrix} \longmapsto X \begin{pmatrix} r \\ s \end{pmatrix},$$

for each $\binom{r}{s}$ in $V^-(m,h,\alpha)$.

An inspection of the definitions of $V$ and $V^-$, in (14) and (15), shows that these spaces depend only on the value that the deriver takes on $R_h$. In turn, (13) and (11) show that the deriver on $R_h$ depends only on the values of the functional $\alpha$ on $R_h^-$. Thus the definition of $\mathbf{V}(m,h,\alpha)$ in (16) depends only on the restriction of $\alpha$ to $R_h^-$.

The pair of maps

$$\begin{array}{ccc} V^- \longrightarrow R_h^- & & V \longrightarrow R_h \\ & \text{and} & \\ \begin{pmatrix} r \\ s \end{pmatrix} \longmapsto r & & \begin{pmatrix} r \\ s \end{pmatrix} \longmapsto r \end{array}$$

establishes a homomorphism from $\mathbf{V}(m,h,\alpha)$ to the torsion-free, rank-1 module $R_h^- \rightrightarrows R_h$, with kernel the torsion-free, rank-1 module $P_{m-1} \rightrightarrows P_m$. Thus our $\mathbf{V}(m,h,\alpha)$'s are rank-2 extensions of finite-dimensional rank-1 modules by rank-1 modules. These modules pick up exactly all such extensions, see [**5**].

The pole space $R_h$ is finite-dimensional exactly when $h$ vanishes except on a finite subset of $K\cup\{\infty\}$ and $h$ is finite-valued on that finite subset. In this case the modules $\mathbf{V}(m,h,\alpha)$ are finite-dimensional and well understood. Henceforth, we make the blanket assumption that in $\mathbf{V}(m,h,\alpha)$:

the pole space $R_h$ is infinite-dimensional over $K$.

The endomorphism algebra of $\mathbf{V}(m,h,\alpha)$ is the object of our interest.

**Endomorphisms as $K(X)$-linear operators on $K(X)^2$.** If $\mathbf{V} = \mathbf{V}(m,h,\alpha)$ we check now that the algebra $\operatorname{End}\mathbf{V}$ of endomorphisms of $\mathbf{V}$ is the same as the $K$-algebra of $K(X)$-linear operators on $K(X)^2$ that leave invariant the $K$-linear subspace $V = V(m,h,\alpha)$ defined in (14).

**Proposition 2.2.** *If a $K(X)$-linear operator $\varphi : K(X)^2 \to K(X)^2$ leaves the subspace $V$ invariant, then $\varphi$ also leaves the space $V^-$ invariant, and the pair*

$$V^- \xrightarrow{\varphi_{|V^-}} V^-, \quad V \xrightarrow{\varphi_{|V}} V$$

*is an endomorphism of $\mathbf{V}$. Conversely, all endomorphisms of $\mathbf{V}$ arise in this way.*

*Proof.* Since $\varphi$ is a $K(X)$-linear operator on $K(X)^2$, the space $X^{-1}V$ is also invariant under $\varphi$. Therefore the space $V\cap X^{-1}V$ is $\varphi$-invariant. Now we just confirm that $V\cap X^{-1}V = V^-$.

From the definition of $V$ in(14), a pair of rational functions $\binom{p}{q}$ belongs to $V\cap X^{-1}V$ if and only if

(17) $p\in R_h, \quad Xp\in R_h, \quad \partial_\alpha(p)+q\in P_m \quad \text{and } \partial_\alpha(Xp)+Xq\in P_m.$

The joint statements $p \in R_h$ and $Xp \in R_h$ say that $p \in R_h^-$. By (10) and (9),

$$\partial_\alpha(Xp) = X\partial_\alpha(p) + \partial_{\alpha * p}(X) = X\partial_\alpha(p) + \langle \alpha, p\rangle,$$

and we see that

$$\partial_\alpha(Xp) + Xq = X(\partial_\alpha(p) + q) + \langle \alpha, p\rangle.$$

By inspecting this last equation it becomes clear that (17) holds if and only if $p \in R_h^-$ and $\partial_\alpha(p) + q \in P_{m-1}$. By (15) this is equivalent to $\binom{p}{q} \in V^-$.

The actions of $a$ and $b$, specified for the modules (16), plus the fact $\varphi$ is $K(X)$-linear on $K(X)^2$, make it clear that the maps $V^- \overset{\varphi_{|V^-}}{\to} V^-$, $V \overset{\varphi_{|V}}{\to} V$ constitute an endomorphism of $\mathbf{V}$.

For the converse, we refer to [**10**, p. 508]. Briefly one can check that every endomorphism $V^- \overset{\psi}{\to} V^-$, $V \overset{\varphi}{\to} V$ comes from a $K(X)$-linear operator $\sigma$ on $K(X)^2$ such that $\psi = \sigma_{|V^-}$ and $\varphi = \varphi_{|V}$. Since both $\binom{1}{0}$ and $\binom{0}{1}$ belong to $V$, the operator $\sigma$ must be the unique $K(X)$-linear operator on $K(X)^2$ that agrees with $\varphi$ on these basis vectors. $\square$

From now on, endomorphisms of $\mathbf{V}(m, h, \alpha)$ will always be taken to be $K(X)$-linear operators on $K(X)^2$, that leave invariant the space $V = V(m, h, \alpha)$ defined in (14). This point of view facilitates the role that linear algebra can play in their study.

**3.** End $\mathbf{V}(m, h, \alpha)$ **when** $h$ **is singular.** It is shown in [**10**, Theorem 2.6] that, for arbitrary height $h$, End $\mathbf{V}(m, h, \alpha)$ is noncommutative if and only if $\mathbf{V}$ decomposes as $\mathbf{V} = \mathbf{P} \oplus \mathbf{T}$ where $\mathbf{P} = P_{n-1} \rightrightarrows P_n$ for some positive integer $n$ and $\mathbf{T} = R_k^- \rightrightarrows R_k$ for some height function $k$. Using the structure of homomorphisms between rank-1, torsion-free modules (see, e.g., [**4**, Section 3] and [**7**, Section 85]), it can be checked that

$$\operatorname{End}\mathbf{V} \cong \begin{bmatrix} A & R_k \\ 0 & K \end{bmatrix},$$

where $A$ is the maximal $K$-algebra contained in $R_k$. Our goal is to present all possible commutative algebras $\operatorname{End}\mathbf{V}(m, h, \alpha)$ in the case where $h$ is singular, that is, when $h$ never assumes the value $\infty$.

**The space $T_x$.** The reader familiar with Abelian groups may see a parallel in the definition that follows, with the notion of pure-closure.

If $V = V(m, h, \alpha)$ as defined in (14) and $x \in V$ while $r \in K(X)$, the vector $rx$ need not belong to $V$. We define

$$T_x = \{r \in K(X) : rx \in V\}. \tag{18}$$

It turns out that such $T_x$ are pole spaces, and to prove it we need the following lemma.

**Lemma 3.1.** *If $\beta$ is a functional on $K(X)$ and $p \in P_m$, then the space*

$$\{r \in K(X) : \partial_\beta(r) + pr \in P_m\},$$

*denoted by $(\partial_\beta + p)^{-1}(P_m)$ is a pole space.*

*Proof.* By (12) all scalars lie in $(\partial_\beta + p)^{-1}(P_m)$. Supposing $t \in (\partial_\beta + p)^{-1}(P_m)$ and that a rational function $s$ is dominated by $t$ as in (5), we must show $s \in (\partial_\beta + p)^{-1}(P_m)$. Thus we have $\partial_\beta(t) + pt \in P_m$, and we require $\partial_\beta(s) + ps \in P_m$.

First we check that $\partial_\beta(s) + ps$ is a polynomial, which amounts to checking that $\partial_\beta(s) + ps$ can only have a pole at $\infty$. Suppose $\theta$ is a pole of $\partial_\beta(s)+ps$ and that $\theta \in K$. We will deduce that such a $\theta$ is a pole of $\partial_\beta(t) + pt$, in contradiction to the fact $\partial_\beta(t) + pt$ is a polynomial. Since $p$ is a polynomial and since $\partial_\beta$, according to Proposition 2.1, leaves pole spaces invariant, $\theta$ must also be a pole of $s$. Write $t = su$, where $u \in K(X)$. Because $t$ dominates $s$, we deduce form (2) that $\operatorname{ord}_\theta(u) \geq 0$. Using (2) and the fact that $\theta$ is a pole of $\partial_\beta(s) + ps$, we have

$$\operatorname{ord}_\theta(u(\partial_\beta(s) + ps)) = \operatorname{ord}_\theta(u) + \operatorname{ord}_\theta(\partial_\beta(s) + ps) > \operatorname{ord}_\theta(u).$$

As $\mathrm{ord}_\theta(u) \geq 0$, Proposition 2.1 gives

$$\mathrm{ord}_\theta(u) \geq \mathrm{ord}_\theta(\partial_{\beta * s}(u)).$$

Therefore

$$\mathrm{ord}_\theta(u(\partial_\beta(s) + ps)) > \mathrm{ord}_\theta(\partial_{\beta * s}(u)).$$

From (4) it follows that

$$\mathrm{ord}_\theta(u(\partial_\beta(s) + ps) + \partial_{\beta * s}(u)) = \mathrm{ord}_\theta(u(\partial_\beta(s) + ps)) > \mathrm{ord}_\theta(u) \geq 0.$$

Using (10) applied to $t = su$ we have $\partial_\beta(su) = u\partial_\beta(s) + \partial_{\beta * s}(u)$. Thus

$$\partial_\beta(t) + pt = \partial_\beta(su) + psu = u(\partial_\beta(s) + ps) + \partial_{\beta * s}(u).$$

The previous inequalities now give $\mathrm{ord}_\theta(\partial_\beta(t) + pt) > 0$, contradicting the fact $\partial_\beta(t) + pt$ is a polynomial.

It remains to prove that the polynomial $\partial_\beta(s) + ps$ lies in $P_m$. Seeking a contradiction, assume

$$\mathrm{ord}_\infty(\partial_\beta(s) + ps) \geq m. \tag{19}$$

Using (3) and then (2), we obtain

$$\mathrm{ord}_\infty(\partial_\beta(s) + ps) \leq \max\{\mathrm{ord}_\infty(\partial_\beta(s)), \mathrm{ord}_\infty(p) + \mathrm{ord}_\infty(s)\}.$$

If $\mathrm{ord}_\infty(s) \leq 0$, Proposition 2.1 yields that $\mathrm{ord}_\infty(\partial_\beta(s)) \leq 0$ too. In that case we see that

$$\mathrm{ord}_\infty(\partial_\beta(s) + ps) \leq \mathrm{ord}_\infty(p) = \deg p < m,$$

against our assumption (19). Thus $\mathrm{ord}_\infty(s) > 0$.

Since $t$ dominates $s$, we can write $t = su$ where $\mathrm{ord}_\infty(u) \geq 0$. From Proposition 2.1, it follows that

$$\mathrm{ord}_\infty(u) \geq \mathrm{ord}_\infty(\partial_{\beta * s}(u)).$$

Since $m > 0$, (19) implies that $\mathrm{ord}_\infty(\partial_\beta(s) + ps) > 0$. Thus (2) plus $\mathrm{ord}_\infty(u) \geq \mathrm{ord}_\infty(\partial_{\beta * s}(u))$ yields

$$\mathrm{ord}_\infty(u(\partial_\beta(s) + ps)) = \mathrm{ord}_\infty(u) + \mathrm{ord}_\infty(\partial_\beta(s) + ps) > \mathrm{ord}_\infty(\partial_{\beta * s}(u)).$$

From this inequality we can use (4), then (2) and then $\mathrm{ord}_\infty(u) \geq 0$ to get $\mathrm{ord}_\infty(u(\partial_\beta(s) + ps) + \partial_{\beta * s}(u)) = \mathrm{ord}_\infty(u(\partial_\beta(s) + ps)) \geq \mathrm{ord}_\infty(\partial_\beta(s) + ps)$.

By (10) applied to $t = su$, we have

$$u(\partial_\beta(s) + ps) + \partial_{\beta * s}(u) = \partial_\beta(t) + pt.$$

Thus

$$\mathrm{ord}_\infty(\partial_\beta(t) + pt) \geq \mathrm{ord}_\infty(\partial_\beta(s) + ps) \geq m.$$

This puts $\partial_\beta(t) + pt$ outside of $P_m$ and yields our desired contradiction. $\square$

If $h$ is a height function and $r \in R_h$, where $r \neq 0$, then $h(\theta) - \mathrm{ord}_\theta(r) \geq 0$ for all $\theta$ in $K \cup \{\infty\}$ so that $k(\theta) = h(\theta) - \mathrm{ord}_\theta(r)$ defines a height function $k$. It follows routinely from (2) that $r^{-1}R_h = R_k$. In particular, $r^{-1}R_h$ is a pole space.

**Proposition 3.2.** *For each nonzero $x$ in $V$, the space $T_x$, given by (18), is a pole space that sits inside a pole space of the form $qR_h$ or $qP_m$ for some rational function $q$.*

*Proof.* Using (14), write $x = \binom{r}{p - \partial_\alpha(r)}$ where $r \in R_h$ and $p \in P_m$, at least one of them nonzero. For a rational function $u$ we use the definition of $T_x$, then the definition of $V$ in (14), followed by (10) applied to $\partial_\alpha(ur)$ we obtain the following equivalencies:

$$\begin{aligned} u \in T_x &\iff ux \in V \\ &\iff ur \in R_h \text{ and } \partial_\alpha(ur) + up - u\partial_\alpha(r) \in P_m \\ &\iff ur \in R_h \text{ and } \partial_{\alpha * r}(u) + up \in P_m \\ &\iff ur \in R_h \text{ and } u \in (\partial_{\alpha * r} + p)^{-1}(P_m). \end{aligned}$$

The space $(\partial_{\alpha * r} + p)^{-1}(P_m)$ is a pole space according to Lemma 3.2. In case $r \neq 0$, the space $r^{-1}R_h$ is a pole space. In light of the above equivalences, $T_x$ is the pole space $r^{-1}R_h \cap (\partial_{\alpha * r} + p)^{-1}(P_m)$, which is certainly inside $qR_h$ where $q = r^{-1}$. In case $r = 0$, then $p \neq 0$ and we get $T_x = p^{-1}P_m$ which is now a pole space of the type $qP_m$. $\square$

**An eigenvector for all endomorphisms.** The next result identifies some fully invariant submodules of $\mathbf{V}(m,h,\alpha)$.

**Proposition 3.3.** *If $x$ is a nonzero element of $V(m,h,\alpha)$ such that space $T_x$ is infinite-dimensional over $K$, then $x$ is an eigenvector for every endomorphism of $\mathbf{V}(m,h,\alpha)$.*

*Proof.* Let $\varphi \in \operatorname{End} \mathbf{V}(m,h,\alpha)$. Thus $\varphi$ is a $K(X)$-linear operator on $K(X)^2$ leaving the space $V(m,h,\alpha)$ invariant. Write

$$x = \begin{pmatrix} p \\ q \end{pmatrix} \text{ and } \varphi(x) = \begin{pmatrix} u \\ v \end{pmatrix},$$

for some $p,q,u,v \in K(X)$. Note that $p \neq 0$, for otherwise $T_x$ would be the finite-dimensional space $q^{-1}P_m$. Since for each $t$ in $T_x$ we have $t\varphi(x) = \varphi(tx) \in V$, therefore $T_x \subseteq T_{\varphi(x)}$. Hence $T_{\varphi(x)}$ is also infinite-dimensional. As is the case with $x$, we deduce likewise that $u \neq 0$.

Letting $k$ be the height function that defines the pole space $T_x$, we put

$$l(\theta) = k(\theta) + \max\{0, \operatorname{ord}_\theta(p), \operatorname{ord}_\theta(u)\} \text{ for all } \theta \text{ in } K \cup \{\infty\}.$$

The height function $l$ defines the pole space $R_l$ that contains all of $T_x, pT_x, uT_x$ as subspaces of finite-codimension.

For each $t$ in $T_x$ we have $tx \in V$ and $t\varphi(x) \in V$. Thus from the definition of $V$ in (14), we get that $\partial_\alpha(tp) + tq \in P_m$ and $\partial_\alpha(tu) + tv \in P_m$. Because $P_m$ is finite-dimensional, it follows that the spaces

$$S_x = \{t \in T_x : \partial_\alpha(tp) + tq = 0\}$$

and

$$S_{\varphi(x)} = \{t \in T_x : \partial_\alpha(tu) + tv = 0\}$$

are each of finite-codimension in $T_x$.

Then $pS_x$ is of finite codimension in $pT_x$, while $uS_{\varphi(s)}$ is of finite-codimension in $uT_x$. As $pT_x$ and $uT_x$ have finite-codimension in $R_l$, it follows that $pS_x$ and $uS_{\varphi(x)}$ have finite-codimension in $R_l$.

Consequently $pS_x$ and $uS_\varphi(x)$ have nonzero intersection. This means that $sp = tu$ for some nonzero $s$ in $S_x$ and nonzero $t$ in $S_{\varphi(x)}$. From the definitions of $S_x$ and $S_{\varphi(x)}$ above, it follows that $sq = tv$ also. Hence $x$ is an eigenvector of $\varphi$ with eigenvalue $s/t$. $\square$

**Structure of** $\operatorname{End}\mathbf{V}$ **when the height is singular.** We now present the possible commutative endomorphisms algebras $\operatorname{End}\mathbf{V}(m,h,\alpha)$ in the case where $h$ is singular.

For a vector space $S$ over $K$, the trivial extension of $K$ by the vector space $S$ may be viewed as the algebra of all $2\times 2$ matrices of the form $\left[\begin{smallmatrix}\lambda & s\\ 0 & \lambda\end{smallmatrix}\right]$ where $\lambda \in K$ and $s \in S$. Following [**6**], we denote these by $K \propto S$.

**Theorem 3.4.** *If* $(m,h,\alpha)$ *is a triplet for which* $h$ *is singular and* $\operatorname{End}\mathbf{V}(m,h,\alpha)$ *is commutative, then the algebra* $\operatorname{End}\mathbf{V}(m,h,\alpha)$ *is one of*

$$K, \text{ the direct product } K\times K \text{ or a trivial extension } K\propto S.$$

*Proof*. We recall that endomorphisms of $\mathbf{V} = \mathbf{V}(m,h,\alpha)$ are $K(X)$-linear operators on $K(X)^2$ that leave the space $V = V(m,h,\alpha)$ invariant.

If every nonzero endomorphism is monic, then [**10**, Corollary 3.7] gives that $\operatorname{End}\mathbf{V} = K$. Thus we may assume that $\mathbf{V}$ has a nonzero, nonmonic endomorphism $\varphi$, in which case nonzero elements $x, y$ in $V$ must exist such that $\varphi(x) = 0$ while $z = \varphi(y) \neq 0$. Let $\ker\varphi$, $\operatorname{im}\varphi$ be the kernel and image of $\varphi$ as an operator on $K(X)^2$. Obviously, $\ker\varphi_{|V} = V\cap\ker\varphi$ and $\operatorname{im}\varphi_{|V} \subseteq V\cap\operatorname{im}\varphi$. From the $K(X)$-linearity of $\varphi$ we get $\ker\varphi = K(X)x$ and $\operatorname{im}\varphi = K(X)z$. Since $V$ is infinite-dimensional over $K$, then either $\ker\varphi_{|V}$ or $\operatorname{im}\varphi_{|V}$ must be a subspace of $V$ infinite-dimensional over $K$. Hence either $V\cap\ker\varphi$ or $V\cap\operatorname{im}\varphi$ is infinite-dimensional over $K$. However, $V\cap\ker\varphi = T_x x$ and $V\cap\operatorname{im}\varphi = T_z z$. It follows that either $T_x$ or $T_z$ is infinite-dimensional over $K$.

We can now assume that, for some nonzero $x$ in $V$, the space $T_x$ is infinite-dimensional over $K$. Put $w = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ in $V$. Note that for every

$r$ in $K(X)$, the space $T_{rw}$ is finite-dimensional. Indeed, if $t \in T_{rw}$, then tr $\in P_m$, which puts $t$ in the finite-dimensional space $r^{-1}P_m$. Since $T_x$ is infinite-dimensional over $K$, it cannot be that $x = rw$ for some rational function $r$. Consequently the pair $\{x, w\}$ forms a basis of $K(X)^2$ as a vector space over the field $K(X)$.

Endomorphisms of $\mathbf{V}$ will hereby be represented as $2 \times 2$ matrices over $K(X)$ with respect to the basis $\{x, w\}$. Proposition 3.3 reveals that every endomorphism $\varphi$ is an upper triangular matrix

$$\varphi = \begin{bmatrix} s_\varphi & t_\varphi \\ 0 & v_\varphi \end{bmatrix} \quad \text{for some } s_\varphi, t_\varphi, v_\varphi \text{ in } K(X).$$

Because $s_\varphi x = \varphi(x) \in V$ for every $\varphi$ in $\operatorname{End}\mathbf{V}$, it follows that each $s_\varphi \in T_x$. Furthermore, the eigenvalue map $\operatorname{End}\mathbf{V} \to K(X)$, whereby $\varphi \mapsto s_\varphi$, is a $K$-algebra homomorphism. Thus the functions $s_\varphi$ constitute a $K$-algebra inside the pole space $T_x$. From Proposition 3.2 the space $T_x$, being infinite-dimensional, sits inside a pole space $qR_h$ for some rational function $q$. Since $h$ is singular, so are the height functions for $qR_h$ and for $T_x$. This means that the only $K$-subalgebra of $T_x$ is $K$. Therefore, $s_\varphi \in K$.

Next we check that $v_\varphi \in K$ as well. The mapping $\operatorname{End}\mathbf{V} \to K(X)$, whereby $\varphi \mapsto v_\varphi$, is also a $K$-algebra homomorphism. Thus the rational functions $v_\varphi$ form a $K$-subalgebra of $K(X)$. According to the definition of $V$ in (14), we may write

$$x = \begin{pmatrix} r \\ p - \partial_\alpha(r) \end{pmatrix} \text{ for some } r \text{ in } R_h \text{ and some } p \text{ in } P_m.$$

Note that $r \neq 0$ because $x$ is independent of $w$ over $K(X)$. The element $\varphi(w) = t_\varphi(x) + v_\varphi w$ belongs to $V$ and therefore, by (14),

$$t_\varphi r \in R_h \text{ and } \partial_\alpha(t_\varphi r) + t_\varphi p - t_\varphi \partial_\alpha(r) + v_\varphi \in P_m.$$

Dividing by $r$ and using (10) applied to $\partial_\alpha(t_\varphi r)$, the above becomes

$$t_\varphi \in r^{-1}R_h \text{ and } \partial_{\alpha * r}(t_\varphi) + t_\varphi p + v_\varphi \in P_m.$$

Since the deriver $\partial_{\alpha * r}$ leaves the pole space $r^{-1}R_h$ invariant, it follows that

$$v_\varphi \in P_m + r^{-1}R_h + pr^{-1}R_h.$$

Thus the $v_\varphi$ constitute a $K$-algebra that sits inside the space $P_m + r^{-1}R_h + pr^{-1}R_h$. The singularity of $h$ means that $R_h$ only contains $K$ as a $K$-subalgebra, and so the same is true of the space $P_m + r^{-1}R_h + pr^{-1}R_h$. We deduce that $v_\varphi \in K$.

If the map $\operatorname{End}\mathbf{V} \to K$, whereby $\varphi \to v_\varphi$ is injective, then $\operatorname{End} E = K$. If this map is not injective, we have a nonzero endomorphism $\psi$ that takes the form

$$\psi = \begin{bmatrix} s_\psi & t_\psi \\ 0 & 0 \end{bmatrix}, \text{ where } s_\psi \in K \text{ and } t_\psi \in K(X).$$

The commutativity of $\operatorname{End}\mathbf{V}$ leads to

$$\begin{bmatrix} s_\psi & t_\psi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s_\varphi & t_\varphi \\ 0 & v_\varphi \end{bmatrix} = \begin{bmatrix} s_\varphi & t_\varphi \\ 0 & v_\varphi \end{bmatrix}\begin{bmatrix} s_\psi & t_\psi \\ 0 & 0 \end{bmatrix}$$

for all $\varphi$ in $\operatorname{End}\mathbf{V}$, which is equivalent to

$$t_\varphi s_\psi + v_\varphi t_\psi = s_\varphi t_\psi \text{ for all } \varphi \text{ in } \operatorname{End}\mathbf{V}. \tag{20}$$

Should it happen that the scalar $s_\psi \neq 0$, we can suppose by rescaling that $s_\psi = 1$. Then $\psi = \left[\begin{smallmatrix} 1 & t_\psi \\ 0 & 0 \end{smallmatrix}\right]$ is a proper idempotent, making $\mathbf{V}$ a decomposable module. In that case (20) becomes

$$t_\varphi = s_\varphi t_\psi - v_\varphi t_\psi \text{ for all } \varphi \text{ in } \operatorname{End}\mathbf{V}.$$

This allows us to say that any $\varphi$ in $\operatorname{End}\mathbf{V}$ can be expressed as

$$\varphi = s_\varphi \psi + v_\varphi(I - \psi), \qquad \text{where } I \text{ is the identity endomorphism.}$$

Here we obtain $\operatorname{End}\mathbf{V} = K \times K$.

Should it happen that $s_\psi = 0$ while $t_\psi \neq 0$, (20) gives $v_\varphi t_\psi = s_\varphi t_\psi$, so that $v_\varphi = s_\varphi$. In this case we obtain that $\operatorname{End}\mathbf{V} = K \propto S$ where $S$ is the space of all nonzero $t_\varphi$ that might come up and all contingencies have been covered. $\square$

**Corollary 3.5.** *If* $\operatorname{End}\mathbf{V}(m, h, \alpha)$ *is an integral domain and* $h$ *is singular, then* $\operatorname{End}\mathbf{V}(m, h, \alpha)$ *is trivial.*

Each of the possible algebras from Theorem 3.4 arise for suitable choices of $(m, h, \alpha)$. For examples realizing $K$ and $K \times K$, see [**10**, Theorem 3.8, Example 3.9]. For $K \propto S$ we give one example and refer to [**12**] for its proof as well as more general realization theorems and constructions.

**Example giving** $K \propto S$**.** Take $m = 1$. Let $h$ be the height function that is identically 2 on $K$ and 0 at $\infty$. Pick an infinite, proper subset $J$ of $K$. Define the functional $\alpha$ by $\langle \alpha, X_\theta^2 \rangle = 1$ when $\theta \in J$ and $\alpha = 0$ on the rest of the standard basis of $K(X)$. Here $S$ turns out to be the span of all $X_\theta$ and $X_\theta^2$ as $\theta$ runs over $K \setminus J$.

The computation of endomorphism algebras of Kronecker modules opens up a line of investigation in the spirit of Reid's work on the structure of Abelian groups as modules over their endomorphism rings, see, e.g., [**13, 3**]. For example, it would be interesting to know when $\mathbf{V}$ is a finitely generated module over $\operatorname{End} \mathbf{V}$.

## REFERENCES

**1.** D.M. Arnold, *Abelian groups and representations of finite partially ordered sets*, Springer-Verlag, New York, 2000.

**2.** M. Auslander, I. Reiten, S.O. Smalø, *Representation theory of Artin algebras*, Cambridge Univ. Press, Cambridge, 1995.

**3.** T. Faticoni, *Categories of modules over endomorphism rings*, Mem. Amer. Math. Soc. **103** (1993).

**4.** U. Fixman, *On algebraic equivalence between pairs of linear transformations*, Trans. Amer. Math. Soc. **113** (1964), 424–453.

**5.** U. Fixman and F. Okoh, *Extensions of modules characterized by finite sequences of linear functionals*, Rocky Mountain J. Math. **21** (1991), 1235–1258.

**6.** R.M. Fossum, P.A. Griffith and I. Reiten, *Trivial extensions of Abelian categories*, Lecture Notes in Math., vol. 456, Springer, New York, 1975.

**7.** L. Fuchs, *Infinite Abelian groups*, Vol. II, Academic Press, New York, 1973.

**8.** R. Göbel and D. Simson, *Embeddings of Kronecker modules into the category of prinjective modules and the endomorphism ring problem*, Colloq. Math. **75** (1998), 213–244.

**9.** ———, *Rigid families and endomorphism algebras of Kronecker modules*, Israel J. Math. **11** (1999), 293–315.

**10.** F. Okoh and F.A. Zorzitto, *A family of commutative endomorphism algebras*, J. Algebra **201** (1998), 501–527.

**11.** ———, *Curves arising from Kronecker modules*, submitted to Linear Algebra Appl.

**12.** ———, *Endomorphism algebras realized through height functions*, in preparation.

**13.** J.D. Reid, *Abelian groups finitely generated over their endomorphism rings*, Lecture Notes in Math., vol. 874, Springer, New York, 1981, pp. 41–53.

**14.** D. Simson, *Linear representations of partially ordered sets and vector space categories*, Gordon and Breach, New York, 1992.

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