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ISOMORPHISM CLASSES OF UNIFORM GROUPS

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ABSTRACT. In this paper we count isomorphism classes of uniform groups within a fixed near-isomorphism class.

1. Preliminaries. An almost completely decomposable group X is an extension of a completely decomposable group R by a finite group X/R. If $\exp(X/R) = h$, denote $^{-} : R \to \overline{R} = h^{-1}R/R$, $x \mapsto \bar{x} = h^{-1}x + R$ the natural epimorphism. Furthermore, $^{-}$ denotes also the induced homomorphism $^{-} : \operatorname{Aut} R \to \operatorname{Aut} \overline{R}$, $\alpha \mapsto \bar{\alpha}$, which is well defined by $\bar{\alpha}(\bar{x}) := \overline{\alpha(x)}$. Recall, cf. [**6**], that

$$\operatorname{Typ} \operatorname{Aut} \overline{R} = \{\xi \in \operatorname{Aut} \overline{R} \mid \forall_{\tau \in T_{\mathrm{cr}}(R)} \xi \overline{R(\tau)} = \overline{R(\tau)}\}$$

is the set of *type automorphisms* of $\overline{R}$. Let $R = \bigoplus_{j=1}^{n} \langle x_j \rangle_*^R$, where $\mathbf{x} = (x_1, \dots , x_n)$ is an h-decomposition basis, i.e., $\operatorname{hgt}_p^R(x_j) \in \{0, \infty\}$ for all j and all primes p dividing h. Then $\bar{\mathbf{x}} = (\bar{x}_1, \dots , \bar{x}_n)$ is called an *induced decomposition basis* of $h^{-1}R/R$. We write $\mathbf{Z}_h := \mathbf{Z}/h\mathbf{Z}$. Let $\mathbf{a} = (a_1, \dots , a_r)$ be a basis of $X/R \subseteq h^{-1}R/R$. Then the basis elements a_i may be written as linear combinations of the induced decomposition basis $a_i = \sum_{j=1}^{n} \alpha_{ij} \bar{x}_j$, for $i = 1, \dots , r$, where $\alpha_{ij} \in \mathbf{Z}_h$. The $(r \times n)$-matrix $M = (\alpha_{ij})_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \in \mathbf{M}^{r\times n}(\mathbf{Z}_h)$ is called *representing matrix* of X over R relative to $\mathbf{a}$ and $\bar{\mathbf{x}}$.

A group X is called *p-local* for a prime p if the regulator quotient $X/R(X)$ is a (finite) p-group.

Definition 1.1. Let p be a prime and e, n, r natural numbers. Let $T = (\tau_1, \dots , \tau_n)$ be an ordered n-tuple of pairwise incomparable types, where $\tau_i(p) \neq \infty$ each i. Then $\mathcal{C}(T, p, e, r)$ denotes the class of almost completely decomposable groups X such that

(1) $T = T_{\mathrm{cr}}(X)$ is the critical typeset of X,

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(2) X is rigid, i.e., $X(\tau)$ has rank 1 for all $\tau \in T$,

(3) the regulator quotient is homocyclic of exponent p^e, i.e., $X/R(X) \cong (\mathbf{Z}_{p^e})^r = \underbrace{\mathbf{Z}_{p^e} \oplus \mathbf{Z}_{p^e} \oplus \cdots \oplus \mathbf{Z}_{p^e}}_{r}$ is a direct sum of r copies of $\mathbf{Z}_{p^e}$.

We call such groups X *uniform*.

Note that a group in $\mathcal{C}(T, p, e, r)$ has rank n and is p-reduced which is equivalent to the assumption $\tau_i(p) \neq \infty$.

The following lemma is folklore.

Lemma 1.2. *Let X be an almost completely decomposable group. Let $R = \oplus_{j=1}^{n} R_{\tau_j}$ be a rigid completely decomposable subgroup of finite index and X/R of exponent $m \in \mathbf{N}$. The following are equivalent*:

(1) $R(X) = R$.

(2) $(R_{\tau_j})_*^X = R_{\tau_j}$ for $j = 1, \ldots, n$.

(3) $X/R \cap (m^{-1}R_{\tau_j} + R)/R = 0$ for $j = 1, \ldots, n$.

In Lemma 3.4 we derive a regulator criterion for the representing matrix of a p-local group.

Let X and Y be groups with a common regulator $R(X) = R(Y) = R$ and $\exp(X/R) = \exp(Y/R) = h$. Let M be the representing matrix of X and N be the representing matrix of Y over R relative to some bases. Then the near-isomorphism criterion [**6**, Theorem 2.15] and the isomorphism criterion [**6**, Theorem 4.2] have the following matrix forms:

(1) The groups X and Y are *nearly isomorphic*, $X \cong_{\mathrm{nr}} Y$, if and only if $N = PMD$, where P is the matrix of an automorphism of Y/R and D is the matrix of a type-automorphism $\xi \in \operatorname{Typ}\operatorname{Aut}\overline{R}$, or equivalently $\xi(X/R) = (Y/R)$.

(2) The groups X and Y are isomorphic if and only if $N = PMF$, where P is the matrix of an automorphism of Y/R and F is the matrix of an induced automorphism $\zeta \in \overline{\operatorname{Aut} R}$ or equivalently $\zeta(X/R) = (Y/R)$.

If we specialize to rigid groups, then these matrix equations simplify, since typ-automorphisms have diagonal matrices in this case. We achieve reductions by using a special form of the representing matrices in Theorem 4.3. In Theorem 5.1 we use such matrix equations to determine the number of distinct isomorphism classes contained in a fixed near-isomorphism class. We will find it easy to calculate upper and lower bounds for that number.

2. Matrix theory.

Definition 2.1. Let S be a commutative ring with 1, let r, n be natural numbers. Let S^* denote the set of units in S. Let the set of $(r \times n)$-matrices over S be denoted by $\mathbf{M}^{r\times n}(S)$. A matrix which is obtained by striking out rows and columns of a matrix A is called a *submatrix* of A. The maximal natural number k such that there is an invertible k-rowed submatrix of A is called *determinantal rank* of A. Write $\mathrm{rk}_{\det}(A) = k$. Abbreviate a diagonal matrix by

$$\mathrm{diag}\,(d_1, \dots, d_n) \in \mathbf{M}^{n\times n}(S).$$

If $r < n$ and $D = \mathrm{diag}\,(d_1, \dots, d_r, d_{r+1}, \dots, d_n)$, then define the submatrices $D_{\le r} := \mathrm{diag}\,(d_1, \dots, d_r)$ and $D_{>r} := \mathrm{diag}\,(d_{r+1}, \dots, d_n)$.

Let $U, U_1, \dots, U_n$ be subgroups of $(S^*, \cdot)$. Write

$$\mathrm{DIAG}\,(n; U) := \{\mathrm{diag}\,(d_1, \dots, d_n) \mid \forall_{j=1,\dots,n} d_j \in U\}$$

for the set of all $(n \times n)$-diagonal matrices over U. This definition can be generalized to

$$\mathrm{DIAG}\,(U_1, \dots, U_n) := \{\mathrm{diag}\,(f_1, \dots, f_n) \mid \forall_{j=1,\dots,n} f_j \in U_j\}.$$

The matrices $M, N \in \mathbf{M}^{r\times n}(S)$ are said to be *diagonally equivalent* if there are invertible diagonal matrices D_1, D_2 such that

$$N = D_1 M D_2.$$

The matrices $A, B \in \mathbf{M}^{r\times r}(S)$ are said to be *diagonally similar* if there is an invertible diagonal matrix D such that $B = D^{-1}AD$.

Remark 2.2. Note that $\mathrm{DIAG}\,(U_1, \dots, U_n) \cong \prod_{j=1}^n U_j$ is an abelian subgroup of $\mathrm{GL}\,(n, S)$. This group acts on $\mathbf{M}^{r\times(n-r)}(S)$ via diagonal equivalence:

$$\begin{aligned}\mathrm{DIAG}\,(U_1,\dots,U_n)\times\mathbf{M}^{r\times(n-r)}(S)&\longrightarrow\mathbf{M}^{r\times(n-r)}(S),\\(F,M)&\longmapsto F_{\le r}^{-1}MF_{>r}.\end{aligned}$$

Let $A\in\mathbf{M}^{r\times(n-r)}(S)$. The *stabilizer* of A in $\mathrm{DIAG}\,(U_1,\dots,U_n)$ is defined as

(2.3)
$$\mathrm{Stab}_{\mathrm{DIAG}\,(U_1,\dots,U_n)}(A)=\{F\in\mathrm{DIAG}\,(U_1,\dots,U_n)\mid F_{\le r}^{-1}AF_{>r}=A\}.$$

The $\mathrm{DIAG}\,(U_1,\dots,U_n)$-*orbits* are known as diagonal equivalence classes in $\mathbf{M}^{r\times(n-r)}(S)$. The orbit of A is

$$\mathrm{Orb}\,(A)=\left\{F_{\le r}^{-1}A\,F_{>r}\;\middle|\;\begin{array}{l}F_{\le r}=\mathrm{diag}\,(f_1,\dots,f_r),\\F_{>r}=\mathrm{diag}\,(f_{r+1},\dots,f_n),\quad\text{where } f_j\in U_j\end{array}\right\}.$$

Lemma 2.4. *The number of matrices which are* $\mathrm{DIAG}\,(U_1,\dots,U_n)$-*diagonally equivalent to* A *is*

$$[\mathrm{DIAG}\,(U_1,\dots,U_n):\mathrm{Stab}_{\mathrm{DIAG}\,(U_1,\dots,U_n)}(A)]=\frac{\prod_{j=1}^n|U_j|}{|\mathrm{Stab}_{\mathrm{DIAG}\,(U_1,\dots,U_n)}(A)|}.$$

Proof. This is the well-known fact that the length of the orbit of A is the index of the stabilizer of A in the group of all $\mathrm{DIAG}\,(U_1,\dots,U_n)$ matrices. $\square$

Definition 2.5. Let p be a prime and $e\in\mathbf{N}$ a natural number. A matrix $C=(\gamma_{ij})_{\substack{1\le i\le r\\1\le j\le k}}$ over $\mathbf{Z}_{p^e}$ is said to be *normed* if all the main submatrices $C_m=(\gamma_{ij})_{\substack{1\le i\\j\le m}}$ for $m=1,2,\dots,\min(r,k)$ have determinant 1.

Let M be an $(r\times n)$-matrix over $\mathbf{Z}_{p^e}$ with determinantal rank r. Then there are invertible submatrices of size $r\times r$. The set of indices of the columns for such an invertible submatrix is called a *pivot set* of the matrix M. A pivot set is not uniquely determined in general.

Example 2.6. We want to determine the cardinality of an arbitrary diagonal equivalence class of normed invertible (2×2)-matrices over $\mathbf{Z}_{p^e} = \mathbf{Z}/p^e\mathbf{Z}$. A 2-rowed matrix A is invertible and normed if and only if

$$A = \begin{pmatrix} 1 & \alpha \\ \beta & 1+\alpha\beta \end{pmatrix},$$

where $\alpha = \lambda p^m$, $\beta = \mu p^l$ and λ, μ are units, $0 \le m,\, l \le e$.

By Lemma 2.4, we have to calculate the cardinality of $\mathrm{Stab}_{\mathrm{DIAG}\,(4;\mathbf{Z}^*_{p^e})}(A)$. By Definition 2.3 we have $D = \mathrm{diag}\,(d_1, d_2, d_3, d_4) \in \mathrm{Stab}_{\mathrm{DIAG}\,(4;\mathbf{Z}^*_{p^e})}(A)$ if and only if

$$\begin{aligned}\begin{pmatrix} d_1^{-1} & 0 \\ 0 & d_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ \beta & 1+\alpha\beta \end{pmatrix}\begin{pmatrix} d_3 & 0 \\ 0 & d_4 \end{pmatrix} &= \begin{pmatrix} d_1^{-1}d_3 & d_1^{-1}d_4\alpha \\ d_2^{-1}d_3\beta & d_2^{-1}d_4(1+\alpha\beta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & \alpha \\ \beta & 1+\alpha\beta \end{pmatrix}.\end{aligned}$$

It can be shown that therefore $d_3 = d_1$ and $d_4 = d_2$ and $(d_2 - d_1)\cdot\alpha = 0$ and $(d_2 - d_1)\cdot\beta = 0$.

We count the possibilities of the solutions to determine $|\mathrm{Stab}_{\mathrm{DIAG}\,(4;\mathbf{Z}^*_{p^e})}(A)|$.

Case 1. $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $|\mathrm{Stab}_{\mathrm{DIAG}\,(4;\mathbf{Z}^*_{p^e})}(A)| = \varphi(p^e)^2$ and

$$|\mathrm{Orb}\,(A)| = \varphi(p^e)^2 = (p^{e-1}(p-1))^2.$$

Case 2. $A \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $|\mathrm{Stab}_{\mathrm{DIAG}\,(4;\mathbf{Z}^*_{p^e})}(A)| = \varphi(p^e)\cdot p^{\min(l,m)}$ and

$$|\mathrm{Orb}\,(A)| = \varphi(p^e)^3 \cdot p^{-\min(l,m)} = p^{3e-3-\min(l,m)}(p-1)^3.$$

Here $|\mathrm{Orb}\,(A)|$ is the number of (2×2)-matrices over $\mathbf{Z}_{p^e}$ which are diagonally equivalent to A. Recall that φ denotes the Euler φ-function. $\square$

3. Representing matrices.

Theorem 3.1 [**5**, Theorem 3.7]. *Let p be a prime and e, n, r natural numbers. Let X be a p-reduced almost completely decomposable group of rank n with completely decomposable subgroup R such that*

$$X/R \cong (\mathbf{Z}/p^{e_1}\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/p^{e_r}\mathbf{Z}), \quad \text{with } e = e_1 \geq \cdots \geq e_r \geq 1.$$

Then there is an ordered induced decomposition basis $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ of $p^{-e}R/R$ and an ordered basis $\mathbf{a} = (a_1, \ldots, a_r)$ of X/R with $\langle a_i \rangle \cong \mathbf{Z}/p^{e_i}\mathbf{Z}$ such that the representing matrix of X over R relative to $\bar{\mathbf{x}}$ and $\mathbf{a}$ is in Hermite *normal form*

$$M = \Lambda(E \mid A), \quad \text{where } \Lambda = \operatorname{diag}(p^{e-e_1}, \ldots, p^{e-e_r}), \quad \text{and}$$

$$E = \begin{pmatrix} 1 & m_{1,2} & \cdots & m_{1,r} \\ 0 & 1 & \cdots & m_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad m_{i,j} \in \mathbf{Z}/p^e\mathbf{Z}. \tag{3.2}$$

If especially $e_i = e_j$, then $m_{i,j} = 0$. In particular, when $e = e_1 = \cdots = e_r$, there are bases $\bar{\mathbf{x}}$ and $\mathbf{a}$ such that X has a representing matrix in Hermite normal form *$M = (I_r \mid A)$ where I_r is the $(r \times r)$-identity matrix.*

Definition 3.3. The matrix $A \in \mathbf{M}^{r\times k}(\mathbf{Z}_{p^e})$ is called *primitive* if each row of A has an entry which is a unit in $\mathbf{Z}_{p^e}$.

Lemma 3.4. *Let p be a prime and $e, n, r \in \mathbf{N}$ natural numbers with $r < n$. Let X be a p-reduced rigid almost completely decomposable group of rank n with a completely decomposable subgroup R such that $X/R \cong \bigoplus_{i=1}^{r}(\mathbf{Z}/p^{e_i}\mathbf{Z})$, where $e = e_1, \geq \cdots \geq e_r \geq 1$. Let $\bar{\mathbf{x}}$ be an ordered basis of $p^{-e}R/R$ and $\mathbf{a} = (a_1, \ldots, a_r)$ an ordered basis of X/R with $\langle a_i \rangle \cong \mathbf{Z}/p^{e_i}\mathbf{Z}$. Let $B \in \mathbf{M}^{r\times n}(\mathbf{Z}_{p^e})$ be some matrix such that*

$$M = \operatorname{diag}(p^{e-e_1}, \ldots, p^{e-e_r}) \cdot B$$

is the representing matrix of X over R relative to $\mathbf{a}$ and $\bar{\mathbf{x}}$.

Then $R = R(X)$ is the regulator of X if and only if any submatrix obtained from B by deleting one column has determinantal rank r.

If in addition $e = e_1 = \cdots = e_r$ and $M = (I_r \mid A)$ is in Hermite normal form, then R is the regulator of X if and only if A is primitive.

Proof. Write $R = \bigoplus_{j=1}^n R_{\tau_j} = \bigoplus_{j=1}^n \langle x_j \rangle_*^R$, where $\mathbf{x} = (x_1, \dots, x_n)$ is an ordered p-decomposition basis with $\operatorname{tp}(x_j) = \tau_j \in T_{\mathrm{cr}}(R)$. Let $^- : R \to \overline{R} = p^{-e}R/R$, $x \mapsto \bar{x} = p^{-e}x + R$ denote the natural epimorphism. Recall that $X/R = \bigoplus_{i=1}^r \mathbf{Z}_{p^e} a_i$ and $\bigoplus_{j=1}^n (p^{-e}R_{\tau_j} + R)/R = (p^{-e}R)/R = \overline{R} = \bigoplus_{j=1}^n \overline{R}_{\tau_j} = \bigoplus_{j=1}^n \mathbf{Z}_{p^e}\bar{x}_j$. Write $B = (\beta_{ij})_{\substack{i=1,\dots,r \\ j=1,\dots,n}}$ and $(\alpha_{ij})_{i,j} = M = \operatorname{diag}(p^{e-e_1}, \dots, p^{e-e_r}) \cdot B = (p^{e-e_i}\beta_{ij})_{i,j}$. Let $B^{(k)}$ denote the $[r \times (n-1)]$-matrix over $\mathbf{Z}_{p^e}$ obtained from B by deleting the k-th column. This matrix $B^{(k)}$ has p-independent rows if and only if $\operatorname{rk}_{\det} B^{(k)} = r$. By the regulator criterion 1.2, we have to show

$$\frac{X}{R} \cap \frac{p^{-e}R_{\tau_k} + R}{R} = 0 \quad \text{for all } k = 1, \dots, n \quad [\overset{1.2}{\Longleftrightarrow} R = R(X)]$$
$$\Longleftrightarrow B^{(k)} \text{ has } p\text{-independent rows for all } k = 1, \dots, n.$$

"$\Leftarrow$." Assume that $k \in \{1, \dots, n\}$ and $B^{(k)}$ has p-independent rows. Let $\sum_{i=1}^r m_i a_i \in (X/R) \cap (p^{-e}R_{\tau_k} + R)/R \subseteq (p^{-e}R_{\tau_k} + R)/R = \mathbf{Z}_{p^e}\bar{x}_k$ be an arbitrary element of the intersection. Then

$$\sum_{i=1}^r m_i a_i = \sum_{i=1}^r m_i \left(\sum_{j=1}^n \alpha_{ij}\bar{x}_j \right) = \sum_{j=1}^n \left(\sum_{i=1}^r m_i \alpha_{ij} \right) \bar{x}_j \in \mathbf{Z}_{p^e}\bar{x}_k.$$

Since the sum $\overline{R} = \bigoplus_{j=1}^n \mathbf{Z}_{p^e}\bar{x}_j$ is direct, we conclude $\sum_{i=1}^r m_i \alpha_{ij} = \sum_{i=1}^r (m_i p^{e-e_i})\beta_{ij} = 0$ in $\mathbf{Z}_{p^e}$ for all $j \neq k$. Hence

$$(m_1 p^{e-e_1}, \dots, m_r p^{e-e_r}) \cdot B^{(k)} = \underbrace{(0, \dots, 0)}_{n-1 \text{ times}}.$$

So $(m_1 p^{e-e_1}, \dots, m_r p^{e-e_r}) = (0, \dots, 0)$ as the rows of $B^{(k)}$ are p-independent. Thus $\sum_{i=1}^r m_i a_i = \sum_{i=1}^r m_i (\sum_{j=1}^n \alpha_{ij}\bar{x}_j) = \sum_{j=1}^n \sum_{i=1}^r \times \underbrace{m_i k p^{e-e_i}}_{=0} \beta_{ij}\bar{x}_j = 0$ and therefore $X/R \cap (p^{-e}R_{\tau_k} + R)/R = 0$. Since this is true for all $k = 1, \dots, n$, $R(X) = R$ follows.

"$\Rightarrow$." Assume that $k \in \{1, \dots, n\}$ and $(X/R) \cap (p^{-e} R_{\tau_k} + R)/R = 0$. Then $(X/R)[p] \cap (p^{-1} R_{\tau_k} + R)/R = 0$. Notice that $(X/R)[p] = (p^{-1} R \cap X)/R = \langle p^{e_i - 1} a_i \mid 1 \leq i \leq r \rangle$ is the p-socle of (X/R) and $(p^{-1} R_{\tau_k} + R)/R = \langle p^{e-1} \bar{x}_k \rangle$ is the p-socle of $\overline{R}_{\tau_k} = (p^{-e} R_{\tau_k} + R)/R$. Let $m_1, \dots, m_r \in \mathbf{Z}_{p^e}$ be such that $(m_1, \dots, m_r) \cdot B^{(k)} \in p(\mathbf{Z}_{p^e})^{n-1}$. Then

$$\begin{aligned}
\frac{X}{R}[p] \ni \sum_{i=1}^{r} m_i p^{e_i - 1} a_i &= \sum_{i=1}^{r} m_i p^{e_i - 1} \Bigg(\sum_{j=1}^{n} \underbrace{\alpha_{ij}}_{= p^{e - e_i} \beta_{ij}} \bar{x}_j \Bigg) \\
&= \sum_{j=1}^{n} p^{e-1} \Bigg(\underbrace{\sum_{i=1}^{r} m_i \beta_{ij}}_{\in p Z_{p^e} \text{ for } j \neq k} \Bigg) \bar{x}_j \\
&= p^{e-1} \Bigg(\sum_{i=1}^{r} m_i \beta_{ik} \Bigg) \bar{x}_k \in \langle p^{e-1} \bar{x}_k \rangle \\
&= \frac{p^{-e} R_{\tau_k} + R}{R} [p].
\end{aligned}$$

Therefore $p^{e-1} (\sum_{i=1}^{r} m_i \beta_{ik}) \bar{x}_k \in (X/R)[p] \cap (p^{-e} R_{\tau_k} + R)/R[p] = 0$, so $(m_1, \dots, m_r). \; (\beta_{1k}, \dots, \beta_{rk})^{\mathrm{tr}} = \sum_{i=1}^{r} m_i \beta_{ik} \in p \mathbf{Z}_{p^e}$, since $\operatorname{ord} \bar{x}_k = p^e$. Hence

$$(m_1, \dots, m_r) \cdot B \in p(\mathbf{Z}_{p^e})^n,$$

and therefore $(m_1, \dots, m_r) \in (p \mathbf{Z}_{p^e})^r$, since B has p-independent rows.

We have shown $(m_1, \dots, m_r) \cdot B^{(k)} \in p(\mathbf{Z}_{p^e})^{n-1}$ implies $(m_1, \dots, m_r) \in (p \mathbf{Z}_{p^e})^r$. For $1 \leq l \leq e$ we get recursively the implication $(n_1, \dots, n_r) \cdot B^{(k)} \in p^l (\mathbf{Z}_{p^e})^{n-1} \Rightarrow (n_1, \dots, n_r) \in (p^l \mathbf{Z}_{p^e})^r$. This shows, by Definition [**2**, 32.], the p-independence of the rows of $B^{(k)}$ in $[(\mathbf{Z}_{p^e})^{n-1}, +]$ for all $k = 1, \dots, n$.

The second statement is clear for a representing matrix in Hermite normal form. $\square$

Definition 3.5. Let R be the regulator of X such that $X/R \cong \bigoplus_{i=1}^{r} (\mathbf{Z}/p^{e_i} \mathbf{Z})$, where $e = e_1 \geq \dots \geq e_r \geq 1$. Let $B = (\beta_{ij})_{\substack{i=1,\dots,r \\ j=1,\dots,n}}$ be some matrix over $\mathbf{Z}_{p^e}$ such that

$$M = \operatorname{diag}(p^{e-e_1}, \dots, p^{e-e_r}) \cdot B$$

is the representing matrix of X over R relative to bases $\mathbf{a}$ and $\bar{\mathbf{x}}$.

Then the subset $\{\tau_{j_1}, \dots, \tau_{j_r}\}$ of T corresponding to some pivot set $\{j_1, \dots, j_r\} \subseteq \{1, \dots, n\}$ of the columns of B is called a *pivot set* of X. A pivot set of B does not depend on the bases $\mathbf{a}$ and $\bar{\mathbf{x}}$. In particular, pivot sets are invariants of X.

The n-tuple $T = (\tau_1, \dots, \tau_r, \tau_{r+1}, \dots, \tau_n)$ is said to be an *admissible indexing* of the critical typeset of X if there is a basis $\mathbf{a}'$ of X/R such that $B = (E \mid A)$ is the Hermite normal form (3.2).

4. Matrix (near-)isomorphism criterion for p-local groups.

The following result is a generalization of [**1**, Theorem 2.10].

Theorem 4.1. *Pivot sets are near-isomorphism invariants for reduced p-local rigid groups, in other words admissible indexings of the critical typeset are near-isomorphism invariants.*

Proof. Fix an admissible indexing of the group X. Let X have a representing matrix $\Lambda(E \mid A)$ in Hermite normal form (3.2), where Λ is a diagonal matrix with p-power entries and $E = (m_{i,j})_{i,j}$ is an upper triangular matrix. Let Y be nearly isomorphic to X. Let $\Lambda(E' \mid B)$ be the representing matrix of Y relative to the same indexing of the critical typeset. Use [**6**, Theorem 4.2] for this situation. Then near-isomorphism means a diagonal equivalence of the representing matrices, where the diagonal matrix D of a type-automorphism is multiplied from the right. By comparison of the left $(r \times r)$-blocks, there is a matrix P of an automorphism of $Y/R(Y)$ and an invertible diagonal matrix $D_{\leq r}$ such that $PE' = ED_{\leq r}$. Then

$$D_{\leq r}^{-1} PE' = D_{\leq r}^{-1} ED_{\leq r} = \begin{pmatrix} 1 & & d_i^{-1} m_{ij} d_j \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

is the upper triangular form (3.2) too. Thus the indexing of the columns is also admissible for Y. $\square$

Definition 4.2. Let m be a natural number and τ any type. Define

$$\mathbf{Z}_m^*(\tau) := \langle -1 + m\mathbf{Z}, q + m\mathbf{Z} \in \mathbf{Z}_m^* \mid q \text{ prime number}, \tau(q) = \infty \rangle_{\text{mult}}.$$

We investigate p-local groups with a simultaneous admissible indexing of the critical typeset:

Theorem 4.3 (Matrix [near-]isomorphism criterion for p-local groups). *Let $r < n$ be natural numbers and $e = e_1 \geq \cdots \geq e_r \geq 1$ integers. Let X and Y be p-reduced rigid groups of rank n with a common regulator R such that*

$$X/R \cong \bigoplus_{i=1}^{r} (\mathbf{Z}/p^{e_i}\mathbf{Z}) \cong Y/R.$$

Let $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ be an induced decomposition basis of $\overline{R} = p^{-e}R/R$ ordered by a simultaneous admissible indexing $T = (\tau_1, \ldots, \tau_n)$ of the critical typeset for X and Y. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a basis of X/R and $\mathbf{b} = (b_1, \ldots, b_r)$ a basis of Y/R with $\langle a_i \rangle \cong \mathbf{Z}/p^{e_i}\mathbf{Z} \cong \langle b_i \rangle$. Set $\Lambda = \operatorname{diag}(p^{e-e_1}, \ldots, p^{e-e_r})$. Let the representing matrix $M = \Lambda(A_{\leq r} \mid A_{>r})$ of X over R and the representing matrix $N = \Lambda(B_{\leq r} \mid B_{>r})$ of Y over R be in the Hermite normal form, where

$$A_{\leq r} = \begin{pmatrix} 1 & m_{12} & \cdots & m_{1r} \\ 0 & 1 & \cdots & m_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \textit{and} \quad B_{\leq r} = \begin{pmatrix} 1 & n_{12} & \cdots & n_{1r} \\ 0 & 1 & \cdots & n_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(1) *The groups X and Y are nearly isomorphic, $X \cong_{\mathrm{nr}} Y$, if and only if there is a matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_j \in \mathbf{Z}_{p^e}^*$ and an upper triangular matrix*

$$P = \begin{pmatrix} d_1^{-1} & & * \\ & \ddots & \\ 0 & & d_r^{-1} \end{pmatrix}$$

such that

$$N = PMD.$$

(2) *The groups X and Y are isomorphic, $X \cong Y$, if and only if there is a matrix $F = \operatorname{diag}(f_1, \ldots, f_n)$ with $f_j \in \mathbf{Z}_{p^e}^*(\tau_j)$ and an upper triangular matrix*

$$P = \begin{pmatrix} f_1^{-1} & & * \\ & \ddots & \\ 0 & & f_r^{-1} \end{pmatrix}$$

such that

$$N = PMF.$$

Proof. (1) "$\Leftarrow$." The invertible matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ defines a type automorphism ξ of $\overline{R}$ via $\xi\bar{x}_j := d_j\bar{x}_j$. The invertible upper triangular matrix P represents an automorphism $\Gamma : (Y/R) \to (Y/R)$ which maps the new basis elements $\xi a_i = b_i'$ to the old ones b_i. Then $N' := MD$ is the representing matrix of Y over R relative to $\mathbf{b}' := (b_1', \dots, b_r')$. We get $(Y/R) = \langle b_1', \dots, b_r'\rangle = \langle \xi a_1, \dots, \xi a_r\rangle = \xi(\langle a_1, \dots, a_r\rangle) = \xi(X/R)$. Therefore, $X \cong_{\mathrm{nr}} Y$.

"$\Rightarrow$." Assume that $X \cong_{\mathrm{nr}} Y$. Then X is type isomorphic to Y and $(X/R) \cong (Y/R)$. By [**6**, Theorem 4.2], there exist a matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ of a type-automorphism $\xi \in \operatorname{Typ}\operatorname{Aut}\overline{R}$ and a matrix P of an automorphism $\Gamma \in \operatorname{Aut}(X/R)$ such that $PMD = N$. We compare the left $(r \times r)$-blocks of this matrix equation

$$P\Lambda \underbrace{\begin{pmatrix} 1 & & m_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{=A_{\leq r}} D_{\leq r} = \Lambda \underbrace{\begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{=B_{\leq r}}. \tag{4.4}$$

All appearing matrices are invertible except Λ. Then $P\Lambda = \Lambda B_{\leq r} D_{\leq r}^{-1} A_{\leq r}^{-1}$ is an upper triangular matrix. Hence we can choose $P = (\gamma_{ij})_{\substack{1\leq i \\ j \leq r}}$ to be an upper triangular matrix, too. Note that the entry γ_{ij} is unique modulo $p^{e_j}\mathbf{Z}_{p^e}$. The coefficient (i, i) of the matrix equation (4.4) is $\gamma_{ii} \cdot p^{e-e_i} \cdot 1 \cdot d_i = p^{e-e_i} \cdot 1$. Hence we can assume that $\gamma_{ii} = d_i^{-1}$ for $P = (\gamma_{ij})_{i,j}$.

(2) Use a matrix $F = \operatorname{diag}(f_1, \dots, d_n)$, $f_j \in \mathbf{Z}_{p^e}^*(\tau_j)$, of an induced automorphism $\zeta \in \overline{\operatorname{Aut} R}$ instead of D in part (1) of this proof. All conclusions are the same. $\square$

Definition 4.5. Let p be a prime and $e, n, r \in \mathbf{N}$ natural numbers with $r < n$. Let $T = (\tau_1, \dots, \tau_n)$ be an ordered n-tuple of types. We set

$$\begin{aligned} \operatorname{DIAG}(T; \mathbf{Z}_{p^e}^*) :&= \operatorname{DIAG}(\mathbf{Z}_{p^e}^*(\tau_1), \dots, \mathbf{Z}_{p^e}^*(\tau_n)) \\ &= \{\operatorname{diag}(f_1, \dots, f_n) \mid \forall_{j=1,\dots,n} f_j \in \mathbf{Z}_{p^e}^*(\tau_j)\}, \end{aligned}$$

the set of T-diagonal matrices over $\mathbf{Z}_{p^e}$.

Let A and B be $[r \times (n-r)]$-matrices over $\mathbf{Z}_{p^e}$. Then A and B are called *T-diagonally equivalent* if there is a T-diagonal matrix $F \in \mathrm{DIAG}\,(T; \mathbf{Z}^*_{p^e})$ such that

$$B = F^{-1}_{\leq r} A F_{>r}.$$

Corollary 4.6 (Matrix [near-] isomorphism criterion for uniform groups). *Let $X, Y \in \mathcal{C}(T, p, e, r)$ be uniform groups with a common regulator R. Let $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ be an induced decomposition basis of $\overline{R} = p^{-e}R/R$ ordered by a simultaneous admissible indexing $T = (\tau_1, \ldots, \tau_n)$ of the critical typeset for X and Y. Let $M = (I_r \mid A)$ and $N = (I_r \mid B)$ be the representing matrices of X and Y over R relative to $\bar{\mathbf{x}}$ in Hermite normal form, where A and B are $[r \times (n-r)]$-matrices over $\mathbf{Z}_{p^e}$.*

(1) *The groups X and Y are nearly isomorphic, $X \cong_{\mathrm{nr}} Y$, if and only if A and B are diagonally equivalent.*

(2) *The groups X and Y are isomorphic, $X \cong Y$, if and only if A and B are T-diagonally equivalent.*

5. Isomorphism classes of uniform groups. We count isomorphism classes of groups within a fixed near-isomorphism class in $\mathcal{C}(T, p, e, r)$. A representative of this class has a representing matrix $(I_r \mid C)$ with a primitive rest block $C \in \mathbf{M}^{r \times (n-r)}(\mathbf{Z}_{p^e})$.

Theorem 5.1. *Let $R = \bigoplus_{j=1}^n R_{\tau_j}$ be a rigid and p-reduced completely decomposable group of rank n with an indexing $T = (\tau_1, \ldots, \tau_n)$ of its critical typeset. Let $C \in \mathbf{M}^{r \times (n-r)}(\mathbf{Z}_{p^e})$ be a primitive matrix. Let* $\mathrm{Stab}_{\cong_{\mathrm{nr}}}(C) = \{D = \mathrm{diag}\,(d_1, \ldots, d_n) \mid d_j \in \mathbf{Z}^*_{p^e}, D^{-1}_{\leq r} C D_{>r} = C\}$ *and* $\mathrm{Stab}_{\cong}(C) = \{F = \mathrm{diag}\,(f_1, \ldots, f_n) \mid f_j \in \mathbf{Z}^*_{p^e}(\tau_j), F^{-1}_{\leq r} C F_{>r} = C\}$.

Each near-isomorphism class is the union of isomorphism classes all of equal length. The number of distinct isomorphism classes contained in the near-isomorphism class of C and with regulator R is

$$\frac{(p^{e-1}(p-1))^n}{\prod_{j=1}^n |\mathbf{Z}^*_{p^e}(\tau_j)| \cdot [\mathrm{Stab}_{\cong_{\mathrm{nr}}}(C) : \mathrm{Stab}_{\cong}(C)]} = \frac{(p^{e-1}(p-1))^n}{|\mathrm{DIAG}\,(T; \mathbf{Z}^*_{p^e}) \cdot \mathrm{Stab}_{\cong_{\mathrm{nr}}}(C)|}.$$

Proof. Here $\mathrm{DIAG}\,(n;\mathbf{Z}_{p^e}^*)$ acts on $\mathbf{M}^{r\times(n-r)}$ via diagonal equivalence. The stabilizer of A under this action is $\mathrm{Stab}_{\cong_{\mathrm{nr}}}(A)$ and the orbit of A is $\mathrm{Orb}_{\cong_{\mathrm{nr}}}(A) = \{D_{\le r}^{-1}AD_{>r} \mid D \in \mathrm{DIAG}\,(n;\mathbf{Z}_{p^e}^*)\}$. In addition $\mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*)$ acts on $\mathbf{M}^{r\times(n-r)}$ via diagonal equivalence, too. The stabilizer of A under this action is $\mathrm{Stab}_{\cong}(A)$ and the orbit of A is $\mathrm{Orb}_{\cong}(A) = \{F_{\le r}^{-1}AF_{>r} \mid F \in \mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*)\}$.

Firstly, we show that the isomorphism classes of near-isomorphic groups have equal length. For that, let C' be diagonally equivalent to C, i.e., $C' = D_{\le r}^{-1}CD_{>r}$, where $D \in \mathrm{DIAG}\,(n;\mathbf{Z}_{p^e}^*)$. Then

$$\begin{aligned}
F \in \mathrm{Stab}_{\cong}(C) \Leftrightarrow F_{\le r}^{-1}CF_{>r} &= C \Leftrightarrow D_{\le r}^{-1}(F_{\le r}^{-1}CF_{>r})D_{>r}\\
&= D_{\le r}^{-1}CD_{>r} \Leftrightarrow F_{\le r}^{-1}\underbrace{(D_{\le r}^{-1}CD_{>r})}_{C'}F_{>r}\\
&= \underbrace{D_{\le r}^{-1}CD_{>r}}_{C'} \Leftrightarrow F \in \mathrm{Stab}_{\cong}(C').
\end{aligned}$$

Hence $\mathrm{Stab}_{\cong}(C) = \mathrm{Stab}_{\cong}(C')$ and therefore $|\mathrm{Orb}_{\cong}(C)| = |\mathrm{Orb}_{\cong}(C')|$.

The number of groups contained in the near-isomorphism class of C is $|\mathrm{Orb}_{\cong_{\mathrm{nr}}}(C)| = [\mathrm{DIAG}\,(n;\mathbf{Z}_{p^e}^*) : \mathrm{Stab}_{\cong_{\mathrm{nr}}}(C)]$. Similarly $|\mathrm{Orb}_{\cong}(C)| = [\mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*) : \mathrm{Stab}_{\cong}(C)]$. We compute

$$\begin{aligned}
|\{\mathrm{Orb}_{\cong}(A) \mid A \in \mathrm{Orb}(C)\}| &= \frac{|\mathrm{Orb}_{\cong_{\mathrm{nr}}}(C)|}{|\mathrm{Orb}_{\cong}(C)|}\\
&= \frac{(p^{e-1}(p-1))^n}{\prod_{j=1}^n |\mathbf{Z}_{p^e}^*(\tau_j)| \cdot [\mathrm{Stab}_{\cong_{\mathrm{nr}}}(C) : \mathrm{Stab}_{\cong}(C)]}.
\end{aligned}$$

We have $\mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*) \cap \mathrm{Stab}_{\cong_{\mathrm{nr}}}(C) = \mathrm{Stab}_{\cong}(C)$. Therefore the denominator simplifies:

$$\begin{aligned}
&|\mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*)| \cdot [\mathrm{Stab}_{\cong_{\mathrm{nr}}}(C) : \mathrm{Stab}_{\cong}(C)]\\
&\qquad = \frac{|\mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*)| \cdot |\mathrm{Stab}_{\cong_{\mathrm{nr}}}(C)|}{|\mathrm{Stab}_{\cong}(C)|}\\
&\qquad = |\mathrm{DIAG}\,(T;\mathbf{Z}_{p^e}^*) \cdot \mathrm{Stab}_{\cong_{\mathrm{nr}}}(C)|,
\end{aligned}$$

and the claim follows. ∎

Corollary 5.2. *There are at most*

$$\prod_{j=1}^{n}[\mathbf{Z}_{p^e}^* : \mathbf{Z}_{p^e}^*(\tau_j)] = \frac{(p^{e-1}(p-1))^n}{\prod_{j=1}^{n}|\mathbf{Z}_{p^e}^*(\tau_j)|}$$

pairwise nonisomorphic groups within the near-isomorphism class of the primitive matrix $C \in \mathbf{M}^{r\times(n-r)}(\mathbf{Z}_{p^e})$.

Proof. Since $[\mathrm{Stab}_{\cong_{\mathrm{nr}}}(C) : \mathrm{Stab}_{\cong}(C)] \geq 1$, Theorem 5.1 shows the claim. □

Remark 5.3. Let us investigate uniform groups of even rank $n = 2r$. Then a fixed near-isomorphism class is represented by a modified diagonal similarity class of normed $(r \times r)$-matrices, cf. [**7**, Theorem 4.3]. These are the rest blocks of the representing matrices.

There are at least

$$\prod_{m=1}^{r}[\mathbf{Z}_{p^e}^* : \mathbf{Z}_{p^e}^*(\tau_m \vee \tau_{r+m})] = \frac{(p^{e-1}(p-1))^r}{\prod_{m=1}^{r}|\mathbf{Z}_{p^e}^*(\tau_m \vee \tau_{r+m})|}$$

pairwise nonisomorphic groups contained in the near-isomorphism class relative to a normed and invertible $(r \times r)$-matrix. This is shown in [**8**, Theorem 9.8].

Example 5.4. Let $\tau_1 = \mathbf{Z}[3^{-1}] = \{\frac{n}{3^k} \mid n \in \mathbf{Z}, k \in \mathbf{N}_0\}$ and $\tau_2 = \mathbf{Z}[5^{-1}]$. Then $R := \tau_1 x_1 \oplus \tau_2 x_2$ is 17-reduced.

Consider the almost completely decomposable group

$$Z = R + \mathbf{Z}\frac{1}{17}(x_1 + x_2)$$

with corresponding representing matrix $M = (1 \mid 1)$.

We compute $3^8 \equiv -1 \pmod{17}$ and $5^8 \equiv -1 \pmod{17}$. Then we obtain $\mathbf{Z}_{17}^*(\tau_1) = \mathbf{Z}_{17}^*(\tau_2) = \mathbf{Z}_{17}^* \cong \mathbf{Z}_{16}$, since $\mathrm{ord}\,(3 + 17\mathbf{Z}) = 16 = \mathrm{ord}\,(5 + 17\mathbf{Z})$. Hence $\mathbf{Z}_{17}^*(\tau_1 \vee \tau_2) = \mathbf{Z}_{17}^*$ and therefore the formulas of 5.2 and 5.3 simplify:

$$\frac{16^2}{\prod_{j=1}^{2}|\mathbf{Z}_{17}^*(\tau_j)|} = \frac{16^2}{16^2} = 1 \quad \text{and} \quad \frac{16^1}{\prod_{m=1}^{1}|\mathbf{Z}_{17}^*(\tau_m \vee \tau_{1+m})|} = 1.$$

This means that the upper and lower bounds of 5.2 and 5.3 are sharp. All groups in the near-isomorphism class of Z are isomorphic.

Corollary 5.5. *Let $R = \tau_1 x_1 \oplus \cdots \oplus \tau_4 x_4$ be a rigid and p-reduced completely decomposable group with critical typeset $T = (\tau_1, \ldots, \tau_4)$. Let $X \in \mathcal{C}(T; p, e, 2)$ be an almost completely decomposable group with regulator R such that $X/R = (\mathbf{Z}_{p^e})^2$. Let $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_4)$ be the induced decomposition basis of $\overline{R} = p^{-e}R/R$ and $\mathbf{a}$ be an ordered basis of X/R. Let the representing matrix M of X over R relative to $\bar{\mathbf{x}}$ and $\mathbf{a}$ be in Hermite normed form with invertible rest block*

$$M = (I_2 \mid A) = \left(\begin{array}{cc|cc} 1 & 0 & 1 & \alpha \\ 0 & 1 & \beta & 1+\alpha\beta \end{array}\right),$$

where $\alpha = \lambda p^m$, $\beta = \mu p^l$ for some units λ, μ and some integers $0 \leq m,\ l \leq e$. Recall that $\mathrm{Stab}_{\cong}(A) = \{F = \mathrm{diag}\,(f_1, \ldots, f_4) \mid f_j \in \mathbf{Z}^*_{p^e}(\tau_j), F^{-1}_{\leq 2} A F_{>2} = A\}$ denotes the stabilizer of A relative to the T-diagonal equivalence.

Let $A \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\begin{aligned}\mathrm{Stab}_{\cong}(A) = \{&\mathrm{diag}\,(f_1, f_2, f_1, f_2) \mid f_1 \in \mathbf{Z}^*_{p^e}(\tau_1) \cap \mathbf{Z}^*_{p^e}(\tau_3),\\ &f_2 \in \mathbf{Z}^*_{p^e}(\tau_2) \cap \mathbf{Z}^*_{p^e}(\tau_r),\ \text{such that } f_2 - f_1 \in p^{e-\min(m,l)} \cdot \mathbf{Z}_{p^e}\}.\end{aligned}$$

The number of distinct isomorphism classes contained in the near-isomorphism class of X is

$$N = \frac{\varphi(p^e)^3 p^{-\min(m,l)} \cdot |\mathrm{Stab}_{\cong}(A)|}{\prod_{j=1}^{4} |\mathbf{Z}^*_{p^e}(\tau_j)|}.$$

If α or β is a unit in $\mathbf{Z}_{p^e}$, then

$$N = \frac{(p^{e-1}(p-1))^3 \cdot |\bigcap_{j=1}^{4} \mathbf{Z}^*_{p^e}(\tau_j)|}{\prod_{j=1}^{4} |\mathbf{Z}^*_{p^e}(\tau_j)|}.$$

Proof. Let $\mathrm{Stab}_{\cong_{\mathrm{nr}}}(A) = \{D = \mathrm{diag}\,(d_1, \ldots, d_4) \mid d_j \in \mathbf{Z}^*_{p^e}, D^{-1}_{\leq 2} \times A D_{>2} = A\}$ denote the stabilizer of A relative to arbitrary diagonal equivalence. Let $\mathrm{Orb}_{\cong}(A) = \{F_{\leq 2}^{-1} A F_{>2} \mid F \in \mathrm{DIAG}\,(T; \mathbf{Z}_{p^e}^*)\}$ denote the T-diagonal equivalence class of A. Recall from Example 2.6 that $\mathrm{Stab}_{\cong_{\mathrm{nr}}}(A) = \{D = \mathrm{diag}\,(d_1, d_2, d_1, d_2) \mid d_1, d_2 \in \mathbf{Z}_{p^e}^*$ such that $d_2 - d_1 \in p^{e-\min(m,l)}\mathbf{Z}_{p^e}\}$ and $|\mathrm{Orb}_{\cong_{\mathrm{nr}}}(A)| = \varphi(p^e)^3 p^{-\min(m,l)} = p^{3e-3-\min(l,m)}(p-1)^3$. We compute $\mathrm{Stab}_{\cong}(A) = \mathrm{DIAG}\,(T; \mathbf{Z}_{p^e}^*) \cap \mathrm{Stab}_{\cong_{\mathrm{nr}}}(A) = \{\mathrm{diag}\,(f_1, f_2, f_3, f_4) \mid f_j \in \mathbf{Z}_{p^e}^*,\ f_1 = f_3,\ f_2 = f_4$ such that $f_2 - f_1 \in p^{e-\min(m,l)}\mathbf{Z}_{p^e}\}$, and we get the claim. Clearly we have $|\mathrm{Orb}_{\cong}(A)| = [\mathrm{DIAG}\,(T; \mathbf{Z}_{p^e}^*) : \mathrm{Stab}_{\cong}(A)] = (\Pi_{j=1}^4 |\mathbf{Z}_{p^e}^*(\tau_j)|)/\mathrm{Stab}_{\cong}(A)$. By Theorem 5.1 the number of isomorphism classes within the near-isomorphism class of X is

$$N = \frac{|\mathrm{Orb}_{\cong_{nr}}(A)|}{|\mathrm{Orb}_{\cong}(A)|} = \frac{\varphi(p^e)^3 p^{-\min(m,l)} \cdot |\mathrm{Stab}_{\cong}(A)|}{\prod_{j=1}^4 |\mathbf{Z}_{p^e}^*(\tau_j)|}.$$

If α or β is a unit in $\mathbf{Z}_{p^e}$, then $\min(m,l) = 0$ and we compute $\mathrm{Stab}_{\cong}(A) = \{\mathrm{diag}\,(f,f,f,f) \mid f \in \cap_{j=1}^4 \mathbf{Z}_{p^e}^*(\tau_j)\}$. This proves the last statement. □

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