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ANOTHER OBSERVATION ON THE DISTRIBUTION OF VALUES OF CONTINUED FRACTIONS $K(a_n/1)$

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Offered in celebration of the 70th birthday of William B. Jones
Dedicated to the memory of Wolfgang J. Thron
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ABSTRACT. Jacobsen, Thron and Waadeland [**1**] determined results about the probability distribution for the values f of convergent continued fractions $K(a_n/1)$ in the case that the elements a_n are uniformly distributed both over the "real Worpitzky inverval" $[-\rho(1-\rho), \rho(1-\rho)]$ and over the complex Worpitzky disk $\{z : |z| \leq \rho(1-\rho)\}$, for $0 < \rho \leq \frac{1}{2}$. This note explores extensions of some of those results in the case that the elements a_n are piecewise linearly distributed, with symmetry about zero, on the real Worpitzky interval.

1. Introduction. We are considering the values of continued fractions of the form

$$K_{n=1}^{\infty}\left(\frac{a_n}{1}\right) = \frac{a_1}{1+}\;\frac{a_2}{1+}\;+\cdots+\;\frac{a_n}{1+}\cdots \tag{1.1}$$

where the $a_n \neq 0$ are taken from the real Worpitsky interval

$$W = [-\rho(1-\rho),\ \rho(1-\rho)] \tag{1.2}$$

for $0 < \rho \leq \frac{1}{4}$. This interval is known to be a convergence region for (1.1). The values, f, of (1.1) are known to fill in the best limit value interval

$$V = [-\rho,\ \rho]. \tag{1.3}$$

Here, however, we assume that the a_n have a known distribution on (1.2); the problem is to determine the distribution of the values,

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f, of (1.1) in (1.3). This issue was explored by Jacobsen, Thron and Waadeland in Theorem 1 of [**1**] and by Waadeland in [**2**] in the case that the a_n are uniformly distributed on (1.2). In [**1**] they determined that the probability density for f, μ, is the constant $\frac{1}{2\rho(1-\rho)}$ over the subinterval $\left[-\frac{\rho(1-\rho)}{(1+\rho)}, \frac{\rho(1-\rho)}{(1+\rho)}\right]$ of (1.3). This yields the result that

$$P\left(|f| \leq \frac{\rho(1-\rho)}{1+\rho}\right) = \frac{1}{1+\rho}$$

and, by symmetry and taking complements, that on (1.3),

$$P\left(-\rho \leq f < -\frac{\rho(1-\rho)}{(1+\rho)}\right) = P\left(\frac{\rho(1-\rho)}{(1+\rho)} < f \leq \rho\right) = \frac{\rho}{2(1+\rho)}.$$

In this note we extend those results in the case that the a_n have a symmetric distribution about zero on (1.2) which is piecewise linear away from zero.

2. The extension. To extend the distribution on a to a linear distribution symmetric about zero, we define the probability density function $D(a)$ by

$$D(-a) = D(a) \quad \text{(symmetry about zero)} \tag{2.1}$$

$$D(a) = b_0 + b_1 a, \tag{2.2}$$

for $0 < a \leq \rho(1-\rho)$, $0 < \rho \leq \frac{1}{2}$, which must satisfy

$$D(a) \geq 0, \tag{2.3}$$

$$\int_{-\rho(1-\rho)}^{\rho(1-\rho)} D(a)\, da = 1, \tag{2.4}$$

and

$$E(a) = \int_{-\rho(1-\rho)}^{\rho(1-\rho)} aD(a)\, da = 0 \tag{2.5}$$

from symmetry about zero, where $E(a)$ denotes the expected value of a.

Condition (2.4) holds if and only if

$$b_0 = \frac{1 - b_1\rho^2(1-\rho)^2}{2\rho(1-\rho)}.$$

Condition (2.3) holds, of course, if and only if

$$D(0) \geq 0 \text{ and } D(\rho(1-\rho)) \geq 0$$

both of which hold if and only if

$$b_0 \geq 0 \quad \text{and} \quad |b_1| \leq \frac{1}{\rho^2(1-\rho)^2}. \tag{2.6}$$

Now we extend to the full interval (1.2) by symmetry to get our preliminary

Theorem 1. *The linear density function for the elements* $a \neq 0$ *on* $[-\rho(1-\rho), \rho(1-\rho)]$ *for the continued fraction* (1.1) *is of the form*

$$D(a) = \frac{1 - b_1\rho^2(1-\rho)^2}{2\rho(1-\rho)} + b_1|a| \tag{2.7}$$

for $|b_1| \leq \frac{1}{\rho^2(1-\rho)^2}$.

3. Special cases. We consider three special cases of linear density functions on (1.2).

1. In the case that $b_1 = 0$, we recover the uniform distribution

$$D(a) = b_0 = \frac{1}{2\rho(1-\rho)} \tag{3.1}$$

dealt with in Theorem 1 of [**1**]. When $\rho = \frac{1}{2}$ we get $D(a) = 2$ on $[-\frac{1}{4}, \frac{1}{4}]$.

2. In the case that $D(0) = b_0 = 0$, a more natural case since we assume that $a \neq 0$, we obtain

$$D(a) = \frac{1}{\rho^2(1-\rho)^2}\,|a| \tag{3.2}$$

which linearly increases the density for a towards the boundaries of the interval. When $\rho = \frac{1}{2}$ we get $D(a) = 16|a|$ on $[-\frac{1}{4}, \frac{1}{4}]$.

3. In the case that $D(\rho(1-\rho)) = 0$, the density is

$$D(a) = \frac{1}{\rho(1-\rho)} + \frac{1}{\rho^2(1-\rho)^2}\,|a| \tag{3.3}$$

which linearly increases the probability of a near zero. When $\rho = \frac{1}{2}$ we get $D(a) = 4 - 16|a|$ on $[-\frac{1}{4}, \frac{1}{4}]$.

4. The technique. To find the probability density function $h(v)$ for values f of the continued fraction (1.1), we follow [**1**] and [**2**] in using the cumulative distribution function technique. We let μ be the probability measure for the continued fraction values, f. Then $d\mu(t) = h(t)\,dt$. We determine $h(v)$ by

$$h(v) = \frac{dH(v)}{dv}$$

where

$$H(v) = P(f \le v) = \int_{-\infty}^{v} d\mu(t)$$

with

$$H(v) = \begin{cases} 0, & v \le -\rho; \\ 1, & v \ge \rho. \end{cases}$$

For fixed a in (1.2), f and the first tail value, g, where

$$f = \frac{a}{1+g} \tag{4.1}$$

we have

$$H(v) = P(f \le v) = P\left(\frac{a}{1+g} \le v\right). \tag{4.2}$$

That the distribution of f must be symmetric about zero follows from the symmetry of $D(a)$ about zero. It can also be seen as follows. The distribution of g must be the same as the distribution of f. Since $|g| \le \rho \le \frac{1}{2}$, clearly $\frac{1}{1+g} > 0$. When $-a$ is substituted for a, we just get $-f$. Therefore, for $0 < v_1 < v_2 \leq \rho(1-\rho)$, we have $P(v_1 \leq f \leq v_2) = P(-v_2 \leq f \leq -v_1)$ and consequently $P(f < 0) = P(f > 0)$.

As a result, in determining $H(v)$ we may assume, without loss of generality, that $a > 0$ (and hence $v > 0$). Now, since

$$P\left(\frac{a}{1+g} \leq v\right) = \iint_{\{(a,g)|a/(1+g)\leq v\}} D(a)\, da\, d\mu(g) \tag{4.3}$$

for $(a, g) \in [-\rho(1-\rho), \rho(1-\rho)] \times [-\rho, \rho]$ we must determine when

$$\frac{a}{1+g} \leq v.$$

We distinguish between two cases.

Case (α). $0 < v \leq \frac{\rho(1-\rho)}{1+\rho}$. In this case $0 < \frac{a}{1+g} \leq v$ if and only if $0 < a \leq v(1+g)$ which is always the case since, with $g \leq \rho$,

$$v(1+g) \leq \frac{\rho(1-\rho)}{1+\rho}(1+\rho) = \rho(1-\rho)$$

is always within the range of a. The bounding equation

$$a = v(1+g) \tag{4.4}$$

or

$$g = \frac{a}{v} - 1 \tag{4.5}$$

determines a level set of v for $(a, g) \in [-\rho(1-\rho), \rho(1-\rho)] \times [-\rho, \rho]$. This linear level set of v with the g intercept of -1 in the (α)-case can be seen in Figure 1 to run through the entire $[-\rho(1-\rho), \rho(1-\rho)] \times [-\rho, \rho]$ box entering at $(a, g) = (v(1-\rho), -\rho)$ and exiting the top at $(a, g) = (v(\rho+1), \rho)$.

Case (β). $\frac{\rho(1-\rho)}{(1+\rho)} < v \leq \rho$. In this case we cannot reach v for every choice of g since we may have

$$\frac{\rho(1-\rho)}{(1+g)} < v$$

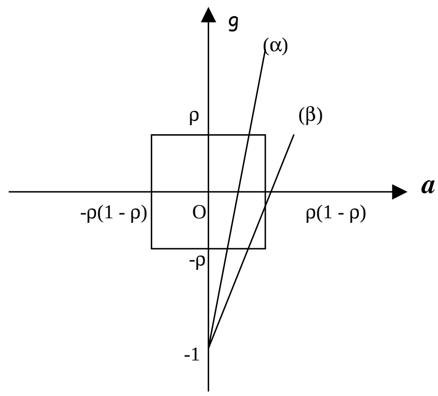

FIGURE 1.

for some $g < \rho$. This means that $0 < \frac{a}{(1+g)} \leq v$ if and only if either

(i) $g \leq \dfrac{\rho(1-\rho)}{v} - 1$ and $0 < a \leq v(1+g)$, or

(ii) $g > \dfrac{\rho(1-\rho)}{v} - 1$ and $0 < a \leq \rho(1+\rho)$.

In this, the (β)-case, the linear level set of v can be seen in Figure 1 to cut through the side of the $[-\rho(1-\rho),\, \rho(1-\rho)] \times [-\rho,\, \rho]$ box where $(a, g) = \big(\rho(1-\rho),\, \frac{\rho(1-\rho)}{v} - 1\big)$.

We now begin extending the results in [**1**] in the (α)-case. We will not deal with the (β)-case in this note. In the (α)-case with $0 < v \leq \frac{\rho(1-\rho)}{(1+\rho)}$, we have

(4.6)
$$H(v) = 1 - \int_{-\rho}^{+\rho} \left(\int_{v(g+1)}^{\rho(1-\rho)} D(a)\, da \right) d\mu(g)$$

(4.7)
$$= 1 - \int_{-\rho}^{+\rho} \left(\int_{v(g+1)}^{\rho(1-\rho)} (b_0 + b_1 a)\, da \right) d\mu(g)$$

(4.8)
$$= 1 - \int_{-\rho}^{+\rho} \Big\{ b_0[\rho(1-\rho) - v(g+1)] + \frac{b_1}{2}[\rho^2(1-\rho)^2 - v^2(g+1)^2] \Big\} \, d\mu(g),$$

$$= 1 - \int_{-\rho}^{+\rho} \Big\{ b_0[\rho(1-\rho) - v] + \frac{b_1}{2}[\rho^2(1-\rho)^2 - v^2] \Big\} \, d\mu(g)$$

(4.9)
$$- \int_{-\rho}^{+\rho} \Big\{ b_0(-vg) + \frac{b_1}{2}[-v^2(g^2+2g)] \Big\} \, d\mu(g).$$

Since only the variable g remains in this integration process, we obtain

(4.10)
$$H(v) = 1 - \Big\{ b_0[\rho(1-\rho) - v] + \frac{b_1}{2}\,[\rho^2(1-\rho)^2 - v^2] \Big\} \int_{-\rho}^{+\rho} d\mu(g) + [b_0 v + b_1 v^2] \int_{-\rho}^{+\rho} g \, d\mu(g) + \frac{b_1}{2}\, v^2 \int_{-\rho}^{\rho} g^2 \, d\mu(g).$$

Recalling that $\int_{-\rho}^{+\rho} d\mu(g) = 1$ and, with μ being symmetric with respect to zero, that

$$\int_{-\rho}^{+\rho} g \, d\mu(g) = E(g) = 0,$$

it follows that, for $0 < v \le \rho\frac{(1-\rho)}{(1+\rho)}$:

(4.11)
$$H(v) = 1 - b_0[\rho(1-\rho) - v] - \frac{b_1}{2}[\rho^2(1-\rho)^2 - v^2] + \frac{b_1}{2}\, E(g^2) v^2$$

(4.12)
$$= \Big[1 - b_0\rho(1-\rho) - \frac{b_1}{2}\rho^2(1-\rho)^2\Big] + b_0 v + \frac{b_1}{2}(1 + E(g^2))v^2.$$

Consequently, for $0 \le v \le \rho\frac{(1-\rho)}{(1+\rho)}$, we obtain $h(v)$ by

(4.13)
$$h(v) = \frac{dH(v)}{dv} = b_0 + b_1(1 + E(g^2))v, = \frac{1 - b_1\rho^2(1-\rho)^2}{2\rho(1-\rho)} + b_1(1 + E(g^2))v.$$

We extend $h(v)$ to $-\rho\frac{(1-\rho)}{(1+\rho)} \le v \le \rho\frac{(1-\rho)}{(1+\rho)}$ by symmetry in defining $h(-v) = h(v)$ for $v > 0$. Thus we get our main:

Theorem 2. *For elements a_n in the continued fraction* (1.1) *with distribution* (2.7) *on the interval* (1.2), *the probability measure, μ, for values f of the continued fraction have the piecewise linear probability density function*

$$\frac{d\mu(t)}{dt} = h(t) = \frac{1 - b_1\rho^2(1-\rho)^2}{2\rho(1-\rho)} + b_1(1 + E(g^2))\,|t| \tag{4.14}$$

on $\left[-\,\rho\,\frac{(1-\rho)}{(1+\rho)},\, \rho\,\frac{(1-\rho)}{(1+\rho)}\right]$ *for* $|b_1| \le \frac{1}{\rho^2(1-\rho)^2}$.

Remark 1. While this result is somewhat disappointing because $h(v)$ depends on $E(g^2)$, we do know from this that $h(v)$ is piecewise linear with a slope greater than $|b_1|$ in absolute value. This may, then, be considered as a qualitative result.

5. Special cases. On the subinterval $\left[-\,\rho\,\frac{(1-\rho)}{(1+\rho)},\, \rho\,\frac{(1-\rho)}{(1+\rho)}\right]$ of (1.3), we get

1. In the case of [**1**] under the condition in (3.1) we recover their Theorem 1:

$$h(v) = b_0 = \frac{1}{2\rho(1-\rho)}. \tag{5.1}$$

In the case that $\rho = \frac{1}{2}$, we have $h(v) = 2$ on $[-\frac{1}{6}, \frac{1}{6}]$.

2. In the case of (3.2) we get

$$h(v) = \left(\frac{1 + E(g^2)}{\rho^2(1-\rho)^2}\right)|v|. \tag{5.2}$$

In the case $\rho = \frac{1}{2}$, we get $h(v) = 16(1 + E(g^2))|v|$ on $[-\frac{1}{6}, \frac{1}{6}]$.

3. In the case of (3.3), we obtain

$$h(v) = \frac{1}{\rho(1-\rho)} - \left(\frac{1 + E(g^2)}{\rho^2(1-\rho)^2}\right)|v|. \tag{5.3}$$

In case $\rho = \frac{1}{2}$, we get $h(v) = 4 - 16(1 + E(g^2))|v|$ on $[-\frac{1}{6}, \frac{1}{6}]$.

6. Approximate bounds on $E(g^2)$. Since $E(g^2) = \int_{-\rho}^{+\rho} v^2 h(v)\, dv$ and $h(v)$ itself depends on $E(g^2)$, while we do not know $E(g^2)$ exactly, we can get an idea of its size compared with the range, $R = 2\rho$, of g and then use the Empirical Rule and Chebyshev's Theorem. Since $\mathrm{Var}\,(g) = \sigma_g^2 = E(g^2) - E(g)^2$ and $E(g) = 0$, we have $\mathrm{Var}\,(g) = \sigma_g^2 = E(g^2)$.

The Empirical Rule derives from the normal distribution, but as an approximating tool, does remarkably well for most distributions. The Empirical Rule states in part that the probability the random variable values lie within three standard deviations of its expected value is approximately 99%. But since $E(g) = 0$, we have $P(-3\sigma_g < g < 3\sigma_g) \approx 0.99$. If we estimate the range of g, $R = 2\rho$, with that interval we obtain the approximation $R \approx 6\sigma_g$ or $\sigma_g \approx \frac{R}{6}$. Applying this, with $R = 2\rho$, we obtain

$$E(g^2) = \sigma_g^2 \approx \left(\frac{\rho}{3}\right)^2 = \frac{\rho^2}{9}. \tag{6.1}$$

On the other hand, Chebyshev's Theorem applies to any distribution for X, having a mean, μ, and standard deviation, σ. It states that

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

It states, then, that the probability of a variable being within $k = 10$ standard deviations of its mean is at least $1 - \frac{1}{10^2} = 0.99$. If we estimate the range with that interval of 20 standard deviations, we find

$$E(g^2) = \sigma_g^2 \approx \left(\frac{R}{20}\right)^2 = \left(\frac{2\rho}{20}\right)^2 = \frac{\rho^2}{100}. \tag{6.2}$$

As a result, upon combining (6.1) and (6.2) we obtain the very rough estimates that

$$E(g^2) \gtrsim \left\{\frac{\rho^2}{9}, \frac{\rho^2}{100}\right\}. \tag{6.3}$$

In the case that $\rho = \frac{1}{2}$, this gives rough lower bounds of

$$E(g^2) \gtrsim \{0.0277, 0.0025\}. \tag{6.4}$$

On the other hand, we know that, at worst, $\sigma_g \le \frac{R}{2} = \rho$ so that

$$E(g^2) \le \rho^2 = \frac{1}{4} = 0.25 \tag{6.5}$$

when $\rho = \frac{1}{2}$.

As a result, we expect that

$$\{0.0277, 0.0025\} \lesssim E(g^2) \le 0.25. \tag{6.6}$$

7. Numerical examples. We now study the special cases (3.2) and (3.3), with $\rho = \frac{1}{2}$, by computing the fourth approximates

$$\frac{a_1}{1+}\,\frac{a_2}{1+}\,\frac{a_3}{1+}\,\frac{a_4}{1}. \tag{7.1}$$

For $\rho = \frac{1}{2}$, the element interval is

$$W = [-\rho(1-\rho),\, \rho(1-\rho)] = \left[-\frac{1}{4},\, \frac{1}{4}\right]$$

while the value interval is

$$V = [-\rho,\, \rho] = \left[-\frac{1}{2},\, \frac{1}{2}\right].$$

Since $\rho\frac{(1-\rho)}{(1+\rho)} = \frac{1}{6}$, the value interval divides into the mid-interval $[-\frac{1}{6}, \frac{1}{6}]$ for the (α)-case over which we have characterized $h(v)$ and the tails $[-\frac{1}{2}, -\frac{1}{6})$ and $(\frac{1}{6}, \frac{1}{2}]$ corresponding to the (β)-case which we have not examined in this paper.

Special case (3.2). We used a systematic draw of the a_n from the distributions by dividing the element interval $[-\frac{1}{4}, \frac{1}{4}]$ into 12 equal parts and listed the midpoint values of the intervals on $D(a) = 16|a|$ in their corresponding relative frequencies. This gave a total listing of 72 values counting negative entries, positive entries and repeats. We then computed all of the possible 72^4 combinations of assignment to each of the four a_n values in the fourth approximants, (7.1).

Table 1 shows the relative frequencies of the computed values (7.1) in cells of width $\frac{1}{18}$. The interval cell frequencies are rounded to *thousands*, while the *totals* are not.

Remark 2. We can compare these results with our findings. First notice the symmetry about zero. Notice also that the relative frequencies in intervals $[0, \frac{1}{18}]$, $(\frac{1}{18}, \frac{2}{18})$ and $(\frac{2}{18}, \frac{1}{6})$ indicate linearity. The relative frequency over those three intervals is 0.2324. We can compare that with areas under $h(v) = 16(1+E(g^2))v$ over $[0, \frac{1}{6}]$ with different choices of $E(g^2)$. If we set $E(g^2) = 0$, the area under $h(v) = 16v$ over $[0, \frac{1}{6}]$ is 0.2222, just under the numerical model's relative frequency. If we use our upper bound of $E(g^2) = 0.25$, the area is 0.2778. Choosing the lower bound $E(g^2) = 0.0277$ gives an area of 0.2284, somewhat closer. These results suggest that the estimated bounds in (6.6) may be good.

Special case (3.3). Using the same method of systematic draw as in special case (3.2), we again studied the relative frequencies of values (7.1) in the total of 72^4 different combinations. See Table 2.

Recall that the interval cell frequencies are rounded to *thousands*, while the *totals* are not.

Remark 3. Again notice the symmetry about zero of the numerical model's relative frequencies. Notice again also that the relative frequencies in intervals $[0, \frac{1}{18}]$, $[\frac{1}{18}, \frac{2}{18})$ and $[\frac{2}{18}, \frac{1}{6})$ suggest linearity. The relative frequency over those three intervals in this case is 0.4409. We can compare that with various areas under $h(v) = 4 - 16(1 + E(g^2))v$ over $[0, \frac{1}{6}]$ again with different choices of $E(g^2)$. If we set $E(g^2) = 0$, the area under $h(v) = 4 - 16v$ over $[0, \frac{1}{6}]$ is 0.4442, just over the numerical model's relative frequency, suggesting again that $E(g^2) > 0$. If we again use our upper bound of $E(g^2) = 0.25$, the area is a bit low at 0.3892. Choosing the lower bound of $E(g^2) = 0.0277$ gives an area of 0.4383, again somewhat closer. These results again suggest that the estimated bounds in (6.6) may be reasonable. In fact they suggest that

TABLE 1.

	$[-\frac{1}{2},-\frac{1}{6})$	$[-\frac{1}{6},-\frac{2}{18})$	$[-\frac{2}{18},-\frac{1}{18})$	$[-\frac{1}{18},0)$	$[0,\frac{1}{18})$	$[\frac{1}{18},\frac{2}{18})$	$[\frac{2}{18},\frac{1}{6})$	$[\frac{1}{6},\frac{1}{2}]$	Total
n	7190	3733	1729	785	785	1729	3733	7190	72^4
$\frac{n}{72^4}$	0.1187	0.1389	0.0643	0.0292	0.0292	0.0643	0.1389	0.1187	1.0000

TABLE 2.

	$[-\frac{1}{2},-\frac{1}{6})$	$[-\frac{1}{6},-\frac{2}{18})$	$[-\frac{2}{18},-\frac{1}{18})$	$[-\frac{1}{18},0)$	$[0,\frac{1}{18})$	$[\frac{1}{18},\frac{2}{18})$	$[\frac{2}{18},\frac{1}{6})$	$[\frac{1}{6},\frac{1}{2}]$	Total
n	1588	2516	4803	4531	4531	4803	2516	1588	72^4
$\frac{n}{72^4}$	0.0591	0.0936	0.1787	0.1686	0.1686	0.1787	0.0936	0.0591	1.0000

the estimate derived from the Empirical Rule perhaps serves better than that offered by the more conservative Chebyshev's Theorem.

Finally notice, however, that in this second case the middle relative frequencies indicate something of a dip at the top, rather than the expected linear peak. This anomaly, however, seems to disappear upon taking cells of width $\frac{1}{24}$ rather than $\frac{1}{18}$ as we have done before. Because of the symmetry, we show this in Table 3 only over the positive interval $[0, \frac{1}{2})$ because of limitations of space.

Again recall that the interval cell frequencies are rounded to *thousands*, while the *totals* are not.

TABLE 3.

	$[0, \frac{1}{24})$	$[\frac{1}{24}, \frac{1}{12})$	$[\frac{1}{12}, \frac{3}{24})$	$[\frac{3}{24}, \frac{1}{6})$	$[\frac{1}{6}, \frac{1}{2}]$	Total
n	4106	3357	2571	1816	1588	72^4
$\frac{n}{72^4}$	0.1528	0.1249	0.0957	0.0676	0.0591	1.0000

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