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CONVERGENCE OF PPC-CONTINUED FRACTION APPROXIMANTS IN FREQUENCY ANALYSIS

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Dedicated to the memory of Wolfgang J. Thron
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1. Introduction. Many natural phenomena can be represented by real-valued functions of the form

$$G(t) = \sum_{j=-I}^{I} \alpha_j e^{i2\pi f_j t}, \quad I \in \mathbf{N}, \tag{1.1a}$$

where t denotes time (sec.), the *frequencies* f_j are in cycles per sec (Hertz) and the *complex amplitudes* α_j satisfy

$$\alpha_0 \geq 0 \neq \alpha_j = \bar{\alpha}_{-j}, \quad f_j = -f_{-j}, \quad \text{for } j = 1, 2, \dots, I \tag{1.1b}$$

and

$$0 = f_0 < f_1 < f_2 < \cdots < f_I. \tag{1.1c}$$

The *frequency analysis problem* (FAP) consists of determining the unknown frequencies f_j by using N values of "observed data"

$$G(t_m), \quad m = 0, 1, \dots, N-1, \quad \text{where } t_m := m\Delta t, \quad \Delta t > 0. \tag{1.2}$$

For convenience we introduce *normalized frequencies*

$$\omega_j := 2\pi f_j \Delta t, \quad j = 0, \pm 1, \pm 2, \dots, \pm I, \tag{1.3a}$$

with the restrictions imposed by

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_I < \pi \tag{1.3b}$$

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and define an *N-truncated discrete signal* $\{x_N(m)\}_{m=-\infty}^{\infty}$ by

$$(1.4) \quad x_N(m) := \begin{cases} G(t_m) = \sum_{j=-I}^{I} \alpha_j e^{im\omega_j} & m = 0, 1, 2, \dots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

A consequence of (1.3b) is that

$$(1.5) \qquad 0 < f_j < \frac{1}{2\Delta t} \quad \text{for } j = 1, 2, \dots, I,$$

which means that frequencies f_j greater than or equal to $(1/2\Delta t)$ cannot be determined with a *time interval* Δt. With this terminology the FAP consists of determining unknown normalized frequencies $\omega_1, \omega_2, \dots, \omega_I$ using the discrete signal $\{x_N(m)\}_{m=-\infty}^{\infty}$.

Among various methods that have been used for frequency analysis (see, e.g., [**4**, pp. 379–386], [**17**], [**20**], [**23**], [**24**]) the one investigated in the present paper (referred to hereafter as the *N-process*) is a reformulation of Wiener-Levinson linear prediction ([**10**], [**18**], [**31**]) in terms of Szegö polynomials and positive Perron-Carathéodory continued fractions (PPC-fractions). Recent research on the N-process and its extensions can be found in [**6**], [**7**], [**10**]–[**15**], [**19**], [**22**], [**25**]–[**28**].

Starting with the signal $\{x_N(m)\}$, the N-process uses an absolutely continuous distribution function $\psi_N(\theta)$ defined by

$$(1.6) \qquad \psi_N'(\theta) := \frac{1}{2\pi}\left|\sum_{m=0}^{N-1} x_N(m)e^{-im\theta}\right|^2, \quad -\pi \le \theta \le \pi.$$

In [**7**] the distributions $(1/N)d\psi_N(\theta)$ were shown to converge in the weak star sense,

$$(1.7) \qquad \frac{1}{N}\, d\psi_N(\theta) \xrightarrow{*} d\psi(\theta), \quad \text{as } N \to \infty,$$

where $d\psi(\theta)$ is a *discrete distribution* with mass $|\alpha_j|^2$ located at $\theta = \omega_j$ for $-I \le j \le I$. The distribution function $\psi(\theta)$ is a nondecreasing step function with jump $|\alpha_j|^2$ at $\theta = \omega_j$, $-I \le j \le I$. Therefore, the unknown frequencies ω_j can be determined by finding $\psi(\theta)$ or, equivalently, its *Herglotz transform*

$$(1.8) \qquad H(\psi;z) := \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\psi(\theta) = \sum_{j=-I}^{I} |\alpha_j|^2 \frac{e^{i\omega_j}+z}{e^{i\omega_j}-z}.$$

The purpose of the present paper is to investigate the convergence to $H(\psi;z)$ as $N\to\infty$, of the approximants

$$R_{2m}(\psi_N;z) \quad \text{and} \quad R_{2m+1}(\psi_N;z)$$

(of order $2m$ and $2m+1$, respectively) of the PPC-fraction associated with $\psi_N(\theta)$. In [**10**, Theorem 3.4], it was shown that, for

$$(1.9) \qquad m \ge n_0 := 2I + L \quad \text{where } L := \begin{cases} 0 & \text{if } \alpha_0 = 0 \\ 1 & \text{if } \alpha_0 > 0, \end{cases}$$

$$(1.10\text{a}) \qquad \lim_{N\to\infty} \frac{1}{N} R_{2m}(\psi_N;z) = H(\psi,z) \quad \text{for } |z|<1$$

and

$$(1.10\text{b}) \qquad \lim_{N\to\infty} \frac{1}{N} R_{2m+1}(\psi_N;z) = H(\psi;z) \quad \text{for } |z|>1,$$

the convergence in both cases being locally uniform on the given regions. Truncation error bounds for (1.10a) were derived in [**15**, Theorem 1]. In the main theorem of the present paper (Theorem 3.1) the result described by (1.10) is extended to include regions obtained by removing neighborhoods of all frequency points $e^{i\omega_j}$, $-I\le j\le I$, from certain disks on the Riemann sphere. Our proof of Theorem 3.1 makes use of known information concerning the location of the poles of the rational functions $R_m(\psi_N;z)$ (Theorem 2.1), knowledge about the structure of the $R_m(\psi_N;z)$ derived from recurrence relations, Lemma 3.3, and properties of normal families of the holomorphic functions $R_m(\psi_N;z)$. Example 4.1 in Section 4 illustrates possibilities for poles of the $R_m(\psi_N;z)$ to be dense in certain subsets of $\mathbf{C}$, suggesting that the convergence regions in Theorem 3.1 may be best in a certain sense. In order to render the present paper self-contained, Section 2 is used to summarize definitions, notation and results that are subsequently used.

2. PPC-fractions and Szegö polynomials. This section is used to summarize basic properties of PPC-fractions and Szegö polynomials associated with $\psi_N(\theta)$ and $\psi(\theta)$. Proofs of these results can be found in [**8**]–[**10**], [**14**] and [**22**]. Moments with respect to $\psi_N(\theta)$ are defined by

$$\mu_m^{(N)} := \int_{-\pi}^{\pi} e^{-\mathrm{im}\theta}\, d\psi_N(\theta), \quad m \in \mathbf{Z} \tag{2.1}$$

and can be computed by the autocorrelation formulas

$$\mu_m^{(N)} = \begin{cases} \displaystyle\sum_{k=0}^{N-m-1} x_N(k)x_N(k+m) & m = 0, 1, 2, \dots, \\ \mu_{-m}^{(N)} & m = -1, -2, -3, \dots . \end{cases} \tag{2.2}$$

Since the trigonometric moment problem (TMP) for $\{\mu_m^{(N)}\}_{m=-\infty}^{(\infty)}$ has a solution $\psi_N(\theta)$ which has infinitely many points of increase, the sequence $\{\mu_m^{(N)}\}$ satisfies

$$\mu_k^{(N)} = \mu_{-k}^{(N)} \quad \text{and} \quad T_{k+1}^{(0)}(\psi_N) > 0 \quad \text{for } k = 0, 1, 2, \dots , \tag{2.3a}$$

where the Toeplitz determinants $T_k^{(m)}(\psi_N)$ are defined by

(2.3b)

$$T_0^{(m)}(\psi_N) := 1, \quad T_k^{(m)}(\psi_N) := \det\,(\mu_{m-\mu+\nu}^{(N)})_{\mu,\nu=0}^{k-1}, \quad k \geq 1,\ m \in \mathbf{Z}.$$

Hence $\{\mu_m^{(N)}\}_{m=-\infty}^{\infty}$ is said to be *Hermitian positive definite.*

Moments μ_m with respect to the step function $\psi(\theta)$ are given by

$$\mu_m := \int_{-\pi}^{\pi} e^{-\mathrm{im}\,\theta}\, d\psi(\theta) = \sum_{j=-I}^{I} |\alpha_j|^2\, e^{im\omega_j}, \quad m \in \mathbf{Z} \tag{2.4}$$

are related to $\mu_m^{(N)}$ by

$$\frac{1}{N}\,\mu_m^{(N)} = \mu_m + O\left(\frac{1}{N}\right), \quad \text{as } N \to \infty, \quad \text{for } m \in \mathbf{Z} \tag{2.5}$$

and to the Herglotz transform $H(\psi; z)$ in (1.8) by

$$H(\psi; z) = \begin{cases} \mu_0 + 2\sum_{k=1}^{\infty} \mu_k z^k & |z| < 1 \\ -\mu_0 - 2\sum_{k=1}^{\infty} \mu_{-k} z^{-k} & |z| > 1. \end{cases} \tag{2.6}$$

The PPC-function associated with $\psi_N(\theta)$ is given by

$$\delta_0^{(N)} - \frac{2\delta_0^{(N)}}{1+}\;\frac{1}{\delta_1^{(N)}z+}\;\frac{(1-|\delta_1^{(N)}|^2)z}{\delta_1^{(N)}+}\;\frac{1}{\delta_2^{(N)}z+}\;\frac{(1-|\delta_2^{(N)}|^2)z}{\delta_2^{(N)}+}\cdots \tag{2.7a}$$

where

$$\delta_0^{(N)} := \mu_0^{(N)} := \int_{-\pi}^{\pi} d\psi_N(\theta) = \sum_{m=0}^{N-1} |x_N(m)|^2 > 0, \quad N \geq 1, \tag{2.7b}$$

and

$$\delta_m^{(N)} := (-1)^m\,\frac{T_m^{(-1)}(\psi_N)}{T_m^{(0)}(\psi_N)}, \quad m \geq 1, \quad N \geq 1. \tag{2.7c}$$

The $\delta_m^{(N)}$, $m \geq 1$, are called *reflection coefficients*, and they satisfy

$$\delta_m^{(N)} \in \mathbf{R} \quad \text{and} \quad |\delta_m^{(N)}| < 1 \quad \text{for } N \geq 1,\ m \geq 1. \tag{2.8}$$

For $m \geq 0$ and $N \geq 1$, we let $R_m(\psi_N; z)$, $P_m(\psi_N; z)$ and $Q_m(\psi_N; z)$ denote the *mth approximant, numerator* and *denominator*, respectively, of the PPC-fraction (2.7). These are defined by

$$R_m(\psi_N; z) := \frac{P_m(\psi_N; z)}{Q_m(\psi_N; z)}, \tag{2.9}$$

where

$$P_0(\psi_N; z) := -P_1(\psi_N; z) := \delta_0^{(N)}, \quad Q_0(\psi_N; z) := Q_1(\psi_N; z) := 1, \tag{2.10a}$$

$$\begin{pmatrix} P_{2m}(\psi_N; z) \\ Q_{2m}(\psi_N; z) \end{pmatrix} := \delta_m^{(N)} z \begin{pmatrix} P_{2m-1}(\psi_N; z) \\ Q_{2m-1}(\psi_N; z) \end{pmatrix} + \begin{pmatrix} P_{2m-2}(\psi_N; z) \\ Q_{2m-2}(\psi_N; z) \end{pmatrix}, \quad m \geq 1, \tag{2.10b}$$

$$\begin{pmatrix} P_{2m+1}(\psi_N; z) \\ Q_{2m+1}(\psi_N; z) \end{pmatrix} := \delta_m^{(N)} \begin{pmatrix} P_{2m}(\psi_N; z) \\ Q_{2m}(\psi_N; z) \end{pmatrix} + (1-|\delta_m^{(N)}|^2) z \begin{pmatrix} P_{2m-1}(\psi_N; z) \\ Q_{2m-1}(\psi_N; z) \end{pmatrix}, \quad m \geq 1. \tag{2.10c}$$

It follows readily that $Q_{2m+1}(\psi_N; z)$ is a monic polynomial of degree m, while $P_{2m}(\psi_N; z)$, $Q_{2m}(\psi_N; z)$ and $P_{2m+1}(\psi_N; z)$ are polynomials of degree at most m. Moreover, $P_m(\psi_N; z)$ and $Q_m(\psi_N; z)$ have no common zeros, the m poles of $R_{2m+1}(\psi_N; z)$ lie in $|z| < 1$ and

$$P_{2m+1}(\psi_N; z) = -z^m \overline{P_{2m}(\psi_N; 1/\bar{z})}, \tag{2.11a}$$

$$Q_{2m+1}(\psi_N; z) = z^m \overline{Q_{2m}(\psi_N; 1/\bar{z})}, \tag{2.11b}$$

$$R_{2m+1}(\psi_N; z) = -\overline{R_{2m}(\psi_N; 1/\bar{z})}. \tag{2.11c}$$

The terminating PPC-fraction associated with $\psi(\theta)$ is given by

$$R_{2n_0}(\psi; z) := \delta_0 - \frac{2\delta_0}{1} + \frac{1}{\delta_1 z} + \frac{(1-\delta_1^2) z}{\delta_1} + \cdots + \frac{1}{\delta_{n_0-1} z} + \frac{(1-\delta_{n_0-1}^2) z}{\delta_{n_0-1}} + \frac{1}{\delta_{n_0} z}, \tag{2.12a}$$

where

$$\delta_0 := \mu_0, \quad \delta_m := (-1)^m \frac{T^{(-1)}(\psi)}{T_m^{(0)}(\psi)} \quad \text{for } 1 \leq m \leq n_0, \tag{2.12b}$$

$$\delta_0 > 0, \quad |\delta_m| < 1 \quad \text{for } 1 \le m \le n_0 - 1 \text{ and } |\delta_{n_0}| = 1 \tag{2.13}$$

and n_0 is defined by (1.9).

The nth approximant, numerator and denominator of (2.12) are denoted by $R_m(\psi; z), P_m(\psi; z)$ and $Q_m(\psi; z)$, respectively. Some useful properties include

$$Q_{2n_0}(\psi; z) = (z-1)^L \prod_{j=1}^{I} (z - e^{i\omega_j})(z - e^{-i\omega_j}), \tag{2.14}$$

where L is as in (1.9),

$$R_{2n_0}(\psi; z) := \frac{P_{2n_0}(\psi; z)}{Q_{2n_0}(\psi; z)} = H(\psi; z) = \sum_{j=-I}^{I} |\alpha_j|^2 \frac{e^{i\omega_j} + z}{e^{i\omega_j} - z}, \tag{2.15}$$

$$\begin{gathered} \delta_0^{(N)} = N\delta_0 + O(1), \quad \delta_m^{(N)} = \delta_m + O\left(\frac{1}{N}\right) \\ \text{for } 1 \le m \le n_0 \quad \text{as } N \to \infty, \end{gathered} \tag{2.16}$$

and, for $1 \le m \le 2n_0$,

$$\lim_{N\to\infty} \frac{1}{N} P_m(\psi_N; z) = P_m(\psi; z), \quad \lim_{N\to\infty} Q_m(\psi_N; z) = Q_m(\psi; z), \tag{2.17}$$

the convergence being locally uniform on $\mathbf{C}$. The asymmetry of the factor $1/N$ in (2.17) is a consequence of the normalization of the distribution functions $\psi_N(\theta)$ and $\psi(\theta)$ in (1.6) and (1.7) which agrees with the notation used in earlier work.

The *monic Szegö polynomial* $\rho_m(\psi_N; z)$ *of degree* m *with respect to* $\psi_N(\theta)$ *and the* m*th reciprocal polynomial* $\rho_m^*(\psi_N; z)$ are given by

$$\rho_m(\psi_N; z) := Q_{2m+1}(\psi_N; z), \; \rho_m^*(\psi_N; z) := Q_{2m}(\psi_N; z), \quad m \ge 0. \tag{2.18}$$

They satisfy by (2.11b)

$$\rho_m^*(\psi_N; z) = z^m \overline{\rho_m(\psi_N; 1/\bar{z})}, \quad m \ge 0, \tag{2.19}$$

the recurrence relations

(2.20a) $$\rho_0(\psi_N; z) = \rho_0^*(\psi_N; z) = 1,$$

(2.20b) $$\rho_m(\psi_N; z) = z\rho_{m-1}(\psi_N; z) + \delta_m^{(N)}\rho_{m-1}^*(\psi_N; z), \; m \geq 1,$$

(2.20c) $$\rho_m^*(\psi_N; z) = \delta_m^{(N)} z\rho_{m-1}(\psi_N; z) + \rho_{m-1}^*(\psi_N; z), \; m \geq 1$$

and *orthogonality relations*

(2.21a)
$$\langle \rho_m(\psi_N; z), z^k\rangle_{\psi_N} = \begin{cases} 0 & \text{if } 0 \leq k \leq m-1 \\ T_{m+1}^{(0)}(\psi_N)/T_m^{(0)}(\psi_N) > 0 & \text{if } k = m, \end{cases}$$

(2.21b)
$$\langle \rho_m^*(\psi_N; z), z^k\rangle_{\psi_N} = \begin{cases} T_{m+1}^{(0)}(\psi_N)/T_m^{(0)}(\psi_N) > 0 & \text{if } k = 0, \\ 0 & \text{if } 1 \leq k \leq m, \end{cases}$$

where
$$\langle f, g\rangle_{\psi_N} := \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\psi_N(\theta), \quad f, g \in \Pi^{\mathbf{R}}$$

is an *inner product* on the linear space of polynomials over $\mathbf{R}$. Since, from (2.20b),

(2.22) $$\delta_m^{(N)} = \rho_m(\psi_N; 0) = -\frac{\langle z\rho_{m-1}(\psi_N; z), 1\rangle_{\psi_N}}{\langle \rho_{m-1}^*(\psi_N; z), 1\rangle_{\psi_N}}, \quad m \geq 1,$$

one can compute the $\delta_m^{(N)}$ and $\rho_m(\psi_N; z)$ recursively by using (2.20) and (2.22). This procedure is known as *Levinson's algorithm.*

We conclude this section with the statement of important properties of the zeros of the Szegö polynomials (poles of PPC-fraction approximants).

Theorem 2.1. *Let* $m \geq n_0 + 1$ *be given, see* (1.9). *Then*

(A) *there exist* n_0 *sequences of zeros of* $\rho_m(\psi_N; z)$ (*poles of* $R_{2m+1}(\psi_N; z)$) *denoted by*

(2.23)
$$\{z(j, m, N)\}_{N=1}^{\infty} \quad \textit{for } j \in \Delta := \begin{cases} [\pm 1, \pm 2, \dots, \pm I] & \textit{if } \alpha_0 = 0, \\ [0, \pm 1, \dots, \pm I] & \textit{if } \alpha_0 > 0, \end{cases}$$

such that

$$\lim_{N\to\infty} z(j,m,N) = e^{i\omega_j} \quad \textit{for all } j \in \Delta. \tag{2.24}$$

(B) *The remaining zeros of* $\rho_m(\psi_N;z)$ (*not considered in* (A)) *are denoted by*

$$z(j,m,N) \quad \textit{for } N \geq 1, \quad j \in \Gamma := [I+1, I+2, \dots, I+m-n_0], \tag{2.25}$$

and satisfy

$$|z(j,m,N)| \leq K_m < 1 \quad \textit{for } N \geq 1,\ j \in \Gamma, \tag{2.26}$$

where K_m *is a constant independent of* N.

The zeros considered in Theorem 2.1(b) are referred to as the *uninteresting zeros*, since they are bounded away from the frequency points $e^{i\omega_j}$. Theorem 2.1 also applies to the zeros of $\rho_m^*(\psi_N;z)$ (poles of $R_{2m}(\psi_N;z)$); in fact, if $z(j,m,N)$ is a zero of $\rho_m(\psi_N;z)$ and $z(j,m,N) \neq 0$, then $z^*(j,m,N) := 1/\bar{z}(j,m,N)$ is a zero of $\rho_m^*(\psi_N;z)$. This follows from (2.11) and (2.18).

3. Convergence of PPC-fractions. We now state and prove the principal results of this paper.

Theorem 3.1. *Let* $m \geq n_0 + 1$ *be given, and let* K_m *be a positive constant* (*see Theorem* 2.1(b)) *such that for the uninteresting zeros* $z(j,m,N)$ *of* $\rho_m(\psi_N;z)$ (*poles of* $R_{2m+1}(\psi_N;z)$) *the following hold*

$$|z(j,m,N)| \leq K_m < 1 \quad \textit{for } N \geq 1,\ I+1 \leq j \leq I+m-n_0. \tag{3.1}$$

Let δ *satisfy*

$$0 < \delta < \frac{1}{2}(1-K_m). \tag{3.2}$$

Then (A)

$$\lim_{N\to\infty} \frac{1}{N} R_{2m}(\psi_N;z) = H(\psi;z) = \sum_{j=-I}^{I} |\alpha_j|^2 \frac{e^{i\omega_j}+z}{e^{i\omega_j}-z}, \tag{3.3a}$$

the convergence being locally uniform on

$$\text{(3.3b)}\quad S^*(K_m,\delta) := \Big[w \in \mathbf{C} : |w| < \frac{1}{K_m} - \delta \ \textit{and}\ |w - e^{i\omega_j}| > \delta,\ j \in \Delta\Big]$$

(*see* (2.23) *for a definition of* Δ).

(B)

$$\text{(3.4a)}\qquad \lim_{N\to\infty} \frac{1}{N} R_{2m+1}(\psi_N; z) = H(\psi; z) = \sum_{j=-I}^{I} |\alpha_j|^2 \frac{e^{i\omega_j}+z}{e^{i\omega_j}-z},$$

the convergence being locally uniform on

$$\text{(3.4b)}\quad S(K_m,\delta) := [w \in \mathbf{C} : |w| > K_m + \delta \text{ and } |w - e^{i\omega_j}| > \delta,\ j \in \Delta].$$

Remark on Theorem 3.1. The conditions (3.2) and $0 < K_m < 1$ imply two inequalities

(a) $1 < (1/K_m) - \delta$ and (b) $0 < K_m + \delta < 1$.

(a) implies that $S^*(K_m,\delta)$ contains the circles $|w| = \rho$ with

$$1 < \rho < \frac{1}{K_m} - \delta$$

and (b) implies that $S(K_m,\delta)$ contains the circles $|w| = \rho$ with

$$0 < K_m + \delta < 1.$$

Our proof of Theorem 3.1 makes use of several lemmas. We begin by defining polynomials

$$U_k^{(N)}(z) \text{ and } V_k^{(N)}(z) \quad \text{for } k \geq 2n_0 \text{ and } N \geq 1,$$

where (N) denotes an index, not a derivative:

$$\text{(3.5a)}\quad U_{2n_0}^{(N)}(z) := V_{2n_0+1}^{(N)}(z) := 1,\ U_{2n_0+1}^{(N)}(z) := \delta_{n_0}^{(N)},\ V_{2n_0}^{(N)}(z) := 0.$$

and, for $m \geq n_0 + 1$,

(3.5b)
$$\begin{pmatrix} U_{2m}^{(N)}(z) \\ V_{2m}^{(N)}(z) \end{pmatrix} := \delta_m^{(N)} z \begin{pmatrix} U_{2m-1}^{(N)}(z) \\ V_{2m-1}^{(N)}(z) \end{pmatrix} + \begin{pmatrix} U_{2m-2}^{(N)}(z) \\ V_{2m-2}^{(N)}(z) \end{pmatrix},$$

(3.5c)
$$\begin{pmatrix} U_{2m+1}^{(N)}(z) \\ V_{2m+1}^{(N)}(z) \end{pmatrix} := \delta_m^{(N)} \begin{pmatrix} U_{2m}^{(N)}(z) \\ V_{2m}^{N}(z) \end{pmatrix} + (1 - |\delta_m^{(N)}|^2) z \begin{pmatrix} U_{2m-1}^{(N)}(z) \\ V_{2m-1}^{(N)}(z) \end{pmatrix}.$$

Remarks on $U_k^{(N)}(z)$ and $V_k^{(N)}(z)$. The polynomials $U_k^{(N)}(z)$ and $V_k^{(N)}(z)$ are introduced in order to obtain relations (3.8) with the factor $(1 - |\delta_{n_0}^{(N)}|^2)$ in the second term on the righthand side since, by (2.13) and (2.16),

$$\lim_{N\to\infty} |\delta_{n_0}^{(n)}| = |\delta_{n_0}| = 1.$$

We note also that the recurrence relations (3.5b) and (3.5c) are identical to (2.10b) and (2.10c) for the polynomials $P_k(\psi_N; z)$ and $Q_k(\psi_N; z)$, but the initial conditions (3.5a) are not the same as (2.10a). The following lemma is an immediate consequence of (3.5).

Lemma 3.2. *For $m \geq n_0 + 1$ and $N \geq 1$, the polynomials* (3.5) *have the forms*

(3.6a) $$U_{2m}^{(N)}(z) = \delta_{n_0}^{(N)} \delta_m^{(N)} z^{m-n_0} + \cdots + 1,$$

(3.6b) $$V_{2m}^{(N)}(z) = \delta_m^{(N)} z^{m-n_0} + \cdots + \delta_{n_0+1}^{(N)} z,$$

(3.6c) $$U_{2m+1}^{(N)}(z) = \delta_{n_0}^{(N)} z^{m-n_0} + \cdots + \delta_m^{(N)},$$

(3.6d) $$V_{2m+1}^{(N)}(z) = z^{m-n_0} + \cdots + \delta_{n_0+1}^{(N)} \delta_m^{(N)}(z).$$

From (3.6) we see that $V_{2m+1}^{(N)}(z)$ is a monic polynomial of degree $(m - n_0)$ and, for N sufficiently large, $U_{2m+1}^{(N)}(z)$ has degree $(m - n_0)$

since, by (2.13) and (2.16),

$$\lim_{N\to\infty} |\delta_{n_0}^{(N)}| = |\delta_{n_0}| = 1. \tag{3.7}$$

The $U_k^{(N)}(z)$ and $V_k^{(N)}(z)$ are related to $P_k(\psi_N; z)$ and $Q_k(\psi_n; z)$ as follows.

Lemma 3.3. *For $k \geq 2n_0 + 1$ and $N \geq 1$,*

$$\begin{aligned}\begin{pmatrix} P_k(\psi_N;z) \\ Q_k(\psi_N;z)\end{pmatrix} &= U_k^{(N)}(z)\begin{pmatrix} P_{2n_0}(\psi_N;z) \\ Q_{2n}(\psi_N;z)\end{pmatrix} \\ &\quad + (1-|\delta_{n_0}^{(N)}|^2) z V_k^{(N)}(z)\begin{pmatrix} P_{2n_0-1}(\psi_N;z) \\ Q_{2n_0-1}(\psi_N;z)\end{pmatrix}.\end{aligned} \tag{3.8}$$

Proof (*by induction*). We prove (3.8) for $\{Q_k(\psi_N;z)\}$ and omit the analogous argument for $\{P_k(\psi_N;z)\}$. For simplicity in the proof we adopt the notation

$$\delta_k := \delta_k^{(N)},\ Q_k := Q_k(\psi_N;z),\ U_k := U_k^{(N)}(z),\ V_k := V_k^{(N)}(z).$$

Thus it suffices to prove that, for all $k \geq 2n_0+1$,

$$Q_k = U_k Q_{2n_0} + (1-\delta_{n_0}^2) z V_k Q_{2n_0-1}. \tag{3.9}$$

To verify (3.9) for $k = 2n_0+1$, we use (2.10c) and (3.5a) with $m = n_0$ to obtain

$$\begin{aligned} Q_{2n_0+1} &= \delta_{n_0} Q_{2n_0} + (1-\delta_{n_0}^2) z Q_{2n_0-1} \\ &= U_{2n_0+1} Q_{2n_0} + (1-\delta_{n_0}^2) z V_{2n_0+1} Q_{2n_0-1}\end{aligned}$$

in agreement with (3.9). Next for $m = n_0+1$ in (2.10b), (3.5) and (3.9) with $k = 2n_0+1$, we have

$$\begin{aligned} Q_{2n_0+2} &= \delta_{n_0+1} z Q_{2n_0+1} + Q_{2n_0} \\ &= \delta_{n_0+1} z [U_{2n_0+1} Q_{2n_0} + (1-\delta_{n_0}^2) z V_{2n_0+1} Q_{2n_0-1}] + Q_{2n_0} \\ &= (\delta_{n_0+1} z \delta_{n_0} + 1) Q_{2n_0} + (1-\delta_{n_0}^2) z \delta_{n_0+1} Q_{2n_0-1} \\ &= U_{2n_0+2} Q_{2n_0} + (1-\delta_{n_0}^2) z V_{2n_0+2} Q_{2n_0-1}\end{aligned}$$

agreeing with (3.9) for $k = 2n_0 + 2$. Now we assume that (3.9) holds for $k = 2n_0 + 2m$ and $k = 2n_0 + 2m + 1$ with $m \geq 2$. Then, by (2.10) and (3.5),

$$\begin{aligned} Q_{2n_0+2m+2} &= \delta_{n_0+m+1} z Q_{2n_0+2m+1} + Q_{2n_0+2m} \\ &= \delta_{n_0+m+1} z[U_{2n_0+2m+1} Q_{2n_0} + (1 - \delta_{n_0}^2) z V_{2n_0+2m+1} Q_{2n_0-1}] \\ &\quad + [U_{2n_0+2m} Q_{2n_0} + (1 - \delta_{n_0}^2) z V_{2n_0+2m} Q_{2n_0-1}] \\ &= U_{2n_0+2m+2} Q_{2n_0} + (1 - \delta_{n_0}^2) z V_{2n_0+2m+2} Q_{2n_0-1} \end{aligned}$$

and

$$\begin{aligned} &Q_{2n_0+2m+3} \\ &\quad = \delta_{n_0+m+1} Q_{2n_0+2m+2} + (1 - \delta_{n_0+m+1}^2) z Q_{2n_0+2m+1} \\ &\quad = \delta_{n_0+m+1}[U_{2n_0+2m+2} Q_{2n_0} + (1 - \delta_{n_0}^2) z V_{2n_0+2m+2} Q_{2n_0-1}] \\ &\qquad + (1 - \delta_{n_0+m+1}^2) z [U_{2n_0+2m+1} Q_{2n_0} + (1 - \delta_{n_0}^2) z V_{2n_0+m+1} Q_{2n_0-1}] \\ &\quad = U_{2n_0+2m+3} Q_{2n_0} + (1 - \delta_{n_0}^2) z V_{2n_0+2m+3} Q_{2n_0-1}. \end{aligned}$$

Therefore (3.9) holds for $k = 2n_0 + 2m + 2$ and $k = 2n_0 + 2m + 3$ and by induction (3.9) holds for all $k \geq 2n_0 + 1$. $\square$

Let $\{N_k\}_{k=1}^{\infty}$ be an arbitrary subsequence of the natural numbers. Then by (2.8) we can obtain a subsequence $\{N_{k_\nu}\}_{\nu=1}^{\infty}$ such that, for $m \geq 1$, $\{\delta_m^{(N_{k_\nu})}\}_{\nu=1}^{\infty}$ is convergent.

We set

$$\delta_m(\{N_{k_\nu}\}) := \lim_{\nu \to \infty} \delta_m^{(N_{k_\nu})} \quad \text{for } m \geq 0 \tag{3.10a}$$

and note that, by (2.16),

$$\delta_m = \delta_m(\{N_{k_\nu}\}) \quad \text{for } 0 \leq m \leq n_0. \tag{3.10b}$$

From the recurrence relations (2.10) and (3.5) one can see that, for each of the polynomials $P_m(\psi_N; z), Q_m(\psi_N; z), U_m^{(N)}(z), V_m^{(N)}(z)$, the coefficients of individual powers of z are continuous functions of the coefficients $\delta_k^{(N)}$. It follows that, for $m \geq 2n_0$, the four sequences

$$\Big\{ \frac{1}{N_{k_\nu}} P_m(\psi_{N_{k_\nu}}; z) \Big\}_{\nu=1}^{\infty}, \quad \{Q_m(\psi_{N_{k_\nu}}; z)\}_{\nu=1}^{\infty},$$
$$\{U_m^{(N_{k_\nu})}(z)\}_{\nu=1}^{\infty}, \quad \{V_m^{(N_{k_\nu})}(z)\}_{\nu=1}^{\infty}$$

converge locally uniformly on $\mathbf{C}$. We write, for $m \geq 2n_0$,

$$P_m(\{N_{k_\nu}\};z) := \lim_{\nu\to\infty} \frac{1}{N_{k_\nu}} P_m(\psi_{N_{k_\nu}};z), \tag{3.11a}$$

$$Q_m(\{N_{k_\nu}\};z) := \lim_{\nu\to\infty} Q_m(\psi_{n_{k_\nu}};z), \tag{3.11b}$$

$$U_m(\{N_{k_\nu}\};z) := \lim_{\nu\to\infty} U_m^{(N_{k_\nu})}(z), \tag{3.11c}$$

$$V_m(\{N_{k_\nu}\};z) := \lim_{\nu\to\infty} V^{(N_{k_\nu})}(z), \tag{3.11d}$$

It follows from this, (3.7), (3.8) and (2.17) that

$$P_m(\{N_{k_\nu}\};z) \equiv U_m(\{N_{k_\nu}\};z)P_{2n_0}(\psi;z), \quad m \geq 2n_0, \tag{3.12a}$$

and

$$Q_m(\{N_{k_\nu}\};z) \equiv U_m(\{N_{k_\nu}\};z)Q_{2n_0}(\psi;z), \quad m \geq 2n_0. \tag{3.12b}$$

For $m \geq 2n_0$ and $\nu \geq 1$, let

$$\varepsilon_{m,\nu}(z) := \frac{1}{N_{k_\nu}} P_m(\psi_{N_{k_\nu}};z) - U_m(\{N_{k_\nu}\};z)P_{2n_0}(\psi;z), \tag{3.13a}$$

$$\eta_{m,\nu}(z) := Q_m(\psi_{N_{k_\nu}};z) - U_m(\{N_{k_\nu}\};z)Q_{2n_0}(\psi;z). \tag{3.13b}$$

Then, for $m \geq 2n_0+1$ and $\nu \geq 1$, by (2.15) and (3.13)

$$\begin{aligned}
&\left|\frac{1}{N_{k_\nu}} R_m(\psi_{N_{k_\nu}};z) - H(\psi;z)\right| \\
&\quad = \left|\frac{1}{N_{k_\nu}} \frac{P_m(\psi_{N_{k_\nu}};z)}{Q_m(\psi_{N_{k_\nu}};z)} - \frac{P_{2n_0}(\psi;z)}{Q_{2n_0}(\psi;z)}\right| \\
&\quad = \left|\frac{U_m(\{N_{k_\nu}\};z)P_{2n_0}(\psi;z)+\varepsilon_{m,\nu}(z)}{U_m(\{N_{k_\nu}\};z)Q_{2n_0}(\psi;z)+\eta_{m,\nu}(z)} - \frac{P_{2n_0}(\psi;z)}{Q_{2n_0}(\psi;z)}\right| \\
&\quad = \left|\frac{Q_{2n_0}(\psi;z)\varepsilon_{m,\nu}(z) - P_{2n_0}(\psi;z)\eta_{m,\nu}(z)}{Q_{2n_0}(\psi;z)[U_m(\{N_{k_\nu}\};z)Q_{2n_0}(\psi;z)+\eta_{m,\nu}(z)]}\right| \\
&\quad = \left|\frac{H(\psi;z)\eta_{m,\nu}(z) - \varepsilon_{m,\nu}(z)}{U_m(\{N_{k_\nu}\};z)Q_{2n_0}(\psi;z)+\eta_{m,\nu}(z)}\right|.
\end{aligned} \tag{3.14}$$

We now replace m by $(2m+1)$ in (3.14) where $m \geq n_0 + 1$, let K_m denote the constant in (2.2b) of Theorem 2.1 and let $S(K_m, \delta)$ denote the open set in (3.4b) of Theorem 3.1, where $0 < \delta < (1 - K_m)/2$. Let K be an arbitrary compact subset of $S(K_m, \delta)$. Let

$$\varepsilon_{m,\nu}^{(K)} := \sup_{z\in K} |\varepsilon_{m,\nu}(z)|, \quad \eta_{m,\nu}^{(K)} := \sup_{z\in K} |\eta_{m,\nu}^{(K)}(z)| \tag{3.15a}$$

so that by (3.11), (3.12) and (3.13),

$$\lim_{\nu\to\infty} \varepsilon_{m,\nu}^{(K)} = 0 \quad \text{and} \quad \lim_{\nu\to\infty} \eta_{m,\nu}^{(K)} = 0. \tag{3.15b}$$

By (2.2b) of Theorem 2.1(B), (3.4b) of Theorem 3.1(B), (3.11) and (3.12)

$$L_m(K) := \inf_{z\in K} |U_{2m+1}(\{N_{k_\nu}\}; z)| > 0, \tag{3.16}$$

and by (2.14)

$$T(K) := \inf_{z\in K} |Q_{2n_0}(\psi; z)| > 0. \tag{3.17}$$

Since $H(\psi; z)$ is holomorphic in $S(K_m, \delta)$ and hence in K,

$$D(K) := \sup_{z\in K} |H(\psi; z)| < \infty. \tag{3.18}$$

Let ε satisfying $0 < \varepsilon < L_m(K)T(K)$ be given. Then there exists $\nu(\varepsilon) > 0$ such that

$$0 \leq \varepsilon_{2m+1,\nu}^{(K)} < \varepsilon \quad \text{and} \quad 0 \leq \eta_{2m+1,\nu}^{(K)} < \varepsilon \quad \text{for } \nu \geq \nu(\varepsilon).$$

Combining these results with (3.14) yields

$$\begin{aligned}
&\sup_{z\in K} \left| \frac{1}{N_{k_\nu}} R_{2m+1}(\psi_{N_{k_\nu}}; z) - H(\psi, z) \right| \\
&\qquad \leq \frac{D(K)\eta_{2m+1,\nu}^{(K)} + \varepsilon_{2m+1,\nu}^{(K)}}{L_m(K)T(K) - \eta_{2m+1,\nu}^{(K)}} \\
&\qquad < \varepsilon \left(\frac{D(K)+1}{L_m(K)T(K) - \varepsilon} \right) \quad \text{for } \nu \geq \nu(\varepsilon).
\end{aligned} \tag{3.19}$$

An analogous argument holds if m is replaced by $2m$ in (3.14). Thus we have proved the following two lemmas. The first gives existence of convergent subsequences $\{(1/N_{k_\nu})R_k(\psi_{N_{k_\nu}}; z)\}_{\nu=1}^{\infty}$, the second asserts the convergence of the whole sequences $\{(1/N)R_k(\psi_N; z)\}_{N=1}^{\infty}$ to $H(\psi; z)$.

Lemma 3.4. *Let $m \geq n_0 + 1$ be given. Let $S(K_m, \delta)$ and $S^*(K_m, \delta)$ be defined as in Theorem* 3.1. *Let $\{N_k\}_{k=1}^{\infty}$ be an arbitrary subsequence of the natural number sequence. Then there exists a subsequence $\{N_{k_\nu}\}_{\nu=1}^{\infty}$ of $\{N_k\}_{k=1}^{\infty}$ such that*: (a) *for $z \in S^*(K_m, \delta)$,*

$$\lim_{\nu\to\infty} \frac{1}{N_{k_\nu}} R_{2m}(\psi_{N_{k_\nu}}; z) = H(\psi; z), \tag{3.20}$$

the convergence being locally uniform on $S^*(K_m, \delta)$.

(b) *For $z \in S(K_m, \delta)$,*

$$\lim_{\nu\to\infty} \frac{1}{N_{k_\nu}} R_{2m+1}(\psi_{N_{k_\nu}}; z) = H(\psi; z), \tag{3.21}$$

the convergence being locally uniform on $S(K_m, \delta)$.

Lemma 3.5. *Let $m \geq n_0 + 1$ and let $S(K_m, \delta)$ and $S^*(K_m, \delta)$ be defined as in Theorem* 3.1. *Then*

(3.22a)
$$\lim_{N\to\infty} \frac{1}{N} R_{2m}(\psi_N; z) = H(\psi; z) \quad \text{for } z \in S^*(K_m, \delta),$$

and

(3.22b)
$$\lim_{N\to\infty} \frac{1}{N} R_{2m+1}(\psi; z) = H(\psi; z) \quad \text{for } z \in S(K_m, \delta).$$

Proof. Assume that there exists a $z_0 \in S(K_m, \delta)$ such that $\{(1/N)R_{2m+1}(\psi_n; z)\}_{N=1}^{\infty}$ does not converge to $H(\psi; z_0)$. Then there exists an $\varepsilon > 0$ and a subsequence $\{N_k\}_{k=1}^{\infty}$ of the natural number sequence such that

$$\left| \frac{1}{N_k} R_{2m+1}(\psi_{N_k}; z_0) - H(\psi; z_0) \right| \geq \varepsilon \quad \text{for } k \geq 1.$$

This contradicts Lemma 3.4(b). Therefore, (3.22b) holds. An analogous argument can be given to prove (3.22a). □

Our proof of Theorem 3.1 makes use of a property of normal families stated here. (See, e.g., [**1**], [**5**], [**30**].) Let $\mathcal{F}$ be a family of functions holomorphic on an open region R. In order for $\mathcal{F}$ to be a *normal family* in R, it suffices that every sequence $\{f_n(z)\}$ in F contains a subsequence $\{f_{n_i}(z)\}$ which converges locally uniformly on R.

Stieltjes-Vitali theorem. *Let R be an open region in* $\mathbf{C}$*, and let Λ be a subset of R having infinitely many elements and having a limit point in R. If $\{f_n(z)\}$ is a normal family in R and if* $\lim_{n\to\infty} f_n(z)$ *exists for all $z \in \Lambda$, then $\{f_n(z)\}$ converges locally uniformly on R.*

Proof of Theorem 3.1. (a) It follows from Theorem 2.1 that there exists an $N^* \geq 1$ such that, for all $N \geq N^*$, $(1/N)R_{2m}(\psi_N; z)$ is holomorphic in $S^*(K_m, \delta)$. Therefore, by Lemma 3.4, $\{(1/N)R_{2m}(\psi_N; z)\}_{N=N^*}^{\infty}$ is a normal family in $S^*(K_m, \delta)$. Assertion (A) of Theorem 3.1 follows from this and Lemma 3.5 and the Stieltjes-Vitali theorem. An analogous proof can be given for (b) of Theorem 3.1. This completes the proof. □

4. Uninteresting zeros (poles). A natural question to raise is the following.

Is it possible to extend the convergence results in Theorem 3.1 even further, to larger domains, possibly to the whole plane minus disks around the uninteresting zeros? Behind this question is this idea of having a discrete set of uninteresting zeros to stay away from. This is, however, an incorrect picture of what may happen, as the following very simple example will show.

Example 4.1. Take a signal with merely the frequencies $\pm\omega$, $0 < \omega < \pi$ and amplitudes 1:

$$x(m) = e^{mi\omega} + e^{-mi\omega} = 2\cos(m\omega). \tag{4.1}$$

The N-process leads to, in limit, the two frequency points $e^{\pm i\omega}$ and, before going to limits, to the uninteresting zero

$$z^{(N)} = -\frac{3\cos\omega + \cos((2N-1)\omega)}{4 + 2\cos^2\omega + 2\cos\omega\cos((2N-1)\omega)} + O\left(\frac{1}{N}\right). \tag{4.2}$$

The limit as $N \to \infty$ does not exist. If ω/π is irrational, the set of points $z^{(N)}$, $N = 1, 2, 3, \dots$ is dense in the interval

$$\left[-\frac{3\cos\omega + 1}{2(\cos^2\omega + \cos\omega + 2)}, \frac{-3\cos\omega + 1}{2(\cos^2\omega - \cos\omega + 2)}\right]. \tag{4.3}$$

This follows from a theorem of Kronecker stating that the set of points $e^{Ni\omega}$, $N = 1, 2, 3, \dots$ is dense on the unit circle when ω/π is irrational.

Actually, this is a special case of Kronecker's theorem [**3**, Chapter 23; see, in particular, 23.2(iii)]. One way of proving this theorem is by using the Pigeonhole principle, also called the Dirichlet drawer (or box) principle: If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more objects. See, e.g., [**29**, 4.2].

This example (and others) exclude the possibility of extending the theorem to larger domains merely by removing disks around a discrete set of points. It does not exclude the possibility of extending it to domains obtained by removing more complicated sets.

5. Final remark (erratum). Lemma 3.4 in the present paper replaces Theorem 3.5(B) in [**14**], which is not correct as it stands. The proof of Theorem 3.8 in [**14**] is based upon Theorem 3.5(B) and is therefore not valid. However, a version where even and odd approximants are separately discussed, and where the domains in Theorem 3.1 of the present paper replace the domain in Theorem 3.8 in [**14**] is correct.

REFERENCES

1. Lars V. Ahlfors, *Complex analysis*, 3rd ed., McGraw-Hill Publ. Co., New York, 1979.

2. N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, Hafner, New York, 1965.

3. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 5th ed., Clarendon Press, Oxford, 1979.

4. F.B. Hildebrand, *Introduction to numerical analysis*, McGraw-Hill Book Co., Inc., New York, 1956.

5. E. Hille, *Analytic function theory*, Vol. II, Ginn and Co., Boston, 1962.

6. W.B. Jones and O. Njåstad, *Applications of Szegö polynomials to digital signal processing*, Rocky Mountain J. Math. **21** (1991), 387–436.

7. W.B. Jones, O. Njåstad and E.B. Saff, *Szegö polynomials associated with Wiener-Levinson filters*, J. Comput. Appl. Math. **32** (1990), 387–406.

8. W.B. Jones, O. Njåstad and W.J. Thron, *Continued fractions associated with the trigonometric and other strong moment problems*, Constr. Approx. **2** (1986), 197–211.

9. ———, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21** (1989), 113–152.

10. W.B. Jones, O. Njåstad, W.J. Thron and H. Waadeland, *Szegö polynomials applied to frequency analysis*, J. Comput. Appl. Math. **46** (1993), 217–228.

11. W.B. Jones, O. Njåstad and H. Waadeland, *An alternative way of using Szegö polynomials in frequency analysis*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Marcel Dekker, Inc., New York, (1994), 141–152.

12. ———, *Asymptotics of zeros of orthogonal and para-orthogonal Szegö polynomials in frequency analysis*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Marcel Dekker, Inc., New York, (1994), 153–190.

13. ———, *Application of Szegö polynomials to frequency analysis*, SIAM J. Math. Anal. **25** (1994), 491–512.

14. W.B. Jones and V. Petersen, *Continued fractions and Szegö polynomials in frequency analysis and related topics*, Acta Appl. Math. **61** (2000), 149–174.

15. W.B. Jones and E.B. Saff, *Szegö polynomials and frequency analysis*, in *Approximation theory* (G. Anastassiou, ed.), Marcel Dekker, Inc., New York, 1992.

16. W.B. Jones and W.J. Thron, *A constructive proof of convergence of the even approximants of positive PC-fractions*, Rocky Mountain J. Math. **19** (1989), 199–210.

17. R. Kumaresam, L.L. Scharf and A.K. Shaw, *An algorithm for pole-zero modeling and spectral analysis*, Trans. ASSP **34** (1986), 637–640.

18. N. Levinson, *The Wiener RMS* (*root mean square*) *error criterion in filter design and prediction*, J. Math. Phys. **25** (1947), 261–278.

19. X. Li, *Asymptotics of columns in the table of orthogonal polynomials with varying measures*, Methods Appl. Anal. **2** (1995), 222–236.

20. J.D. Markel and A.H. Gray, Jr., *Linear prediction of speech*, Springer-Verlag, New York, 1976.

21. O. Njåstad and H. Waadeland, *Generalized Szegö theory in frequency analysis*, J. Math. Anal. Appl. **206** (1997), 280–307.

22. K. Pan and E.B. Saff, *Asymptotics for zeros of Szegö polynomials associated with trigonometric polynomial signals*, J. Approx. Theory **71** (1992), 239–251.

23. A.K. Paul, *Anharmonic frequency analysis*, Math. Comp. **26** (1972), 437–447.

24. ———, *Anharmonic frequency analysis*: *The complement of Fourier analysis*, Annal. Tel. **35** (1979), 154–157.

25. V. Petersen, *A theorem on Toeplitz determinants containing Tchebycheff polynomials of the first kind*, The Royal Norwegian Soc. Sci. Letters Trans. **4** (1996).

26. ———, *A combination of two methods in frequency analysis*: *The* $R(N)$-*process*, in *Orthogonal functions, moment theory and continued fractions*: *Theory and applications* (W.B. Jones and A. Sri Ranga, eds.), Marcel Dekker, Inc., New York, 1998.

27. V. Petersen, *Zeros of Szegö polynomials used in frequency analysis*, in *Orthogonal functions, moment theory, and continued fractions*: *Theory and applications* (W.B. Jones and A. Sri Ranga, eds.), Marcel Dekker, Inc., New York, 1998.

28. ———, *On measures in frequency analysis*, J. Comput. Appl. Math. **105** (1999)

29. K.H. Rosen, *Discrete mathematics and its applications*, 4th ed., McGraw-Hill, New York, 1999.

30. W.J. Thron, *The theory of functions of a complex variable*, John Wiley and Sons, Inc., New York, 1953.

31. N. Wiener, *Extrapolation, interpolation and smoothing of stationary time series*, published jointly by The Technology Press of the Massachusetts Institute of Tech. and John Wiley and Sons, Inc., New York, 1949.

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