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ON DISTORTION UNDER QUASICONFORMAL MAPPING

ALEXANDER VASIL'EV

ABSTRACT. In the paper we study the range of the system of functionals $(|f(z_1)|, |f(z_2)|)$ over the class of K-quasiconformal homeomorphisms of the Riemann sphere with standard three point normalization $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, and for different real values of z_1 and z_2. Extremal functions are given in terms of the complex dilatation dependent only on z_1, z_2. As a corollary, we derive some sharp estimates for the functional $|f(z_2)| \pm |f(z_1)|$ and $|f(z_2) \pm f(z_1)|$. The main tool of the proofs is the extremal partition of a Riemann surface by doubly connected domains.

1. Introduction. Let S_0 be a Riemann surface given by the punctured Riemann sphere $\mathbf{C} \setminus \{0, 1\}$. We shall investigate the class Q_K, of all functions $w = f(z)$ univalent and K-quasiconformal in S_0, such that $f(S_0) = S_0$ with $f(0) = 0$, $f(1) = 1$. These functions are Sobolev generalized homeomorphic solutions of the Beltrami equation

$$w_{\bar{z}} = \mu_f(z) w_z, \quad z \in S_0, \tag{1}$$

with the complex distributional derivatives

$$w_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right), \quad \text{and} \quad w_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right), \quad z = x + iy,$$

where the complex dilatation (or the Beltrami coefficient) $\mu_f(z) = f_{\bar{z}}/f_z$ is a measurable function with the norm

$$\|\mu_f\|_\infty = \text{ess sup } |\mu_f(z)| \le k < 1, \quad z \in S_0, \quad k = \frac{K-1}{K+1}.$$

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A quasiconformal mapping is said to be Teichműller's if its Beltrami coefficient is of the form

$$\mu_f(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad k = \frac{K-1}{K+1}, \tag{2}$$

almost everywhere in S_0, where $\varphi(z)dz^2$ is a holomorphic or meromorphic quadratic differential on S_0 of finite L_1-norm. Such a differential has at most simple poles in the Riemann sphere. The inverse mapping is also a Teichműller map, i.e., there exists a quadratic differential $\psi(w)dw^2$ on $S_0 = f(S_0)$ such that the inverse mapping is given by its Beltrami coefficient $\mu_{f^{-1}}(w) = k\overline{\psi(w)}/|\psi(w)|$. Teichműller mappings are locally affine and map infinitesimal circles onto infinitesimal ellipses having their bigger semi-axes along or orthogonal to a trajectory of the differential $\psi(w)dw^2$. The ratio of the bigger and smaller axes of ellipses is equal to K for any regular point of ψ.

Let $F(f)$ be a continuous functional (or system of functionals) over Q_K. The extremal problem for $F(f)$, i.e.

$$\max_{f\in Q_K} \pm F(f) \quad \text{(or finding the range of } F(f)),$$

has been studied deeply by Lehto, Virtanen [**15**], Belinskii [**3, 4**], Lavrent'ev [**14**], Krushkal' [**10**–**12**], as well as Schiffer [**18**]. An important case of the functional $F(f)$ in the theory of distortion and its applications is

$$F(f) = |F(f(z_1), \dots, f(z_n))|, \tag{3}$$

where $z_1, \dots, z_n$ are fixed points on S_0 and $F(w_1, \dots, w_n)$ is a holomorphic function defined in the space $\{\mathbf{C}^n,\, w_j \neq w_k,\, j \neq k\}$.

The approaches to solution of this extremal problem have gone primarily in three directions. The first one is based on the variational method by Belinskii [**4**], Schiffer [**18**], and has been developed further in, e.g., [**4, 6, 16**]. The second employs the parametric representation of quasiconformal mappings by Shah Dao-Shing [**17**]. Krushkal' [**11, 12**] has developed the method of invariant metrics which gives generally non-sharp estimates, however, in particular cases solves the problem completely. It is known [**3, 10**] that the extremal mapping f^* for the extremal problem for $F(f)$ is a Teichmüller map and the inverse one satisfies the Beltrami equation $z_{\bar{w}} = \mu_{f^{-1}}(w) z_w$, with the dilatation

$$\mu_{f^{-1}}(w) = ke^{it}\frac{\overline{\psi(w)}}{|\psi(w)|}, \quad k = \frac{K-1}{K+1},$$
$$\psi(w) = \sum_{j=1}^{n-3} \frac{\partial F}{\partial w_j} \frac{w_j(w_j-1)}{w(w-1)(w-w_j)},$$
$$\text{for } w = f^*(z),\ w_j = f^*(z_j).$$

This Beltrami coefficient contains a lot of unknown parameters so the extremal problem is resolved only qualitatively.

Therefore, our main effort aims at creating a method to find a representation of extremal mappings in terms of the dilatation of the direct mapping.

Among particular cases of this functional we consider "two point" distortion under quasiconformal mappings from Q_K. This can be reduced to estimation of functionals dependent on two fixed points of S_0. There are some results devoted to such estimation. In the case of the class Q_K there is a result by Krushkal' [**12**]. He has obtained that there is $K_0 > 1$ such that for all $f \in Q_K$, $1 \le K \le K_0$ and for fixed points $z_1, z_2 \in \mathbf{C} \setminus \{0,1\}$, $z_1 + z_2 = 1$ the extremal mapping for the functional $\max_{f \in Q_K} |f(z_1) - f(z_2)|$ is the Teichmüller mapping with the dilatation

$$\mu_f(z) = ke^{it}\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad k = \frac{K-1}{K+1},$$

where $\varphi(z) = z_1 z_2 [z(1-z)(z-z_1)(z-z_2)]^{-1}$.

Among two point distortion theorems we also refer to Agard [**1**]. His result [**1**, formula (3.1)] implies the sharp estimate of the ratio $|f(r_2)|/|f(-r_1)|$ over the class Q_K, $r_2 > 1$, $r_1 > 0$. We use the standard notation, see [**15**],

$$\mu(k) = \frac{\pi}{2} \cdot \frac{\mathbf{K}'(k)}{\mathbf{K}(k)},$$

where

$$\mathbf{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad \text{and} \quad \mathbf{K}'(k) := \mathbf{K}(\sqrt{1-k^2}),$$
$$k \in (0,1)$$

are the complete elliptic integrals. Denote by $p_K(k) = \mu^{-1}(1/K\mu(k))$, $K \geq 1$. For this quantity the useful estimate $(p_K(k))^2 \leq 16^{1-1/K} \cdot k^{1/K}$ can be found in, e.g., [**2, 15**].

If $f \in Q_K$, then [**1**] the sharp estimate

$$(4) \qquad \frac{|f(r_2)|}{|f(-r_1)|} \leq \frac{u^2}{1-u^2} \leq \frac{16^{1-1/K} \cdot r_2^{1/(2K)}}{(r_1+r_2)^{1/(2K)} - 16^{1-1/K} \cdot r_2^{1/(2K)}}, \quad u = p_K\left(\sqrt{\frac{r_2}{r_2+r_1}}\right).$$

obeys.

Our paper deals with the application of the method of the extremal partition of a Riemann surface by doubly connected domains to the extremal problem about two point distortion in the class Q_K. In terms of our method the previous result by Agard invokes the partition by only one doubly connected domain. We use the partition by two doubly connected domains to evaluate the range of the system of functionals $(|f(r_1)|, |f(r_2)|)$ for fixed real values of r_1 and r_2. Our main result is the following theorem.

Theorem A. *Let r_1 and r_2 be fixed real points, $f \in Q_K$. Then*

(i) *The unique extremal mapping f^* gives the maximum to $|f(r_2)| - |f(-r_1)|$, $r_1 > 0$, $r_2 > 1$ in the class Q_K. This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient* (2) *where*

$$\varphi(z) = \frac{c-z}{z(z-1)(z+r_1)(z-r_2)},$$

and

$$c = \frac{r_1 r_2 (r_1 - r_2 + 2)}{r_2(r_2-1) + r_1(1+r_1)}.$$

(ii) *The unique extremal mapping f^{**} gives the maximum to $|f(r_2) - f(-r_1)|$ and $|f(r_2)| + |f(-r_1)|$, $r_1 > 0$, $r_2 > 1$ in the class Q_K. This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient*

(2) *with*

$$\varphi(z) = \frac{c-z}{z(z-1)(z+r_1)(z-r_2)}, \quad \textit{for } r_2 - r_1 > 1,$$
$$\varphi(z) = \frac{z-c}{z(z-1)(z+r_1)(z-r_2)}, \quad \textit{for } r_2 - r_1 < 1,$$
$$\varphi(z) = \frac{-1}{z(z-1)(z+r_1)(z-r_2)}, \quad \textit{for } r_2 - r_1 = 1,$$

where

$$c = \frac{r_1 r_2}{1 + r_1 - r_2}.$$

(iii) *The unique extremal mapping* f^* *gives the maximum to* $|f(r_2) + f(r_1)|$ *and* $|f(r_2)| + |f(r_1)|$, $1 < r_1 < r_2$ *in the class* Q_K. *This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient* (2) *with*

$$\varphi(z) = \frac{z-c}{z(z-1)(z-r_1)(z-r_2)},$$

and

$$c = \frac{r_1 r_2 (r_1 + r_2 - 2)}{r_2(r_2-1) + r_1(r_1-1)}.$$

(iv) *The unique extremal mapping* f^{**} *gives the maximum of* $|f(r_2)| - |f(r_1)|$, $1 < r_1 < r_2$ *in the class* Q_K. *This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient* (2) *with* $\varphi(z)$ *as in* (iii) *and*

$$c = \frac{r_1 r_2}{r_1 + r_2 - 1}.$$

Some initial results have been obtained by the author in [**22**]. In particular, the upper boundary curve of the range of the system of functionals $(|f(r_1)|, |f(r_2)|)$, $0 < r_1 < r_2 < 1$, was described in the class of K-quasiconformal homeomorphisms of the unit disk, $f(0) = 0$. In conformal case there are results by Jenkins [**9**] and by Vasil'ev and Fedorov [**21**], who evaluated the range of the system of functionals $(|f(\pm r_1)|, |f(r_2)|)$ in the class S of all univalent holomorphic normalized in the unit disk functions. More detail information about this will be given in Section 6.

2. Extremal quadratic differentials.

1. Let S be a Riemann surface represented by a multiply connected domain in $\mathbf{C}$ with n punctures and with possibly l hyperbolic boundary components, $2n + 3l > 6$. We define on S an admissible system of curves, following the terminology by Strebel [**20**], $\gamma = (\gamma_1, \dots, \gamma_m)$ that satisfy the following property. The curves from this system are not freely homotopic to each other in pairs and not homotopic to a point or a puncture of S. All curves from the admissible system are not intersected. In the case $m = 1$ we will speak about one admissible curve γ.

A doubly connected hyperbolic domain D_j on S is said to be of homotopic type γ_j if any simple loop on S separating the boundary components of D_j is freely homotopic to the curve γ_j from the given admissible system.

A system of nonoverlapping doubly connected hyperbolic domains $(D_1, \dots, D_m)$ on S is said to be associated with the admissible system $\gamma = (\gamma_1, \dots, \gamma_m)$ if for any $j \in \{1, \dots, m\}$ the domain D_j is of homotopy type γ_j.

Denote by $M(D)$ the conformal modulus of a doubly connected hyperbolic domain D. A general theorem by Jenkins [**8**] asserts that any collection of non-overlapping doubly connected domains $\{D_j\}$ associated with the admissible system $\gamma = (\gamma_1, \dots, \gamma_m)$ satisfies the following inequality

$$\sum_{j=1}^{m} \alpha_j^2 M(D_j) \leq \sum_{j=1}^{m} \alpha_j^2 M(D_j^*) = m(S, \gamma, \alpha) \tag{5}$$

for a nonzero vector $\alpha = (\alpha_1, \dots, \alpha_m)$ with nonnegative coordinates. The equality sign occurs only for $D_j = D_j^*$. Not all of D_j^*, $j = 1, \dots, m$, degenerate. Each D_j^*, if it does not degenerate, is a ring domain in the trajectory structure of a unique quadratic differential $\varphi(z)dz^2$ with closed trajectories. There is a conformal map $g_j(z)$, $z \in D_j^*$ which satisfies the differential equation

$$\alpha_j^2 \left(\frac{g_j'(z)}{g_j(z)} \right)^2 = -4\pi^2 \varphi(z)$$

and maps D_j^* onto the ring $1 < |w| < \exp(2\pi M(D_j^*))$. We normalize the vector α by $\alpha_1 = 1$. If $m = 1$ and the vector α degenerates, $\alpha = (1)$, then we denote by $m(S, \gamma) \equiv m(S, \gamma, \alpha)$. The quantity $m(S, \gamma, \alpha)$ is the modulus, see e.g. [**8,9**], in the modulus problem for the free family of homotopy classes of curves generated by the admissible system γ. So, we call it also "modulus."

The quadratic differential which is extremal in such a problem of the extremal partition has closed trajectories, in Strebel's terminology [**20**], and at most ring domains in its trajectory structure. Vice versa, each differential φ with closed trajectories and only ring domains in its trajectory structure defines a problem of the extremal partition where the admissible system of curves may be defined by noncritical nonhomotopic trajectories of φ and the vector α consists of their length in the metric $\sqrt{|\varphi(z)|}|dz|$.

2. Now we consider some special moduli $m(S, \gamma, \alpha)$ generated by certain quadratic differentials. Assume $r_1 > 0$, $r_2 > 1$.

We set the following one-parametric families of holomorphic quadratic differentials (6–8) on $S = S_0 \setminus \{r_1, r_2\}$.

$$\varphi_1(z)dz^2 = A_1(\alpha)\,\frac{z - c_1(\alpha)}{z(z-1)(z+r_1)(z-r_2)}\,dz^2, \tag{6}$$

where $c_1(\alpha) \in [1, r_2]$, $A_1(\alpha) < 0$. These values are calculated by the equations

$$\int_{-\infty}^{-r_1} \sqrt{\varphi_1(x)}\,dx = 1/2, \quad \int_0^1 \sqrt{\varphi_1(x)}\,dx = \alpha/2;$$

α is a fixed number from the segment $[\alpha_0, 1]$, where

$$\alpha_0 = \int_0^1 \frac{dx}{\sqrt{x(r_2 - x)(r_1 + x)}} \left(\int_{-\infty}^{-r_1} \frac{dx}{\sqrt{x(r_2 - x)(r_1 + x)}} \right)^{-1},$$

$c_1(\alpha_0) = 1$, $c_1(1) = r_2$. For each $\alpha \in (\alpha_0, 1)$ this differential has two ring domains in its trajectory structure $D_1 = D_1(\alpha)$ and $D_2 = D_2(\alpha)$. The domain D_1 is bounded by the ray $(-\infty, r_1]$, the segment $[c_1(\alpha), r_2]$ and the smooth arc $p = p(\alpha)$ of the critical trajectory of the differential φ_1 with two endpoints at $c(\alpha)$ enclosing the segment $[0,1]$. The domain D_2 is bounded by the segment $[0,1]$ and the arc $p(\alpha)$. The corresponding problem of the extremal partition is defined for the admissible system γ of two simple loops γ_1 and γ_2. The loop γ_1 is homotopic on S to the slit along $[-\infty, -r_1]$, the loop γ_1 is homotopic on S to the slit along $[0,1]$. Here the sign at $(-\infty)$ simply shows the direction. Varying α from α_0 to 1 one can learn the dynamics of the trajectory structure from two ring domains $D_1(\alpha_0) = S \setminus \{[-\infty, -r_1] \cup [0, r_2]\}$, $D_2(\alpha_0) = \varnothing$ up to two ring domains $D_1(1) = S \setminus \{[-\infty, -r_1] \cup \overline{\operatorname{int} p(1)}\}$, $D_2(\alpha_0) = \operatorname{int} p(1) \setminus [0,1]$. Here $(\operatorname{int} p)$ means the domain lying left to the clock-wise direction of p and we denote by $\overline{\operatorname{int} p}$ its closure.

For the differentials φ_2 and φ_3 one can easily learn a similar dynamics from the above description.

We set

$$\varphi_2(z)dz^2 = A_2(\alpha)\,\frac{z - c_2(\alpha)}{z(z-1)(z+r_1)(z-r_2)}\,dz^2, \tag{7}$$

where $c_2(\alpha) \in [-r_1, 1]$, $c_2(1) = 0$, $A_2(\alpha) < 0$. These values are calculated by the equations

$$\int_{-\infty}^{-r_1} \sqrt{\varphi_1(x)}\,dx = 1/2, \quad \int_0^1 \sqrt{\varphi_1(x)}\,dx = \alpha/2;$$

α is a fixed number from the segment $[\alpha_1, \alpha_2]$, where

$$\alpha_1 = \int_1^{r_2} \frac{dx}{\sqrt{x(r_2-x)(r_1+x)}} \left(\int_{-\infty}^{-r_1} \frac{dx}{\sqrt{x(r_2-x)(r_1+x)}} \right)^{-1},$$
$$\alpha_2 = \int_1^{r_2} \frac{dx}{\sqrt{x(r_2-x)(x-1)}} \left(\int_{-\infty}^{-r_1} \frac{dx}{\sqrt{x(r_2-x)(x-1)}} \right)^{-1}.$$

Let $\alpha \neq 1$. Then we set

$$\varphi_3(z)\,dz^2 = A_3(\alpha)\,\frac{z - c_3(\alpha)}{z(z-1)(z+r_1)(z-r_2)}\,dz^2, \tag{8}$$

where $c_3(\alpha) \in (\infty, -r_1] \cup [r_2, \infty)$, $A_2(\alpha) < 0$ for $\alpha > 1$ and $A_2(\alpha) > 0$ for $\alpha < 1$. These values are calculated by the equations

$$\int_{-r_1}^{0} \sqrt{\varphi_1(x)}\, dx = 1/2, \qquad \int_{1}^{r_2} \sqrt{\varphi_1(x)}\, dx = \alpha/2;$$

α is a fixed number from the set $[\alpha_3, 1) \cup (1, \alpha_4]$, where

$$\alpha_3 = \int_1^{r_2} \frac{dx}{\sqrt{x(x-1)(r_1+x)}} \left(\int_{-r_1}^{0} \frac{dx}{\sqrt{x(1-x)(r_1+x)}} \right)^{-1},$$
$$\alpha_4 = \int_1^{r_2} \frac{dx}{\sqrt{x(r_2-x)(x-1)}} \left(\int_{-r_1}^{0} \frac{dx}{\sqrt{x(r_2-x)(x-1)}} \right)^{-1}.$$

For $\alpha = 1$ we have

$$\varphi_3(z)\, dz^2 = \frac{-1}{z(z-1)(z+r_1)(z-r_2)}\, dz^2.$$

From (6–8) one can see the dynamics of the zero and trajectory variation dependent on the parameter α. The values $\alpha_0, \dots, \alpha_4$ of the parameter α correspond to the degeneracy of one of $D_1(\alpha)$ and $D_2(\alpha)$ in the trajectory structure of the differentials (6–8). The differentials are of finite L_1-norms and have closed trajectories. For each of them one can define a problem on the extremal partition where certain differential is extremal. This means that the admissible system of curves consists at most of two nonhomotopic noncritical trajectories of the differential given with the weight vector $(1, \alpha)$.

3. Extremal mappings. Now we construct the Teichműller mappings f_j defined on $\mathbf{C}$ satisfying the Beltrami equation (1) with the complex dilatation

$$\mu_{f_j}(z) = k \frac{\overline{\varphi_j(z)}}{|\varphi_j(z)|}, \quad k = \frac{K-1}{K+1}, \quad j = 1, 2, 3,$$

keeping the points $z = 0, 1, \infty$ motionless. These mappings satisfy the condition $f(\bar{z}) = \overline{f(z)}$. It is easily seen by the symmetry of the corresponding Beltrami coefficients. Besides, the mappings are symmetrically normalized. These mappings are extremal with respect to the following problem of the extremal partition. Namely, we consider the differential $\varphi_1(z)\,dz^2$, $\alpha \in (\alpha_0, 1)$ and its two nonhomotopic trajectories as an admissible system of curves on S. Then we construct the admissible system γ on S and define the problem of the extremal partition for this admissible system and the vector $(1,\alpha)$, $\alpha \in [\alpha_0, 1]$. Let $m = m(D, \gamma, (1,\alpha))$ be the solution of this problem. In order to simplify the notations further we shall use the same letter α both for the vector $\alpha := (1,\alpha)$ and for its second coordinate. So, we rewrite $m = m(D,\gamma,\alpha)$.

Let us assume f to be an arbitrary quasiconformal homeomorphism from Q_K and define the modulus $m_f = m(f(S), f(\gamma), \alpha)$. The Teichműller homeomorphism f_1 is the unique extremal homeomorphism in the problem of $\min_{f\in Q_K} m_f$. In fact, $m_f \geq m/K = m(S,\gamma,\alpha)/K$ and the extremal mapping exists. We consider the mapping f_1. The trajectory structure of $\varphi_1(z)\,dz^2$ contains two ring domains D_1 and D_2 described in Section 2. The Teichműller homeomorphism f_1 maps D_1 and D_2 onto a couple of ring domains in the trajectory structure of some quadratic differential $\psi(w)\,dw^2$ in $\mathbf{C}$ with singularities relevant to those for $\varphi_1(z)\,dz^2$. We induce the local parameters $\zeta = \exp(\int^z \sqrt{\varphi_1(z)}\,dz)$ and $\zeta' = \exp(\int^w \sqrt{\psi(w)}\,dw)$ and one can see that in each domain D_j the mapping f_1 being considered in S acts affine in these coordinates $\zeta' = z + k\bar{\zeta}$. For $\alpha \in [\alpha_0, 1]$ the ratio of the length of trajectories is constant for φ_1 and for ψ, therefore, the normalization of ψ implies $m_{f_1} = M(f_1(D_1)) + \alpha^2 M(f_1(D_2)) = m/K$. The uniqueness follows from [**8**]. Similar assertions are proved for the differentials φ_j and the mappings f_j for $j = 2, 3$.

The following result is known and one can find it in, e.g., [**2**]. We adduce its proof here to clarify our method and the following proofs.

Proposition 1. *The unique mapping $f_1\big|_{\alpha=1}$ gives the absolute minimum and $f_3\big|_{\alpha=\alpha_3}$ gives the absolute maximum to $|f(-r_1)|$ in the class Q_K.*

Proof. We prove Proposition 1 for f_3. If $\alpha = \alpha_3$, then $c_3(\alpha_3) = r_2$ and the admissible system of curves γ consists of a curve γ which separates the points $(-r_1), 0$ from $1, \infty$, and homotopic on S to the slit along the segment $[-r_1, 0]$. The corresponding problem of the extremal partition has the domain $D_z = \mathbf{C} \setminus \{[-r_1, 0] \cup [1, \infty)\}$ as the extremal one, see Section 2.

We assume the contrary. Let there be a map $f \in Q_K$ such that $f \not\equiv f_3$ for $\alpha = \alpha_3$ and $|f(-r_1)| \geq -f_3(-r_1)\big|_{\alpha=\alpha_3}$. Then $m_f > m/K$.

Let us denote by D^* the result of circular symmetrization of the doubly connected domain $f(D_z)$ with respect to the origin and the positive real axis. The domain D^* is admissible for the problem of the extremal partition with respect to the admissible curve γ' that separates on $S' = \mathbf{C} \setminus \{0, 1, f_3(-r_1)\big|_{\alpha=\alpha_3}\}$ the punctures $f_3(-r_1)\big|_{\alpha=\alpha_3}, 0$ from others being homotopic on S' to the slit along $\big[f_3(-r_1)\big|_{\alpha=\alpha_3}, 0\big]$. Let D' be the extremal ring domain in this problem of the extremal partition with $m(S', \gamma') = M(D')$. Then the chain of the inequalities

$$m_f \leq M(D^*) \leq M(D') = M\big(f_3\big|_{\alpha=\alpha_3}(D_z)\big) = \frac{1}{K} M(D_z) = m/K$$

holds. This chain is not inconsistent only when $f \equiv f_3\big|_{\alpha=\alpha_3}$. This contradiction proves Proposition 1. For the mapping $f_1\big|_{\alpha=1}$ the proof is similar. $\square$

Now we use the extremal partition of S by two ring domains to derive a result about the range of the system of functionals $(|f(-r_1)|, |f(r_2)|)$ in the class Q_K.

Theorem 1. *For any real $u \geq 0$ with*

$$\min_{f \in Q_K} |f(-r_1)| = -f_1(-r_1)\big|_{\alpha=1} \leq u \leq -f_3(-r_1)\big|_{\alpha=\alpha_3} = \max_{f \in Q_K} |f(-r_1)|$$

among all functions $f_j(z)$, $j = 1, 2, 3$, there is a unique function $f(z, u)$ (more precisely, there are such j and α^* that $f_j(z)\big|_{\alpha=\alpha^*} = f(z, u)$), such that $f(-r_1) = -u$. If $f \in Q_K$ satisfies the condition $|f(-r_1)| = u$, then $|f(r_2)| \leq f(r_2, u)$ with the equality sign only for $f(z) \equiv f(z, u)$.

Proof. The result by Solynin [**19**] implies the continuous dependence of $\varphi_1, \ldots, \varphi_3$ on α and, hence, the same for $f_1, \ldots, f_3$. Moreover, $f_1\big|_{\alpha=\alpha_0} = f_2\big|_{\alpha=\alpha_1}$, $f_2\big|_{\alpha=\alpha_2} = f_3\big|_{\alpha=\alpha_4}$. Therefore, Proposition 1 implies the existence of $f(z, u)$.

We choose j and α^* such that $|f(-r_1)| = -f_j(-r_1)\big|_{\alpha=\alpha^*} = u$ for a function $f \in Q_K$. Assume, for instance, $j = 3$, $\alpha^* \in [\alpha_3, \alpha_4]$ and $f(z,u) \equiv f_3(z)\big|_{\alpha=\alpha^*}$. Then we generate the problem of the extremal partition of $S = \mathbf{C} \setminus \{-r_1, 0, 1, r_2\}$ by the differential $\varphi_3(z)\,dz^2$ with $m = m(S, \gamma, \alpha^*)$ for the admissible system $\gamma = (\gamma_1, \gamma_2)$ with the homotopy defined by the slit along $[-r_1, 0]$. Denote by D_1, D_2 the pair of extremal domains in this problem,

$$m = m(S, \gamma, \alpha^*) = M(D_1) + (\alpha^*)^2 M(D_2).$$

Now we assume the contrary. Let $f \in Q_K$ and suppose that with $f(z,u)$ as above $|f(r_2)| \geq f(r_2, u)$ but $f(z)$ is not identical with $f(z,u)$. We define a problem of the extremal partition of $S' = \mathbf{C} \setminus \{0, 1, -u, f(r_2, u)\}$ for the admissible systems γ' of two loops (γ_1', γ_2') and the vector $(1, \alpha^*)$, where γ_1' is a simple loop which separates the points $f(-r_1, u), 0$ from $1, f(r_2, u), \infty$. The loop γ_1' is homotopic on S' to the slit along $[f(-r_1, u), 0]$. The curve γ_2' is a simple loop which separates the points $1, f(r_2, u)$ from $0, f(-r_1, u), \infty$ with the homotopy defined by γ_1'. Now we apply circular symmetrization [**7, 9**] to the pair of domains $f(D_1), f(D_2)$ with the center at the origin and with the direction along the positive and negative real axes respectively. Denote by D_1^* and D_2^* the result of this symmetrization. This pair of domains is admissible in the problem of the extremal partition of S' for the admissible system γ' and the vector α^* given. Moreover,

$$M(D_1^*) \geq M(f(D_1)), \quad M(D_2^*) \geq M(f(D_2))$$

with the equality only in the case of $D_1^* = f(D_1)$, $D_2^* = f(D_2)$. Rotation is negligible. Therefore,

$$\begin{aligned} \frac{1}{K}\, m &= \frac{1}{K}\,(M(D_1) + (\alpha^*)^2 M(D_2)) \\ &< M(f(D_1)) + (\alpha^*)^2 M(f(D_2)) \\ &\leq M(D_1^*) + (\alpha^*)^2 M(D_2^*) \leq m(S', \gamma', \alpha^*). \end{aligned} \tag{9}$$

The strict inequality sign is because of the uniqueness of $f(z,u)$ in the extremal problem of $\min m_f$. For this extremal function we have

$$\begin{aligned} \frac{1}{K}\,(M(D_1) + (\alpha^*)^2 M(D_2)) &= M(f(D_1, u)) + (\alpha^*)^2 M(f(D_2, u)) \\ &= m_{f(z,u)}. \end{aligned} \tag{10}$$

Taking into account (9)–(10) we observe that this chain of inequalities is not inconsistent only if $f(z) \equiv f(z, u)$ which contradicts our assumption.

The cases $j = 1, 2$ can be considered analogously.

Now we can prove the uniqueness of the choice of α^* and $j = 1, \dots, 3$. For this we assume that there are two different pairs of these parameters which lead to the same point of the boundary of the range of the system of functionals $(|f(-r_1)|, |f(r_2)|)$. One of these pairs we choose as a basic pair and the Teichműller mapping defined by the other one we denote f. Then we can repeat the previous proof and come to the same contradiction. This ends the whole proof. $\square$

4. Boundary parameterization. We construct a new parameterization replacing $\alpha \to t$, $t \in [0, 1]$, to simplify notations. Set

$$x(t) = \begin{cases} -f_1(-r_1)\big|_{\alpha=3t(-1+\alpha_0)+1}, & \text{for } t \in [0, 1/3] \\ -f_2(-r_1)\big|_{\alpha=3t(\alpha_2-\alpha_1)+2\alpha_1-\alpha_2}, & \text{for } t \in [1/3, 2/3] \\ -f_3(-r_1)\big|_{\alpha=3t(\alpha_3-\alpha_4)+3\alpha_4-2\alpha_3}, & \text{for } t \in [2/3, 1], \end{cases}$$

$$y(t) = \begin{cases} f_1(r_2)\big|_{\alpha=3t(-1+\alpha_0)+1}, & \text{for } t \in [0, 1/3] \\ f_2(r_2)\big|_{\alpha=3t(\alpha_2-\alpha_1)+2\alpha_1-\alpha_2}, & \text{for } t \in [1/3, 2/3] \\ f_3(r_2)\big|_{\alpha=3t(\alpha_3-\alpha_4)+3\alpha_4-2\alpha_3}, & \text{for } t \in [2/3, 1]. \end{cases}$$

Theorem 2. *Let $f \in Q_K$, then the upper boundary curve Γ^+ of the range of the system of functionals $(|f(-r_1)|, |f(r_2)|)$, $r_1 > 0$, $r_2 > 1$, i.e., the curve of $\max_{f \in Q_K} |f(r_2)|$ for $|f(-r_1)|$ fixed, is assigned parameterically by $(x(t), y(t))$, $t \in [0, 1]$. This curve is smooth and being considered in the plane (x, y) increases in $t \in [0, 2/3]$ and decreases in $t \in [2/3, 1]$. The tangent vector to Γ^+ is vertical at $t = 0, 1$ and horizontal at $t = 2/3$.*

Proof. By (6)–(8) the functions $A_j(\alpha)$ and $c_j(\alpha)$, for $j = 1, 2, 3$, are differentiable with respect to α in the corresponding intervals; therefore, we have the same for φ_j, f_j. Thus, the functions $x(t)$, $y(t)$ are piecewise differentiable and there are left-side and right-side derivatives at the points 1/3, 2/3. Let us consider the interval $(2/3, 1)$. We have the equality $m_{f_3} = m/K$ in this interval where the modulus m is defined by the extremal differential φ_3. From [**5, 19**] we know that

$$\frac{dm}{d\alpha} = 2\alpha M(D_2), \qquad \frac{\partial m_{f_3}}{\partial \alpha} = 2\alpha M(f_3(D_2));$$
$$\frac{\partial m_{f_3}}{\partial x} = -\pi \mathrm{Res}_{w=-x}\psi(w), \quad \frac{\partial m_{f_3}}{\partial y} = \pi \mathrm{Res}_{w=y}\psi(w).$$

Here we consider $(-x), y$ as simple poles of the extremal differential ψ which is extremal for $m(f_3(S), f_3(\gamma), \alpha)$

$$\psi(w)dw^2 = B\,\frac{w-C}{w(w-1)(w+x)(w-y)}\,dw^2,$$

$C \in (-\infty, -x] \cup [y, \infty)$ when $\alpha \neq 1$ or

$$\psi(w)dw^2 = \frac{-dw^2}{w(w-1)(w+x)(w-y)},$$

otherwise. For $\alpha < 1$ we have $B > 0$, for $\alpha > 1$ we have $B < 0$.

Thus, we obtain the equality

$$\frac{\partial m_{f_3}}{\partial \alpha} + \left(\frac{\partial m_{f_3}}{\partial x}\frac{dx}{dt} + \frac{\partial m_{f_3}}{\partial y}\frac{dy}{dt}\right)\frac{1}{3(\alpha_3 - \alpha_4)} = \frac{1}{K}\frac{dm}{d\alpha}.$$

The mapping f_3 is extremal for the modulus m_f; hence, it maps the extremal configuration in the trajectory structure of the differential φ_3 onto the extremal configuration in the trajectory structure of the differential ψ. Therefore,

$$\frac{\partial m_{f_3}}{\partial \alpha} = \frac{1}{K}\frac{dm}{d\alpha}.$$

The differentiation leads to the derivative

$$\frac{dy}{dx} = \frac{y(y-1)(x+C)}{x(x+1)(y-C)}$$

which exists and negative in all points of (2/3,1). Moreover $dy/dx \to 0$ as $t \to 2/3 \pm 0$ and $dy/dx \to \infty$ as $t \to 1 - 0$. The consideration of the cases $j = 1, 2$ is similar. This ends the proof. $\square$

Remark. From the results by Agard [**1**] and Theorem 1 it easily follows that at the point $t = 1/3$ the argument of the vector $(x(t), y(t))$ admits its maximal value, see the estimate (4).

5. Estimation of functionals.

Theorem 3. *The unique extremal mapping* f^* *gives the maximum to* $|f(r_2)| - |f(-r_1)|$, $r_1 > 0$, $r_2 > 1$ *in the class* Q_K. *This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient* (2) *where*

$$\varphi(z) = \frac{c - z}{z(z-1)(z+r_1)(z-r_2)},$$

and

$$c = \frac{r_1 r_2 (r_1 - r_2 + 2)}{r_2(r_2 - 1) + r_1(1 + r_1)}.$$

Proof. **1.** We look for the extremal functions among those which give the points $(x(t), y(t))$ of the upper boundary curve Γ^+ of the range of the system of functionals $(|f(-r_1)|, |f(r_2)|)$.

Let us consider the point $(x(1/3), y(1/3))$ of Γ^+. Theorem 2 and the remark thereafter imply that

$$\beta = \tan^{-1} \frac{y(1/3)}{x(1/3)} = \tan^{-1} \frac{y'(1/3)}{x'(1/3)}.$$

2. First we suppose that $\beta \leq \pi/4$. Then the extremal function is f_1 for some $\alpha^* \in [\alpha_0, 1)$, which satisfies the necessary condition of extremality for the functional, given by

$$\frac{y'(t^*)}{x'(t^*)} = 1, \quad \alpha^* = 3t^*(-1 + \alpha_0) + 1, \quad t^* \in (0, 1/3].$$

For the function f_1 we have

$$\frac{\partial m_{f_1}}{\partial x} = -\pi \mathrm{Res}_{w=-x} \psi_1(w); \quad \frac{\partial m_{f_1}}{\partial y} = \pi \mathrm{Res}_{w=y} \psi_1(w). \tag{11}$$

Here we consider $(-x), y$ as simple poles of the extremal differential ψ_1,

$$\psi_1(w)\, dw^2 = B \frac{w-C}{w(w-1)(w+x)(w-y)}\, dw^2,$$

$C \in [1, y]$, and $B < 0$. This clearly forces the equality as in Theorem 2

$$\frac{\partial m_{f_1}}{\partial x} \frac{dx}{dt} + \frac{\partial m_{f_1}}{\partial y} \frac{dy}{dt} = 0.$$

By (11) we obtain that for all points α^* and, consequently, t^*, satisfying the necessary condition

$$\frac{y'(t^*)}{x'(t^*)} = \frac{y(y-1)(x+C)}{x(x+1)(y-C)} = 1,$$

we have

$$C = \frac{xy(x-y+2)}{y(y-1)+x(x+1)}, \tag{12}$$

where $C = C(\alpha^*, x, y)$ is a function defined by the conditions for the differential ψ_1

$$\int_{-\infty}^{x} \sqrt{\psi_1(s)}\, ds = 1/2, \quad \int_0^1 \sqrt{\psi_1(s)}\, ds = \alpha^*/2.$$

3. Now we claim

$$c(\alpha^*) = \frac{r_1 r_2 (r_1 - r_2 + 2)}{r_2(r_2-1) + r_1(1+r_1)}. \tag{13}$$

The equality (13) implies that there exist a unique α^* and t^* satisfying the necessary condition which, therefore, becomes sufficient, because the extremal function $f(z,u)$ is unique for each point of Γ^+. To prove this we need more refined observations.

For α^* fixed we consider the quadratic differential

$$\widetilde{\varphi}_1(z)\, dz^2 = \tilde{A}_1 \frac{z - \tilde{c}_1(u,v)}{z(z-1)(z+u)(z-v)}\, dz^2, \tag{14}$$

where $\tilde{c}_1 \in [1, v]$, $\tilde{A} < 0$. These values are calculated by the equations

$$\int_{-\infty}^{-u} \sqrt{\widetilde{\varphi}_1(s)}\, ds = 1/2, \quad \int_0^1 \sqrt{\widetilde{\varphi}_1(s)}\, ds = \alpha^*/2,$$

where the real valued differentiable functions $u = u(\mu)$, $v = v(\mu)$ accept their values from some neighborhood of (r_1, r_2), $\mu \in (-\varepsilon, \varepsilon)$, $u(0) = r_1$, $v(0) = r_2$. Now let us construct the Teichműller mapping $\tilde{f}$ as the solution of the Beltrami equation (1) with the Beltrami coefficient (2) for the quadratic differential (14) with the normalization of the class Q_K. Denote by $x(\alpha^*, u, v) = |\tilde{f}(-r_1)|$, $y(\alpha^*, u, v) = |\tilde{f}(r_2)|$. These functions are differentiable with respect to u and v as the values of $\tilde{f}$ which is the solution of the Beltrami equation with the Beltrami coefficient that is differentiable with respect to u and v. Moreover, we have that $\tilde{c}_1(r_1, r_2) = c_1(\alpha^*)$, $x(\alpha^*) = x(\alpha^*, r_1, r_2)$, $y(\alpha^*) = y(\alpha^*, r_1, r_2)$. Since $\tilde{f} \in Q_K$, the points $(x(\alpha^*, u(\mu), v(\mu)), y(\alpha^*, u(\mu), v(\mu)))$ form a curve assigned by the parameter μ which touches the boundary curve Γ^+ at the point $(x(\alpha^*), y(\alpha^*))$.

Denote by $m(u, v)$ the modulus defined by the differential (14) and by $m_{\tilde{f}}$ the modulus defined by the Teichműller mapping $\tilde{f}$. Of course, $m_{\tilde{f}} = \frac{1}{K} m(u, v)$. Now we differentiate this equality with respect to u and v

$$\frac{1}{K}\frac{\partial m(u,v)}{\partial u} = \frac{\partial m_{\tilde{f}}}{\partial x} \cdot \frac{\partial x(\alpha^*, u, v)}{\partial u} + \frac{\partial m_{\tilde{f}}}{\partial y} \cdot \frac{\partial y(\alpha^*, u, v)}{\partial u}, \tag{15}$$

$$\frac{1}{K}\frac{\partial m(u,v)}{\partial v} = \frac{\partial m_{\tilde{f}}}{\partial x} \cdot \frac{\partial x(\alpha^*, u, v)}{\partial v} + \frac{\partial m_{\tilde{f}}}{\partial y} \cdot \frac{\partial y(\alpha^*, u, v)}{\partial v}. \tag{16}$$

Here in the righthand sides of these equalities the partial derivatives are taken at the simple poles x, y of the extremal differential for $m_{\tilde{f}}$. Taking into account the rule of differentiating of the modulus we obtain that

$$\begin{aligned} &-\frac{u(u+1)(v - \tilde{c}_1(u,v))}{v(v-1)(u + \tilde{c}_1(u,v))} \\ &\quad = \frac{[(\partial m_{\tilde{f}}/\partial x)(\partial x(\alpha^*, u, v)/\partial v)] + [(\partial m_{\tilde{f}}/\partial y)(\partial y(\alpha^*, u, v)/\partial v)]}{[(\partial m_{\tilde{f}}/\partial x)(\partial x(\alpha^*, u, v)/\partial u)] + [(\partial m_{\tilde{f}}/\partial y)(\partial y(\alpha^*, u, v)/\partial u)]}. \end{aligned} \tag{17}$$

Observe, that

$$\frac{\partial m_{\tilde f}}{\partial x}\Big|_{u=r_1,\, v=r_2} = \frac{\partial m_{f_1}}{\partial x}\Big|_{\alpha=\alpha^*}, \quad \frac{\partial m_{\tilde f}}{\partial y}\Big|_{u=r_1,\, v=r_2} = \frac{\partial m_{f_1}}{\partial y}\Big|_{\alpha=\alpha^*}.$$

Then the equalities (11) and (12) imply

$$\frac{\partial m_{\tilde f}}{\partial x}\Big|_{u=r_1,\, v=r_2} = -\frac{\partial m_{\tilde f}}{\partial y}\Big|_{u=r_1,\, v=r_2}.$$

Now we choose the parameterization by $u = r_1 + \mu/2$, $v = r_2 - \mu/2$. Since the curve $(x(\alpha^*, u(\mu), v(\mu)), y(\alpha^*, u(\mu), v(\mu)))$, $\mu \in (-\varepsilon, \varepsilon)$, given by the parameter μ touches the boundary curve of Theorem 3 at the point $\mu = 0$, we have

$$\begin{aligned}
&\frac{d(y(\alpha^*, u, v) - x(\alpha^*, u, v))}{d\mu}\Big|_{\mu=0} \\
&\quad = \frac{1}{2}\left(\frac{\partial(y(\alpha^*, u, v) - x(\alpha^*, u, v))}{\partial u} - \frac{\partial(y(\alpha^*, u, v) - x(\alpha^*, u, v))}{\partial v}\right)\Big|_{\mu=0} \\
&\quad = 0,
\end{aligned}$$

and the righthand side of the equality (17) reduces to 1 at $\mu = 0$, $(u, v) = (r_1, r_2)$. This leads to the value $\tilde c_1(r_1, r_2) = c(\alpha^*)$ given by (13).

4. Now let $\beta > \pi/4$. Then we repeat all previous observations for the function f_2 and $x(t), y(t)$ for $t^* \in (1/3, 2/3)$. As a result, we obtain that the value of $c(\alpha^*)$ is given by the same formula (13). Thus, the extremal function f^* is either f_1 or f_2, dependent on the angle β, but is given by the same Beltrami coefficient. This completes the proof. $\square$

If $|f(r_2)| \le |f(-r_1)|$ for all $f \in Q_K$, then the result of Theorem 3 gives us the sharp lower estimate of $||f(r_2)| - |f(-r_1)||$. We have the equivalence $\mathbf{K}'(1/\sqrt{2}) = \mathbf{K}(1/\sqrt{2})$. Thus, Theorem 3 and the estimate (4) imply the following useful corollary.

Corollary. *Let $f \in Q_K$ and $r_1 > 0$, $r_2 > 1$, and*

$$\frac{r_2}{r_1} \le \frac{(\mu^{-1}(K/2))^2}{1 - (\mu^{-1}(K/2))^2}.$$

Then $|f(r_2)+f(-r_1)| \geq -(f^*(r_2)+f^*(-r_1))$. *This estimate is sharp for the unique function* f^* *of Theorem* 3.

Theorem 4. *Let us set*

$$c = \frac{r_1 r_2}{1+r_1-r_2}.$$

The unique extremal mapping f^{**} *delivers the maximum of* $|f(r_2) - f(-r_1)|$, $r_1 > 0$, $r_2 > 1$ *in the class* Q_K. *This mapping satisfies the Beltrami equation (1) with the Beltrami coefficient (2) where*

$$\varphi(z) = \frac{c-z}{z(z-1)(z+r_1)(z-r_2)}, \quad \text{for } r_2 - r_1 > 1,$$
$$\varphi(z) = \frac{z-c}{z(z-1)(z+r_1)(z-r_2)}, \quad \text{for } r_2 - r_1 < 1,$$
$$\varphi(z) = \frac{-1}{z(z-1)(z+r_1)(z-r_2)}, \quad \text{for } r_2 - r_1 = 1.$$

Proof. **1.** Observe that the inequality $|f(r_2) - f(-r_1)| \leq |f(r_2)| + |f(-r_1)|$ implies that we have to look for the extremal functions among those which give the points $(x(t), y(t))$ of the upper boundary curve for the system of functionals $(|f(-r_1)|, |f(r_2)|)$ for $t \in (2/3, 1)$.

Then the extremal function is f_3 at some $\alpha^* \in (\alpha_4, \alpha_3)$ that satisfies the necessary condition of the extremality

$$\frac{y'(t^*)}{x'(t^*)} = -1, \quad \alpha^* = 3t^*(-1+\alpha_0)+1, \quad t^* \in (2/3, 1).$$

For the function f_3 we have the relation (11) for the extremal differential ψ_3,

$$\psi_3(w)\, dw^2 = B\, \frac{w-C}{w(w-1)(w+x)(w-y)}\, dw^2,$$

$C \in (-\infty, -x) \cup (y, \infty)$ if $\alpha^* \neq 1$, and

$$\psi_3(w)\, dw^2 = -\frac{1}{w(w-1)(w+x)(w-y)}\, dw^2,$$

if $\alpha^* = 1$. $B > 0$ for $\alpha < 1$ and $B < 0$ for $\alpha > 1$. By this, we obtain the equality as in Theorem 3

$$\frac{\partial m_{f_3}}{\partial x}\frac{dx}{dt} + \frac{\partial m_{f_3}}{\partial y}\frac{dy}{dt} = 0.$$

Suppose $\alpha^* \neq 1$. With (11) we obtain that the equality

$$\frac{y'(t^*)}{x'(t^*)} = \frac{y(y-1)(x+C)}{x(x+1)(y-C)} = -1$$

holds for all α^* which satisfy the necessary condition of the extremality, or

$$C = \frac{xy}{1+x-y},$$

where $C = C(\alpha^*, x, y)$ is a function defined by the conditions for the differential ψ_3

$$\int_{-x}^{0} \sqrt{\psi_3(s)}\, ds = 1/2, \quad \int_{1}^{y} \sqrt{\psi_3(s)}\, ds = \alpha^*/2.$$

2. Now we claim

$$c(\alpha^*) = \frac{r_1 r_2}{1+r_1-r_2}.$$

To prove this we repeat the proof of Theorem 4 with the same parameterization dependent on μ. The case $\alpha^* = 1$ one can consider as the limiting case as $\alpha^* \to 1$. Negative or positive constant in front of φ_3 leads to different cases of the function φ in Theorem 4. This completes the proof. □

The same observations for positive fixed points of a function $f \in Q_K$ lead to the following results.

Theorem 5. *The unique extremal mapping f^* gives the maximum to $|f(r_2) + f(r_1)|$, $1 < r_1 < r_2$ in the class Q_K. This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient* (2) *with*

$$\varphi(z) = \frac{c-z}{z(z-1)(z-r_1)(z-r_2)},$$

where

$$c = \frac{r_1 r_2 (r_1 + r_2 - 2)}{r_2(r_2 - 1) + r_1(r_1 - 1)}.$$

The unique extremal mapping f^{**} *gives the maximum to* $|f(r_2)| - |f(r_1)|$, $1 < r_1 < r_2$ *in the class* Q_K. *This mapping satisfies the Beltrami equation* (1) *with the Beltrami coefficient* (2) *with the same function* $\varphi(z)$ *and*

$$c = \frac{r_1 r_2}{r_1 + r_2 - 1}.$$

Theorems 3, 4 and 5 is the contents of Theorem A in the introduction.

6. Conclusions and unsolved problems. We start with the conformal case. Let S be a class of all holomorphic and univalent functions in the unit disk $U = \{z : |z| < 1\}$ with the normalization $f(0) = 0$, $f'(0) = 1$. Denote by S_R its subclass of symmetric functions satisfying the additional condition $\overline{f(z)} = f(\bar{z})$. The earlier result by Jenkins [**9**] asserts that the upper boundary curve of the range of the system of functionals $(|f(-r_1)|, |f(r_2)|)$, $0 < r_1, r_2 < 1$, over the class S coincides with that over the class S_R. By the words "upper" and "lower" we say that the points $(|f(-r_1)|, |f(r_2)|)$ are considered as points of real plane $\mathbf{R}^2$. In Jenkins' proof it was important that the fixed points $(-r_1)$ and r_2 were situated in different legs of the real diameter. Later on the author and Fedorov [**21**] have proved the same result for the system of functionals $(|f(r_1)|, |f(r_2)|)$, $0 < r_1 < r_2 < 1$. As for the lower boundary curves for these ranges, they are different in the classes S and S_R. In S_R the points of the lower boundary curves are delivered by the simple function $z(1 - uz + z^2)^{-1}$, $-2 \le u \le 2$. The points of the upper boundary curves are delivered by functions with more complicated structure. They have been described in [**9**] and [**21**].

Now in the quasiconformal case we can introduce a subclass Q_K^R of the class Q_K of functions satisfying the same symmetry condition $\overline{f(z)} = f(\bar{z})$. This class plays the same role for the class Q_K as the subclass S_R for the class S. One can easily see that the upper boundary curve of the range of the system of functionals $(|f(-r_1)|, |f(r_2)|)$, $r_1 > 0$, $r_2 > 1$ in the class Q_K coincides with that in the class Q_K^R. So, this result is close to [**9**] and [**21**]. Of course, the method of the proof is different. The extra normalization $f(1) = 1$ yields another difficulty.

So, we are not able by now to prove analogous result for the system of functionals $(|f(-r_1)|, |f(r_2)|)$, $r_1 > 0$, $0 < r_2 < 1$ or for the system of functionals $(|f(r_1)|, |f(r_2)|)$, $0 < r_1 < 1$, $r_2 > 1$. But for the system of functionals $(|f(r_1)|, |f(r_2)|)$, $1 < r_1 < r_2$ this is the case as the author talked at the International Congress of Mathematicians, Berlin, 1998 [**23**]. In [**22**] the author has shown this for K-quasiconformal homeomorphisms of the unit disk U.

From these results we deduce (Theorem A) the sharp estimates for the functionals $|f(r_2)| \pm |f(\pm r_1)|$ and $|f(r_2) \pm f(\pm r_1)|$. The proofs open a way to present extremal functions in terms of Beltrami coefficient of direct mapping. But the solution of the problem for $|f(r_2) \pm f(\mp r_1)|$ is still unknown. The same is for the lower estimations of the functional $|f(r_2)|/|f(\pm r_1)|$.

So we consider the following problems as interesting and difficult to resolve:

1) To find the lower boundary curve of the range of the system of functionals $(|f(r_1)|, |f(r_2)|)$ in the class Q_K for different real r_1, r_2.

2) To find the upper boundary curve of the range of the system of functionals $(|f(r_1)|, |f(r_2)|)$ in the class Q_K for different r_1, r_2 so that one of r_1, r_2 lies in the segment $(0, 1)$ (for $r_1, r_2 \in (0, 1)$ this can be obtained the same way it was presented in this paper and, therefore, it is not so interesting).

3) To obtain the sharp estimates of $|f(r_2) - f(r_1)|$ and $|f(r_2) + f(-r_1)|$ for $r_2, r_1 > 0$ in terms of Beltrami coefficient of direct mapping.

4) To obtain the lower sharp estimates of $|f(r_2)|/|f(\pm r_1)|$ for different real values of r_1 and r_2.

Here we do not speak about nonreal fixed points because our method is based on symmetric structures.

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Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
E-mail address: `alexander.vasiliev@mat.utfsm.cl`