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WEAK SYMMETRY IN NATURALLY REDUCTIVE HOMOGENEOUS NILMANIFOLDS

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ABSTRACT. We prove that within the class of naturally reductive homogeneous nilmanifolds, the notions of weak symmetry, i.e., any two points can be interchanged by an isometry, and commutativity, i.e., isometry invariant differential operators commute, are equivalent.

A connected Riemannian manifold M is said to be *weakly symmetric* if for any two points $p, q \in M$ there exists an isometry of M mapping p to q and q to p. These spaces were introduced by Selberg in the framework of his development of the trace formula, see [**12**], where it is proved that in a weakly symmetric space M, the algebra of all invariant (with respect to the full isometry group $I(M)$) differential operators on M is commutative, that is, M is a *commutative* space. Selberg asks in [**12**] whether the converse holds. The answer is negative, the known counterexamples arise in certain homogeneous nilmanifolds, so-called H-type groups, see [**8, 9**]. On the other hand, it has been proved by Akhiezer and Vinberg [**1**] that, for homogeneous spaces of reductive algebraic groups, the answer is affirmative.

In such a way, a natural question takes place: under what extra conditions on the homogeneous Riemannian manifold M, the weak symmetry is necessary for the commutativity of $I(M)$-invariant differential operators?

With such a question in mind, the first observation we make is that none of the counterexamples found is naturally reductive. A Riemannian manifold M is said to be *naturally reductive*, if there exists a transitive Lie group G of isometries with isotropy subgroup K at $p \in M$, and an $\mathrm{Ad}\,(K)$-invariant vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary

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to $\mathfrak{l}$, such that all the geodesics emanating from p are of the form $\exp(tx).p$ for some $x \in \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{l}$ denote the Lie algebras of G and K, respectively, see [**5, 7**] for definitions and properties of these and others symmetric-like Riemannian spaces.

In this paper we take advantage of the classification of naturally reductive homogeneous nilmanifolds which are commutative spaces given in [**10, 11**], to study via a case-by-case method the weak symmetry condition on these spaces. We prove that all of them are weakly symmetric as well, obtaining that within the class of naturally reductive homogeneous nilmanifolds, the converse of Selberg's theorem is valid.

We note that the commutativity of a space is actually defined considering invariance with respect to the connected component $I(M)^0$ of the full isometry group $I(M)$ instead of $I(M)$ itself. As far as we know, the equivalence of these two notions is still an open problem. However in the class of homogeneous nilmanifolds, both notions coincide, see [**3**].

A *homogeneous nilmanifold* is a real nilpotent Lie group N endowed with a left-invariant Riemannian metric, denoted by $(N, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the inner product on the Lie algebra $\mathfrak{n}$ of N determined by the metric. If we assume that N is simply connected, then the full group of isometries of $(N, \langle \cdot, \cdot \rangle)$ is given by

$$I(N, \langle \cdot, \cdot \rangle) = K \ltimes N \quad \text{(semi-direct product)}, \tag{1}$$

where $K = \operatorname{Aut}(\mathfrak{n}) \cap O(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is the isotropy subgroup of the identity and N acts on itself by left translations, see [**13**].

Theorem 1 [**2, 6**]. *Let $(N, \langle \cdot, \cdot \rangle)$ be a homogeneous nilmanifold satisfying any of the following symmetric-like conditions: weak symmetry, commutativity or natural reductivity. Then N must be two-step nilpotent or abelian.*

In [**10**], it has been proved that any naturally reductive homogeneous nilmanifold can be constructed as follows. Consider a data set $(\mathfrak{g}, V, \langle \cdot, \cdot \rangle)$, where

(i) $\mathfrak{g}$ is a compact Lie algebra, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is a compact semi-simple Lie algebra,

(ii) (π, V) is a real faithful representation of $\mathfrak{g}$ without trivial subrepresentations, i.e., $\cap_{x \in \mathfrak{g}} \operatorname{Ker} \pi(x) = 0$,

(iii) $\langle\cdot,\cdot\rangle$ is an inner product (positive definite) on $\mathfrak{n}=\mathfrak{g}\oplus V$ satisfying that $\langle\cdot,\cdot\rangle_{\mathfrak{g}} := \langle\cdot,\cdot\rangle|_{\mathfrak{g}\times\mathfrak{g}}$ is $\mathrm{ad}\,\mathfrak{g}$-invariant, $\langle\cdot,\cdot\rangle_V := \langle\cdot,\cdot\rangle|_{V\times V}$ is $\pi(\mathfrak{g})$-invariant and $\langle\mathfrak{g},V\rangle = 0$.

Such a data set determines a two-step nilpotent Lie group denoted by $N(\mathfrak{g},V)$ having Lie algebra $\mathfrak{n}=\mathfrak{g}\oplus V$, with Lie bracket defined by

$$\begin{cases} [\mathfrak{g},\mathfrak{g}]_{\mathfrak{n}} = [\mathfrak{g},V]_{\mathfrak{n}} = 0 & [V,V]_{\mathfrak{n}} \subset \mathfrak{g}, \\ \langle [v,w]_{\mathfrak{n}}, x\rangle_{\mathfrak{g}} = \langle \pi(x)v, w\rangle_V & \forall\, x\in\mathfrak{g}, v,w\in V. \end{cases} \tag{2}$$

Finally we endow $N(\mathfrak{g},V)$ with the left-invariant metric determined by $\langle\cdot,\cdot\rangle$.

Note that $[\mathfrak{n},\mathfrak{n}]_{\mathfrak{n}} = \mathfrak{g}$ is the center of $\mathfrak{n}$. The construction of the group $N(\mathfrak{g},V)$ does not depend on the chosen $\mathfrak{g}$-invariant inner product $\langle\ ,\ \rangle$, up to Lie group isomorphism.

Theorem 2 [**10**]. *A simply connected homogeneous nilmanifold without Euclidean factor is naturally reductive if and only if it is isometric to a space* $N(\mathfrak{g},V)$ *for some data set* $(\mathfrak{g},V\langle\ ,\ \rangle)$ *satisfying conditions* (i), (ii) *and* (iii) *above.*

We shall now describe the isotropy subgroup K of the isometry group of $N(\mathfrak{g},V)$. It is easy to prove that

(3)

$$K = \{(\phi,T)\in \mathbf{O}(\mathfrak{g},\langle\cdot,\cdot\rangle)\times\mathbf{O}(V,\langle\cdot,\cdot\rangle) : T\pi(x)T^{-1} = \pi(\phi x),\ x\in\mathfrak{g}\}.$$

This implies that $G\times U\subset K$, where $U=\mathrm{End}_{\mathfrak{g}}(V)\cap\mathbf{O}(V,\langle\cdot,\cdot\rangle)$ is the group of orthogonal intertwining operators of V and G is any Lie group with Lie algebra $[\mathfrak{g},\mathfrak{g}]$. The group U acts trivially on $\mathfrak{g}$ and each $g\in G$ acts on $\mathfrak{n}=\mathfrak{g}\oplus V$ by $(\mathrm{Ad}\,(g),\pi(g))$, where we also denote by π the corresponding representation of G on V. Moreover, it is proved in [**10**] that the connected component of the identity of K is given by

$$K^0 = G\times U^0,$$

where $G=\overline{G}/\mathrm{Ker}\,\pi$ and $\overline{G}$ is the simply connected Lie group with Lie algebra $[\mathfrak{g},\mathfrak{g}]$.

The notion of commutativity in the class of naturally reductive homogeneous nilmanifolds has been studied in [**11**], where the following classification was obtained.

Theorem 3 [**11**]. *The two-step nilpotent Lie groups $N(\mathfrak{g}, V)$ which are commutative spaces are*

(i) $N(\mathfrak{su}(2), (\mathbf{C}^2)^n)$, *$n \geq 1$, Heisenberg-type, where $(\mathbf{C}^2)^n$ denotes the direct sum of n copies of the standard representation $\mathbf{C}^2$ of $\mathfrak{su}(2)$ regarded as a real representation.*

(ii) $N(\mathfrak{su}(2), \mathbf{R}^3 \oplus (\mathbf{C}^2)^n)$, *$n \geq 0$, where $\mathbf{R}^3$ is the standard representation of $\mathfrak{so}(3) = \mathfrak{su}(2)$.*

(iii) $N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), (\mathbf{C}^2)^{k_1} \oplus \mathbf{R}^4 \oplus (\mathbf{C}^2)^{k_2})$, *$k_1, k_2 \geq 0$, where the first copy of $\mathfrak{su}(2)$ acts only on $(\mathbf{C}^2)^{k_1}$ and the second one only on $(\mathbf{C}^2)^{k_2}$ and $\mathbf{R}^4$ denotes the standard representation of $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.*

(iv) $N(\mathfrak{sp}(2), (\mathbf{C}^4)^k)$, *$k \geq 1$, where $\mathbf{C}^4$ denotes the standard representation of $\mathfrak{sp}(2)$.*

(v) $N(\mathfrak{su}(n), \mathbf{C}^n)$, $n \geq 3$.

(vi) $N(\mathfrak{so}(n), \mathbf{R}^n)$, *$n \geq 5$, free two-step nilpotent.*

(vii) $N(\mathbf{R}, \mathbf{C}^k)$, *$k \geq 1$, Heisenberg groups.*

(viii) $N(\mathfrak{u}(2), (\mathbf{C}^2)^k \oplus (\mathbf{C}^2)^n)$, *$k \geq 1$, $n \geq 0$, where the center of $\mathfrak{u}(2)$ acts nontrivially only on $(\mathbf{C}^2)^k$.*

(ix) $N(\mathfrak{u}(n), \mathbf{C}^n)$, $n \geq 3$.

(x) $N(\mathfrak{su}(m_1) \oplus \cdots \oplus \mathfrak{su}(m_r) \oplus \mathfrak{c},\ V_1 \oplus \cdots \oplus V_r)$ *with the following actions*: *$\mathfrak{su}(m_i)$ acts trivially on V_j for all $i \neq j$ and $\dim \mathfrak{c}_i = 1$ where $\mathfrak{c}_i$ denotes the maximal subspaces of the center $\mathfrak{c}$ acting nontrivially on V_i. Moreover, if $m_i = 2$, then $V_i = (\mathbf{C}^2)^{k_i} \oplus (\mathbf{C}^2)^{n_i}$ as in (ii), and if $m_i \geq 3$, then $V_i = \mathbf{C}^{m_i}$.*

(xi) *A direct product of some of the spaces given above.*

The weakly symmetric homogeneous nilmanifolds are characterized as follows.

Theorem 4 [**8**]. *$(N, \langle \cdot, \cdot \rangle)$ is a weakly symmetric space if and only if, for any $x \in \mathfrak{n}$, there exists $\varphi \in K$ such that $\varphi x = -x$ where K denotes the isotropy subgroup of the isometry group of $(N, \langle \cdot, \cdot \rangle)$.*

We shall prove now the main result of this paper.

Theorem 5. *All the commutative spaces listed in Theorem* 3 *are weakly symmetric as well. Consequently, in the class of naturally reductive homogeneous nilmanifolds, see Theorem* 2*, the notions of commutativity and weak symmetry coincide.*

Proof. To prove the weak symmetry of most of the commutative spaces in Theorem 3, excluding cases (iv) and (viii), we will make use of the following sufficient condition

(C) for each $x \in \mathfrak{g}$ there exists $\phi \in K$ such that $\phi x = -x$ and ϕ leaves invariant a decomposition $V = V_1 \oplus \cdots \oplus V_r$ for which the stabilizer subgroup

$$K_x = \{\varphi \in K : \varphi x = x\}$$

acts transitively on the product $S_1 \times \cdots \times S_r$, where S_i denotes any sphere of V_i.

Indeed, if condition (C) holds, we decompose each $v \in V$ as $v = v_1 + \cdots + v_r$ with $v_i \in V_i$ and thus $k\phi(x,v) = (-x,-v)$, where $k \in K_x$ satisfy $k\phi v_i = -v_i$, $i = 1, \ldots, r$. The weak symmetry thus follows from Theorem 4.

In what follows we shall prove the weak symmetry of the commutative spaces listed in Theorem 3 by analyzing the different cases. Note that, for any $x \in \mathfrak{g}$ we always have $C_G(x) \times U \subset K_x$ where $C_G(x)$ denotes the centralizer of x in G.

Case (i), (vii). We note that the weak symmetry of these spaces has already been proved in [**4**], in the context of Heisenberg-type Lie groups.

Case (v). For $x \in \mathfrak{su}(n)$, let β be a basis of $\mathbf{C}^n$ such that in terms of β,

$$x = \begin{bmatrix} ia_1 & & \\ & \vdots & \\ & & ia_n \end{bmatrix}, \quad a_i \in \mathbf{R}.$$

Consider $\phi : \mathfrak{su}(n) \to \mathfrak{su}(n)$ the outer automorphism of $\mathfrak{su}(n)$ given by $\phi(A) = \overline{A} = -A^t$, where each $A \in \mathfrak{su}(n)$ is written in terms of the basis β. We extend ϕ to $\mathfrak{n} = \mathfrak{su}(n) \oplus \mathbf{C}^n$ defining $\phi|_{\mathbf{C}^n}$ by $\phi(c_1, \ldots, c_n) = (\overline{c_1}, \ldots, \overline{c_n})$. Using (3), it is easy to see that

$\phi \in K$. In view of the above paragraph, since ϕ leaves invariant the decomposition $V = \mathbf{C} \oplus \cdots \oplus \mathbf{C}$ and the group of diagonal unitary matrices $\mathfrak{U}(1) \times \cdots \times \mathfrak{U}(1) \subset K_x$ acts transitively on the product of spheres $S^1 \times \cdots \times S^1$, we obtain that condition (C) holds and thus $N(\mathfrak{su}(n), \mathbf{C}^n)$ is a weakly symmetric space.

Case (ii). If

$$x = \begin{bmatrix} ia & \\ & -ia \end{bmatrix} \in \mathfrak{su}(2), \quad a \in \mathbf{R},$$

we consider the same ϕ as that in the above case and such that the extension on $\mathbf{R}^3$ is given by

$$\phi|_{\mathbf{R}^3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $\mathbf{R}^3 = \mathbf{R}^2 \oplus \mathbf{R}$ is the eigenspace decomposition of $\pi(x)|_{\mathbf{R}^3}$. We then have the decomposition $V = \mathbf{R}^2 \oplus \mathbf{R} \oplus (\mathbf{C}^2)^n$, where the subgroup $\mathfrak{U}(1) \times \mathrm{Sp}\,(n) \subset K_n$ acts transitive on the product of spheres $S^1 \times S^{4n-1}$. We may prove case (i), respectively (vii), in the same way with $V = (\mathbf{C}^2)^n$ and $\mathrm{Sp}\,(n) \subset K_x$, respectively $V = \mathbf{C}^n$ and $\mathfrak{U}(n) \subset K_x$.

Case (iii). If the element

$$x = \begin{bmatrix} 0 & -a & & \\ a & 0 & & \\ & & 0 & -b \\ & & b & 0 \end{bmatrix} = \left(\begin{bmatrix} ia & 0 \\ 0 & -ia \end{bmatrix}, \begin{bmatrix} ib & 0 \\ 0 & -ib \end{bmatrix} \right)$$
$$\in \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

for some $a, b \in \mathbf{R}$, then the complex conjugation map ϕ considered in the above cases coincides with $\mathrm{Ad}\,(T)$ where

$$T = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}.$$

Thus the extension of ϕ to an element of K is given on V by conjugation on the copies of $\mathbf{C}^2$ and on $\mathbf{R}^4$ by $\phi|_{\mathbf{R}^4} = T$. The decomposition in condition (C) is given by $V = (\mathbf{C}^2)^{k_1} \oplus \mathbf{R}^2 \oplus \mathbf{R}^2 \oplus (\mathbf{C}^2)^{k_2}$ and we have a subgroup $\mathrm{Sp}\,(k_1) \times \mathfrak{U}(1) \times \mathfrak{U}(1) \times \mathrm{Sp}\,(k_2) \subset K_x$ acting transitively on the corresponding product of spheres.

Case (vi). The weak symmetry of this space has been already proved in [**14**]. With respect to a suitable basis we write

$$x = \begin{bmatrix} 0 & -a_1 & & & & \\ a_1 & 0 & & & & \\ & & \vdots & & & \\ & & & 0 & -a_k & \\ & & & a_k & 0 & \\ & & & & & 0 \end{bmatrix} \in \mathfrak{so}(n), \quad a_i \in \mathbf{R},$$

thus we may consider $\phi : \mathfrak{so}(n) \to \mathfrak{so}(n)$ given by $\phi(A) = TAT^{-1}$, extended by $\phi|_{\mathbf{R}^n} = T$, where

$$T = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \vdots & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & -1 \end{bmatrix},$$

with the last column and row deleted if n is even. Using (3) it is easy to see that $\phi \in K$. Hence ϕ leaves invariant the decomposition $V = \mathbf{R}^2 \oplus \cdots \oplus \mathbf{R}^2$ and the subgroup of 2×2 blocks orthogonal matrices $\mathrm{SO}\,(2) \times \cdots \times \mathrm{SO}\,(2) \subset K_x$ acts transitively on the product of spheres $S^1 \times \cdots \times S^1$.

Case (iv). It is convenient to consider the realization of $\mathfrak{sp}(2)$ as the 2×2 skew-Hermitian matrices with coefficients in $\mathbf{H}$, the quaternionic numbers. Thus $V = (\mathbf{C}^4)^k$ can be viewed as the space of matrices $M_{2\times k}(\mathbf{H})$, on which $\mathfrak{sp}(2)$ and $\mathrm{Sp}\,(2)$ acts by left multiplication and the group of orthogonal intertwining operators $U = \mathrm{Sp}\,(k)$ acts by right multiplication. As a generic $x \in \mathfrak{sp}(2)$ we can consider

$$x = \begin{bmatrix} ia & \\ & ib \end{bmatrix}, \quad a \neq b \in \mathbf{R},$$

and we can assume that

$$v = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ c_1 & c_2 & \cdots & c_k \end{bmatrix}, \quad c_i \in \mathbf{H},$$

since Sp (k) acts transitively on the sphere of $\mathbf{H}^k$. If $\phi = \left[\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix}\right] \in \mathrm{Sp}\,(2)$, then $\phi x \phi^{-1} = -x$. Now it is easy to see that there exists $\alpha, \beta \in \mathbf{R}$ such that $e^{i\beta} j c_1 j e^{-i\alpha} - c_1$ where $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$. This implies that

$$\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{bmatrix} \phi v T = -v, \quad \mathrm{Ad}\,(\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{bmatrix} \phi) x = -x,$$

where $T = \left[\begin{smallmatrix} je^{-i\alpha} & \\ & T_1 \end{smallmatrix}\right]$ and $T_1 \in \mathrm{Sp}\,(k-1)$ satisfies $[e^{i\beta} j c_2, \dots, e^{j\beta} j c_k] T_1 = [-c_2, \dots, -c_k]$. It follows from Theorem 4 that $N(\mathfrak{sp}(2), (\mathfrak{c}^4)^k)$ is a weakly symmetric space.

Case (viii). Let us consider the realization of $(\mathbf{C}^2)^k$ as the space of matrices $M_{2\times k}(\mathbf{C})$, on which $\mathfrak{u}(2)$ acts by left multiplication and the group of orthogonal intertwining operators $\mathbf{U}(k) \subset U$ acts by right multiplication. With respect to some basis β of $\mathbf{C}^2$, a generic $x \in \mathfrak{u}(2)$ is given by

$$x = \begin{bmatrix} ia & \\ & ib \end{bmatrix}, \quad a, b \in \mathbf{R},$$

and we can assume that $v \in V = M_{2\times k}(\mathbf{C}) \oplus (\mathbf{C}^2)^n$ is of the form

$$v = \left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ c_1 & c_2 & \cdots & c_k \end{bmatrix}, v_2 \right), \quad c_i \in \mathbf{C}, \quad v_2 \in (\mathbf{C}^2)^n,$$

since $\mathfrak{U}(k)$ acts transitively on the sphere of $\mathbf{C}^k$. Recall that we do not have to pay attention to v_2 since Sp $(n) \subset U$ acts transitively on the sphere of $(\mathbf{C}^2)^n$. Consider as in case (v), $\phi : \mathfrak{u}(2) \to \mathfrak{u}(2)$ the outer automorphism of $\mathfrak{u}(2)$ given by $\phi(A) = \overline{A} = -A^t$, where each $A \in \mathfrak{u}(2)$ is written in terms of the basis β. We also extend ϕ to an element of K, making it act by complex conjugation on V. Note that $\phi x = -x$. Now, if we take $\alpha \in S^1$ such that $\alpha^2 = \overline{c_1}/c_1$ ($\alpha = 1$ if $c_1 = 0$), then

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \phi v T = -v \quad \text{and} \quad \mathrm{Ad}\,(\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}) \phi x = -x,$$

where

$$T = \begin{bmatrix} -\alpha^{-1} & 0 \\ 0 & T_1 \end{bmatrix}$$

and $T_1 \in \mathfrak{U}(k-1)$ satisfies $[\alpha^{-1}\overline{c_2}, \dots, \alpha^{-1}\overline{c_k}]T_1 = [-c_2, \dots, -c_k]$. Thus, the weak symmetry of $N(\mathfrak{u}(2), (\mathbf{C}^2)^k \oplus (\mathbf{C}^2)^n)$ follows from Theorem 4.

Case (ix). Idem to Case (v).

Case (x). The proof of this case follows easily from the already proved weak symmetry of each one of the spaces $N(\mathfrak{su}(m_i), V_i)$. $\square$

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