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A CRITERION FOR LINEAR INDEPENDENCE OF SERIES

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ABSTRACT. The paper establishes a criterion for linear independence of infinite series which consist of rational numbers. A criterion for irrationality is obtained as a consequence.

1. Introduction. There are many papers concerning the algebraic independence of infinite series. Among them we can cite Töpfer [**14**], Loxton and Poorten [**11**] and Kubota [**10**]. A nice survey of results of this kind can be found in the book of Nishioka [**12**].

Other results of this nature include the linear independence of logarithms of special rational numbers which can be found in Sorokin [**13**] and Bezivin's result in [**3**] which proves linear independence of roots of special functional equations.

A special case of linear independence is irrationality. In [**1**] Badea proved the following theorem.

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that, for every large n,*

$$a_{n+1} > \frac{b_{n+1}}{b_n}\, a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1.$$

Then the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is an irrational number.

This result is improved in [**2**]. Another criterion of irrationality was proved by Duverney in [**6**]. In 1992 in [**4**] Borwein proved that the series $\sum_{n=1}^{\infty} \frac{1}{q^n+r}$ is irrational and not Liouville whenever q is an integer $(q \neq 0, \pm 1)$ and r is a nonzero rational number $(r \neq q^n)$. The same author together with Zhou in [**5**] proved the following theorem.

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Theorem 1.2. *Let q be an integer greater than one and r and s any positive rationals such that $1+q^m r-q^{2m}s\neq 0$ for all integers $m\geq 0$. Then the series*

$$\sum_{j=0}^{\infty}\frac{1}{1+q^j r-q^{2j}s}$$

is irrational and is not a Liouville number.

In 1968 in [**8**] Erdös and Strauss proved the following two theorems.

Theorem 1.3. *Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. Assume that*

$$\limsup_{k\to\infty}\frac{n_k^2}{n_{k+1}}\leq 1$$

and

$$\limsup_{k\to\infty}\frac{N_k}{n_{k+1}}\left(\frac{n_{k+1}^2}{n_{k+2}}-1\right)\leq 0.$$

Then $\sum_{k=1}^{\infty}1/n_k$ is irrational except when $n_{k+1}=n_k^2-n_k+1$ for all $k\geq k_0$ where N_k is the least common multiple of $n_1,\dots,n_k$.

Theorem 1.4. *Let $\{a_n\}_{n=1}^{\infty}$, $n\geq 1$, be a sequence of positive integers such that*

$$a_{n+1}\geq a_1a_2\dots a_n$$

for each n. Furthermore, assume that, for every $C>0$ there is a natural number $n>C$ with the property that

$$a_{n+1}\neq a_n^2-a_n+1.$$

Then $\sum_{n=1}^{\infty}1/a_n$ is an irrational number.

Later Erdös in [**7**] proved

Theorem 1.5. *Let $n_1<n_2<\cdots$ be an infinite sequence of positive integers satisfying*

$$\limsup_{k\to\infty} n_k^{1/2^k}=\infty$$

and

$$n_k>k^{1+\varepsilon}$$

for fixed $\varepsilon > 0$ *and for every* $k > k_0(\varepsilon)$. *Then*

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{n_k}$$

is irrational.

If the series tends to infinity very fast, then we can define the so-called linearly unrelated sequences.

Definition 1.1. Let $\{a_{i,n}\}_{n=1}^{\infty}$, $i = 1, \dots, K$, be the sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\sum_{n=1}^{\infty} 1/(a_{1,n}c_n), \sum_{n=1}^{\infty} 1/(a_{2,n}c_n), \dots, \sum_{n=1}^{\infty} 1/(a_{K,n}c_n)$ and 1 are linearly independent, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $i = 1, \dots, K$, are linearly unrelated.

This definition can be found in [**9**] where we also find the following theorem.

Theorem 1.6. *Let* $\{a_{i,n}\}_{n=1}^{\infty}, \{b_{i,n}\}_{n=1}^{\infty}$, $i = 1, \dots, K-1$, *be sequences of positive integers, and let* $\varepsilon > 0$ *be a real number such that*

$$\frac{a_{1,n+1}}{a_{1,n}} \geq 2^{K^{n-1}}, \; a_{1,n}/a_{1,n+1} \quad (a_{1,n} \text{ divides } a_{1,n+1})$$

$$b_{i,n} < 2^{K^{n-(\sqrt{2}+\varepsilon)\sqrt{n}}}, \quad i = 1, \dots, K-1,$$

$$\lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0 \quad \text{for all } j, i \in \{1, \dots, K-1\}, \quad i > j,$$

and

$$a_{i,n}2^{-K^{n-(\sqrt{2}+\varepsilon)\sqrt{n}}} < a_{1,n} < a_{i,n}2^{K^{n-(\sqrt{2}+\varepsilon)\sqrt{n}}}, \quad i = 1, \dots, K-1$$

hold for every sufficiently large natural number n. *Then the sequences* $\{\frac{a_{i,n}}{b_{i,n}}\}_{n=1}^{\infty}$, $i = 1, \dots, K-1$, *are linearly unrelated.*

The main result of this paper is a criterion for linear independence of series of rational numbers and one which is in Section 2. In Section 3 we give reasons why it is impossible to prove that the relevant sequences are linearly unrelated, and we also give a criterion for a series to be irrational.

2. Main result.

Theorem 2.1. *Let K be a positive integer, and let α, ε, A_1 and A_2 be positive real numbers such that $0 < \alpha < 1$, $1 \le A_1 < A_2$. Let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$, $i = 1, \dots, K$, be sequences of positive integers such that $\{a_{1,n}\}_{n=1}^{\infty}$ is nondecreasing and*

$$\limsup_{n\to\infty} a_{1,n}^{1/(K+1)^n} = A_2, \tag{1}$$

$$\liminf_{n\to\infty} a_{1,n}^{1/(K+1)^n} = A_1, \tag{2}$$

$$a_{1,n} \ge n^{1+\varepsilon}, \tag{3}$$

$$b_{i,n} < 2^{(\log_2 a_{1,n})^{\alpha}}, \quad i = 1, \dots, K, \tag{4}$$

$$\lim_{n\to\infty} \frac{a_{i,n} b_{j,n}}{b_{i,n} a_{j,n}} = 0 \quad \textit{for all } j, i \in \{1, \dots, K\}, \quad i > j, \tag{5}$$

and

$$a_{i,n} 2^{-(\log_2 a_{1,n})^{\alpha}} < a_{1,n} < a_{i,n} 2^{(\log_2 a_{1,n})^{\alpha}}, \quad i = 2, \dots, K \tag{6}$$

hold for every sufficiently large natural number n. Then the series $\sum_{n=1}^{\infty} \frac{b_{1,n}}{a_{1,n}}, \dots, \sum_{n=1}^{\infty} \frac{b_{K,n}}{a_{K,n}}$ and the number 1 are linearly independent over the rational numbers.

Proof. We start in the usual way. Assume that there is a K-tuple of integers $\beta_1, \beta_2, \dots, \beta_K$ (not all equal to zero) such that the sum

$$\beta = \sum_{j=1}^{K} \beta_j \sum_{n=1}^{\infty} \frac{b_{j,n}}{a_{j,n} c_n} \tag{7}$$

is a rational number. Let R be a maximal index such that $\beta_R \neq 0$. This and (7) imply

$$\beta = \sum_{j=1}^{K} \beta_j \sum_{n=1}^{\infty} \frac{b_{j,n}}{a_{j,n} c_n} = \sum_{n=1}^{\infty} \sum_{j=1}^{R} \beta_j \frac{b_{j,n}}{a_{j,n} c_n} = \sum_{n=1}^{\infty} \frac{b_{R,n}}{a_{R,n} c_n} \left(\sum_{j=1}^{R-1} \beta_j \frac{b_{j,n} a_{R,n}}{a_{j,n} b_{R,n}} + \beta_R \right). \tag{8}$$

From this and (5) we obtain that the number

$$\sum_{j=1}^{R-1} \beta_j \frac{b_{j,n} a_{R,n}}{a_{j,n} b_{R,n}}$$

is sufficiently small. From this and (8) we can assume, without loss of generality, that

$$\sum_{i=1}^{K} \beta_i \frac{b_{i,n}}{a_{i,n}} > 0 \tag{9}$$

for every sufficiently large n. Let a and b be integers such that $b > 0$ and $\beta = a/b$. Then, from (7) and (9), we obtain that

$$B_N = \left(a - b \sum_{i=1}^{K} \beta_i \sum_{n=1}^{N-1} \frac{b_{i,n}}{a_{i,n}} \right) \prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} = b \left(\prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} \right) \sum_{i=1}^{K} \beta_i \sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}}$$

is a positive integer for every sufficiently large N. This implies that

$$1 \leq Q_1 \left(\prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} \right) \sum_{i=1}^{K} \sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}} \tag{10}$$

holds for every sufficiently large N, where Q_1 is a suitable positive real constant, which does not depend on N. From (1) we obtain that, for every sufficiently large n,

$$a_{1,n} < (2A_2)^{(K+1)^n}. \tag{11}$$

Now (4), (6), (10) and (11) imply

$$
\begin{aligned}
1 &\le Q_1\Big(\prod_{n=1}^{N-1}\prod_{i=1}^{K} a_{i,n}\Big)\sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}}\\
&\le Q_2\Big(\prod_{n=1}^{N-1}\prod_{i=1}^{K} a_{1,n}2^{(\log_2 a_{1,n})^\alpha}\Big)\sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{2^{(\log_2 a_{1,n})^\alpha}}{a_{1,n}2^{-(\log_2 a_{1,n})^\alpha}}\\
&\le Q_2\Big(\prod_{n=1}^{N-1} a_{1,n}\Big)^K 2^{K\sum_{n=1}^{N-1}(\log_2 a_{1,n})^\alpha} K\sum_{n=N}^{\infty}\frac{2^{2(\log_2 a_{1,n})^\alpha}}{a_{1,n}}\\
&\le Q_3\Big(\prod_{n=1}^{N-1} a_{1,n}\Big)^K 2^{K\sum_{n=1}^{N-1}(\log_2(2A_2)^{(K+1)^n})^\alpha}\sum_{n=N}^{\infty}\frac{2^{2(\log_2 a_{1,n})^\alpha}}{a_{1,n}}\\
&\le Q_3\Big(\prod_{n=1}^{N-1} a_{1,n}\Big)^K 2^{\log_2(2A_2)(K+1)^{N\alpha}}\sum_{n=N}^{\infty}\frac{2^{2(\log_2 a_{1,n})^\alpha}}{a_{1,n}}\\
&\le \Big(\prod_{n=1}^{N-1} a_{1,n}\Big)^K 2^{(K+1)^{N\gamma}}\sum_{n=N}^{\infty}\frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}},
\end{aligned}
\tag{12}
$$

where Q_2, Q_3 and γ are suitable positive real constants which do not depend on N and $1>\gamma>\alpha$. Let $S_n = a_{1,n}^{1/(K+1)^n}$. Now the proof falls into two cases.

1. First assume that, for every sufficiently large n,

$$a_n \ge 2^n. \tag{13}$$

Then (13) and the fact that the function $2^{(\log_2 x)^\gamma}x^{-1}$ is decreasing for sufficiently large x imply

(14)

$$
\begin{aligned}
\sum_{n=N}^{\infty}\frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} &= \sum_{n\le\log_2 a_{1,N}}\frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} + \sum_{n>\log_2 a_{1,n}}\frac{2^{(\log_2 a_{1,N})^\gamma}}{a_{1,n}}\\
&\le \frac{2^{2(\log_2 a_{1,N})^\gamma}}{a_{1,N}} + \sum_{n>\log_2 a_{1,N}}\frac{2^{(\log_2 2^n)^\gamma}}{2^n}\\
&= \frac{2^{2(\log_2 a_{1,N})^\gamma}}{a_{1,N}} + \sum_{n>\log_2 a_{1,N}}\frac{1}{2^{n-n^\gamma}}\\
&\le \frac{2^{2(\log_2 a_{1,N})^\gamma}}{a_{1,N}} + C\frac{1}{2^{\log_2 a_{1,N}-(\log_2 a_{1,N})^\gamma}} \le \frac{2^{(\log_2 a_{1,N})^\omega}}{a_{1,N}},
\end{aligned}
$$

for sufficiently large N, where ω and C are positive real constants which do not depend on N and such that $1 > \omega > \gamma$.

For a sufficiently small positive real number δ, it follows from (1) and (2) that there exists a positive integer s_0 which is sufficiently large such that for every $n \geq s_0$,

$$\max(1, A_1 - \delta) < S_n < A_2 + \delta.$$

This implies that for every $n \geq s_0$

$$\max(1, (A_1 - \delta))^{(K+1)^n} < a_{1,n} < (A_2 + \delta)^{(K+1)^n}. \tag{15}$$

Let s_1 be the least positive integer greater than $(K+1)^{s_0+1}$ such that

$$\max(1, A_1 - \delta) < S_{s_1} < A_1 + \delta.$$

Then

$$\max(1, (A_1 - \delta))^{(K+1)^{s_1}} < a_{1,s_1} < (A_1 + \delta)^{(K+1)^{s_1}}. \tag{16}$$

Let s_2 be the least positive integer greater than s_1 such that

$$A_2 - \delta < S_{s_2} < A_2 + \delta \tag{17}$$

and s_3 be the least positive integer greater than s_1 such that

$$S_{s_3} > (1 + (1/s_3^2)) \max_{s_1 \leq j < s_3} (S_j, A_2 - 2\delta) \tag{18}$$

and $s_1 < s_3 \leq s_2$. Such a number s_3 must exist since otherwise using (17) we obtain

$$\begin{aligned} A_2 - \delta < S_{s_2} &< \left(1 + \frac{1}{s_2^2}\right) \max_{s_1 \leq j < s_2} (S_j, A_2 - 2\delta) \\ &< \left(1 + \frac{1}{s_2^2}\right)\left(1 + \frac{1}{(s_2-1)^2}\right) \max_{s_1 < j < s_2 - 1} (S_j, A_2 - 2\delta) < \cdots \\ &< \prod_{j=s_1}^{s_2} \left(1 + \frac{1}{j^2}\right)(A_2 - 2\delta), \end{aligned}$$

a contradiction for a sufficiently large s_0.

From (11), (15), (16), (18) and the fact that δ is a sufficiently small positive number, we obtain

(19)

$$
\begin{aligned}
a_{1,s_3} &= S_{s_3}^{(K+1)^{s_3}} > \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} (\max_{s_1\le j<s_3}(S_j, A_2-2\delta))^{(K+1)^{s_3}} \\
&\ge \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} \max_{s_1\le j<s_3}(S_j, A_2-2\delta)^{K((K+1)^{s_3-1}+(K+1)^{s_3-2}+\cdots+1)} \\
&\ge \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} \left(\prod_{j=s_1+1}^{s_3-1} a_{1,j}\right)^K (A_2-2\delta)^{K((K+1)^{s_1}+(K+1)^{s_1-1}+\cdots+1)} \\
&\ge \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} \left(\prod_{j=1}^{s_3-1} a_{1,j}\right)^K \\
&\quad \times \prod_{j=s_0}^{s_1} \left(\frac{(A_2-2\delta)^{(K+1)^j}}{a_{1,j}}\right)^K \frac{1}{(\prod_{j=1}^{s_0-1} a_{1,j})^K} \\
&\ge \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} \left(\prod_{j=1}^{s_3-1} a_{1,j}\right)^K \left(\frac{A_2-2\delta}{A_1+\delta}\right)^{K(K+1)^{s_1}} \\
&\quad \times \prod_{j=s_0}^{s_1-1} \left(\left(\frac{A_2-2\delta}{A_2+\delta}\right)^{(K+1)^j}\right)^K \frac{Q_4}{\prod_{j=1}^{s_0-1}(2A_2)^{K(K+1)^j}} \\
&\ge \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} \left(\prod_{j=1}^{s_3-1} a_{1,j}\right)^K \\
&\quad \times \left(\prod_{j=s_0}^{s_1-1} \left(\frac{(A_2-2\delta)^2}{(A_1+\delta)(A_2+\delta)}\right)^{(K+1)^j}\right)^K (3A_2)^{-(K+1)^{s_0+1}} \\
&\ge \left(1+\frac{1}{s_3^2}\right)^{(K+1)^{s_3}} \left(\prod_{j=1}^{s_3-1} a_{1,j}\right)^K (3A_2)^{-s_3},
\end{aligned}
$$

where Q_4 is a positive real constant which does not depend on s_0. Now from (11), (12), (14) and (19), we obtain

$$\begin{aligned}1 &\le \Big(\prod_{n=1}^{s_3-1} a_{1,n}\Big)^K 2^{(K+1)^{\gamma s_3}} \sum_{n=s_3}^{\infty} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}}\\ &\le \Big(\prod_{n=1}^{s_3-1} a_{1,n}\Big)^K 2^{(K+1)^{\gamma s_3}} \frac{2^{(\log_2 a_{1,s_3})^\omega}}{a_{1,s_3}}\\ &\le \Big(\prod_{n=1}^{s_3-1} a_{1,n}\Big)^K 2^{(K+1)^{\gamma s_3}} \frac{2^{(\log_2 (2A_2)^{(K+1)^{s_3}})^\omega}}{(1+(1/s_3^2))^{(K+1)^{s_3}}(\prod_{j=1}^{s_3-1} a_{1,j})^K (3A_2)^{-s_3}}\\ &= 2^{-(\log_2(1+(1/s_3^2)))(K+1)^{s_3}+(K+1)^{\gamma s_3}+(\log_2(2A_2))^\omega (K+1)^{\omega s_3}+\log_2(3A_2)s_3},\end{aligned}$$

a contradiction for a sufficiently large number s_3.

2. Now assume that there exist infinitely many n such that

$$a_n < 2^n. \tag{20}$$

Then (3) and the fact that the function $2^{(\log_2 x)^\gamma} x^{-1}$ is decreasing for a sufficiently large x imply

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} &= \sum_{n<a_{1,N}^\alpha} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} + \sum_{n>a_{1,N}^\alpha} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}}\\ &\le \frac{2^{(\log_2 a_{1,n})^\gamma} a_{1,N}^\alpha}{a_{1,N}} + \sum_{n>a_{1,N}^\alpha} \frac{2^{(\log_2 n^{1+\varepsilon})^\gamma}}{n^{1+\varepsilon}}\\ &\le a_{1,N}^{\frac{\alpha-1}{2}} + \sum_{n>a_{1,N}^\alpha} \frac{1}{n^{1+\varepsilon/2}}\\ &\le a_{1,N}^{\frac{\alpha-1}{2}} + \frac{1}{(a_{1,N}^\alpha)^{\varepsilon/3}} \le a_{1,N}^{-B}\end{aligned} \tag{21}$$

for a sufficiently large N, where B is a suitable positive real constant, which does not depend on N. On the other hand, let $A = (1+A_2)/2 = (A_1 + A_2)/2$. From this and (1) we obtain that there is a sufficiently large k such that

$$a_{1,k} > A^{(K+1)^k}. \tag{22}$$

Let k_0 be a greatest positive integer less than k such that (20) holds. Let k_1 be a least positive integer such that

$$S_{k_1} > \left(1+\frac{1}{k_1^2}\right)\max_{k_0\le j<k_1} S_j, \tag{23}$$

and $k_0 < k_1 \le k$. As in the previous case such a k_1 must exist, since, otherwise,

$$\begin{aligned} 1 < A \le S_k &< \left(1+\frac{1}{k_1^2}\right)\max_{k_0\le j<k_1} S_j \\ &< \left(1+\frac{1}{k_1^2}\right)\left(1+\frac{1}{(k_1-1)^2}\right)\max_{k_0\le j<k_1-1} S_j \\ &< \cdots < \prod_{j=k_1}^{k}\left(1+\frac{1}{j^2}\right)S_{k_0}, \end{aligned}$$

a contradiction for a sufficiently large number k_0. From (23) and the fact that the sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is nondecreasing we obtain

(24)

$$\begin{aligned} a_{1,k_1} = S_{k_1}^{(K+1)^{k_1}} &> \left(1+\frac{1}{k_1^2}\right)^{(K+1)^{k_1}}(\max_{k_0\le j<k_1} S_j)^{(K+1)^{k_1}} \\ &\ge \left(1+\frac{1}{k_1^2}\right)^{(K+1)^{k_1}}(\max_{k_0\le j<k_1} S_j)^{K((K+1)^{k_1-1}+(K+1)^{k_1-2}+\cdots+1)} \\ &\ge \left(1+\frac{1}{k_1^2}\right)^{(K+1)^{k_1}}\left(\prod_{j=1}^{k_1-1} a_{1,j}\right)^{K}\left(\prod_{j=1}^{k_0} a_{1,j}\right)^{-K} \\ &\ge \left(1+\frac{1}{k_1^2}\right)^{(K+1)^{k_1}}\left(\prod_{j=1}^{k_1-1} a_{1,j}\right)^{K} 2^{-k_1^2}. \end{aligned}$$

The definition of k_1 implies that, for every N, $k_0 < N < k_1$,

$$S_N \le \left(1+\frac{1}{N^2}\right)\max_{k_0\le j<N} S_j.$$

Thus

$$S_N \le \left(\prod_{j=k_0}^{N}\left(1+\frac{1}{j^2}\right)\right)S_{k_0} < C, \tag{25}$$

where C is a constant which depends on k_0 and C tends to 1 as k_0 tends to infinity. From (25) we obtain that for every $N = k_0, \dots, k_1 - 1$,

$$a_{1,N} \le C^{(K+1)^n}.$$

This implies

$$\left(\prod_{j=1}^{k_1-1} a_{1,j}\right)^K = \left(\prod_{j=1}^{k_0-1} a_{1,j}\right)^K \left(\prod_{j=k_0}^{k_1-1} a_{1,j}\right)^K \le 2^{Kk_0^2} C^{(K+1)^{k_1}}. \tag{26}$$

Inequalities (14) and (21) and the definitions of k_1 and k imply

$$\begin{aligned}\sum_{n=k_1}^{\infty} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} &= \sum_{n=k_1}^{k-1} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} + \sum_{n=k}^{\infty} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} \\ &\le \frac{2^{(\log_2 a_{1,k_1})^\omega}}{a_{1,k_1}} + \frac{1}{a_{1,k}^B}.\end{aligned} \tag{27}$$

Now from (11), (12), (22), (24), (26) and (27), we obtain

$$\begin{aligned}1 &\le \left(\prod_{n=1}^{k_1-1} a_{1,n}\right)^K 2^{(K+1)^{\gamma k_1}} \sum_{n=k_1}^{\infty} \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} \\ &\le \frac{(\prod_{n=1}^{k_1-1} a_{1,n})^K 2^{(K+1)^{\gamma k_1}} 2^{(\log_2 a_{1,k_1})^\omega}}{a_{1,k_1}} + \frac{(\prod_{n=1}^{k_1-1} a_{1,n})^K 2^{(K+1)^{\gamma k_1}}}{a_{1,k}^B} \\ &\le \frac{(\prod_{n=1}^{k_1-1} a_{1,n})^K 2^{(K+1)^{\gamma k_1}} 2^{(\log_2 a_{1,k_1})^\omega}}{(1+(1/k_1^2))^{(K+1)^{k_1}} (\prod_{j=1}^{k_1-1} a_{1,j})^K 2^{-k_1^2}} + \frac{C^{(K+1)^{k_1}} 2^{(K+1)^{\gamma k_1}}}{A^{B(K+1)^k}} \\ &\le \frac{2^{(K+1)^{\gamma k_1}} 2^{(\log_2((2A_2)^{(K+1)^n}))^\omega}}{(1+(1/k_1^2))^{(K+1)^{k_1}} 2^{-k_1^2}} + \frac{C^{(K+1)^{k_1}} 2^{(K+1)^{\gamma k_1}}}{A^{B(K+1)^k}} \\ &\le 2^{-\log_2(1+(1/k_1^2))(K+1)^{k_1} + (K+1)^{\gamma k_1} + (\log_2(2A_2))^\omega (K+1)^{n\omega} + k_1^2} \\ &\quad + 2^{(-B\log_2 A + \log_2 C)(K+1)^k + (K+1)^{\gamma k}},\end{aligned}$$

a contradiction for a sufficiently large k_0. □

3. Comments and examples.

Theorem 3.1. *Let* α, ε, A_1 *and* A_2 *be positive real numbers such that* $0 < \alpha < 1$ *and* $1 \leq A_1 < A_2$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of positive integers where* $\{a_n\}_{n=1}^{\infty}$ *is nondecreasing and*

$$\limsup_{n\to\infty} a_n^{1/2^n} = A_2,$$
$$\liminf_{n\to\infty} a_n^{1/2^n} = A_1,$$
$$a_n \geq n^{1+\varepsilon},$$

and

$$b_n \leq 2^{(\log_2 a_n)^{\alpha}}$$

hold for every sufficiently large n. *Then the series* $\sum_{n=1}^{\infty} b_n/a_n$ *is irrational.*

By putting $K = 1$ in Theorem 2.1, we immediately obtain Theorem 3.1.

Remark 3.1. The problem in Theorem 2.1 and Theorem 3.1 remains open for $A_1 = A_2 > 1$. If a_1 is a positive integer greater than 1 and for every $n > 1$ $a_{n+1} = a_n^2 - a_n + 1$, then the series $\sum_{n=1}^{\infty} 1/a_n$ is rational and $\lim_{n\to\infty} a_n^{1/2^n} > 1$. On the other hand, the series $\sum_{n=1}^{\infty} 1/2^{2^n}$ is an irrational number.

Open problem 3.1. Is it the case that for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series

$$\sum_{n=1}^{\infty} \frac{2^{2^n}+1}{(3^{2^n}+n!)c_n}, \quad \sum_{n=1}^{\infty} \frac{3^{2^n}+1}{(4^{2^n}+n!)c_n}$$

and the number 1 are linearly independent?

Open problem 3.2. Is it the case that for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series

$$\sum_{n=1}^{\infty} \frac{1}{(3^{2^n}+2^n)c_n}$$

is an irrational number?

Example 3.1. Let $\pi(x)$ be the number of primes less than or equal to x, $[x]$ the greatest integer less than or equal to x, and K a positive integer greater than 1. Then the series

$$\sum_{n=1}^{\infty} \frac{3^{j2\pi([n/4])} + n!}{2^{K^{2^{[\log_2 n]}}} + 3^n},$$

$j = 1, \dots, K$, and the number 1 are linearly independent over rational numbers.

Example 3.2. Let $[x]$ and $\pi(x)$ be defined as in the previous case. Then the series

$$\sum_{n=1}^{\infty} \frac{3^{\pi(n)} + 1}{2^{2^{2^{[\log_2 \log_2 n]}}} + n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2^{\pi(n)} + 3}{2^{2^{2^{[\log_2 \log_2 n]}}} + 2n}$$

are irrational.

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