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ARCHIMEDEAN CLOSED LATTICE-ORDERED GROUPS

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ABSTRACT. We show that, if an abelian lattice-ordered group is archimedean closed, then each principal l-ideal is also archimedean closed. This has given a positive answer to the question raised in 1965 and hence proved that the class of abelian archimedean closed lattice-ordered groups is a radical class. We also provide some conditions for lattice-ordered group $F(\Delta, R)$ to be the unique archimedean closure of $\sum(\Delta, R)$.

Introduction. Throughout, let G be a lattice-ordered group (l-group).

Let Γ be a *root system*, that is, Γ is a partially ordered set for which $\{\alpha \in \Gamma \mid \alpha \geq \gamma\}$ is totally ordered, for any $\gamma \in \Gamma$. Let $\{H_\gamma \mid \gamma \in \Gamma\}$ be a collection of abelian totally-ordered groups indexed by Γ. $V(\Gamma, H_\gamma)$ is the set of all functions v on Γ for which $v(\gamma) \in H_\gamma$ and the support of each v satisfies ascending chain condition. $V(\Gamma, H_\gamma)$ is an abelian group under addition. Furthermore, if we define an element of $V(\Gamma, H_\gamma)$ to be positive, if it is positive at each maximal element of its support, then $V(\Gamma, H_\gamma)$ is an abelian l-group, which we call a *Hahn group* on Γ. $\sum(\Gamma, H_\gamma)$ is the l-subgroup of $V(\Gamma, H_\gamma)$ whose elements have finite supports. A *root* in a root system Γ is a totally ordered subset of Γ. $F(\Gamma, H_\gamma)$ is the l-subgroup of $V(\Gamma, H_\gamma)$ such that the support of each element is contained in a finite number of roots in Γ.

A convex l-subgroup which is maximal with respect to not containing some $g \in G$ is called *regular* and is a *value* of g. Element g is *special* if it has a unique value, and in this case the value is called a *special value*. A convex l-subgroup P of G is *prime* if $a \wedge b = 0$ in G implies that either $a \in P$ or $b \in P$. Regular subgroups of G are prime and form a root system under inclusion, written $\Gamma(G)$. A subset $\Delta \subseteq \Gamma(G)$ is *plenary* if $\cap\Delta = \{0\}$ and Δ is a dual ideal in $\Gamma(G)$; that is, if $\delta \in \Delta$, $\gamma \in \Gamma(G)$ and $\gamma > \delta$, then $\gamma \in \Delta$. If G is an abelian l-group, then G is l-isomorphic to

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an l-subgroup of $V(\Gamma(G), R)$ such that if $\gamma \in \Gamma(G)$ is a value of $g \in G$, then γ is a maximal component of g after the embedding, where R is the set of real numbers. Such a value-preserving l-isomorphism is called a *v-isomorphism*. This is the result of the Conrad-Harvey-Holland embedding theorem for abelian lattice-ordered groups. In fact, we do not need the entire root system to obtain such an embedding. For any abelian l-group G, there exists a Conrad-Harvey-Holland embedding into $V(\Delta, R)$, where Δ is any plenary subset of $\Gamma(G)$ [**13**]. An *n-automorphism* of $V(\Gamma, R)$ is a v-automorphism that induces identity on each V^γ/V_γ, where $V^\gamma = \{v \in V(\Gamma, R) \mid v_\alpha = 0 \text{ for all } \alpha > \gamma\}$, and $V_\gamma = \{v \in V(\Gamma, R) \mid v_\alpha = 0 \text{ for all } \alpha \geq \gamma\}$.

For any $g \in G$, $G(g) = \{h \in G \mid |h| \leq n|g|, \text{ for some positive integer } n\}$, the *principal convex l-subgroup of G generated by g* is the least convex l-subgroup of G that contains g.

An element b of G is *basic* if the set $\{g \in G \mid 0 < g \leq b\}$ is totally-ordered. An l-group G has a basis if G possesses a maximal pairwise disjoint set of elements g_λ and, in addition, each $G(g_\lambda)$ is a totally-ordered l-subgroup.

An l-group is *archimedean* if for any elements g and h, $ng \leq h$ for all positive integers n implies that $g \leq 0$. An archimedean l-group is necessarily abelian. Given two positive elements g and h of an l-group G, we say that they are *archimedean equivalent* (*a-equivalent*) if there exists a positive integer n so that $g \leq nh$ and $h \leq ng$. If G is an l-subgroup of H and, for each $h \in H^+$, there exists $g \in G^+$ so that h and g are a-equivalent, then we say that H is an *archimedean extension* (*a-extension*) of G. H is *a-closed* if H admits no a-extensions. H is an a-closure of G if H is an a-closed a-extension of G.

For any subset X of an l-group G, $X' = \{g \in G \mid |g| \wedge |x| = 0, \text{ for all } x \in X\}$ is a *polar subgroup* of G. We denote by g' and g'' the polar subgroups $\{g\}'$ and $\{g\}''$.

An l-group G is *finite-valued* if every element of G has only a finite number of values; this is equivalent to the statement that every element of G can be expressed as a finite sum of disjoint special elements. An l-group G is *special-valued* if G has a plenary subset of special values; this is equivalent to the statement that each positive element of G can be expressed as the join of a set of pairwise disjoint positive special elements.

An l-group G has *property F*, if each $0 < g \in G$ exceeds at most a finite number of disjoint elements, or equivalently each bounded disjoint subset of G is finite.

Let Γ be a root system. For α, β in Γ, we define $\alpha \sim \beta$ if α and β lie on the same roots of Γ. This is an equivalence relation and we shall denote the equivalence class that contains α by $\overline{\alpha}$ and the set of all equivalence classes by $\overline{\Gamma}$. Define $\overline{\beta} > \overline{\alpha}$ if $\bar{\beta} \neq \overline{\alpha}$ and $\beta > \alpha$, or equivalently if $\beta > \alpha$ and $\beta > \gamma$, with $\alpha \| \gamma$. Then $\overline{\Gamma}$ is also a root system consisting of "branch points" of Γ [**5**].

Let G be an l-group and $\Gamma(G)$ the root system of regular subgroups of G. For $\alpha, \beta \in \Gamma(G)$, $\alpha \sim \beta$ if and only if G_α and G_β contain the same set of minimal primes if and only if $\{G_\delta \mid \delta \text{ and } \alpha \text{ are comparable}\} = \{G_\delta \mid \delta \text{ and } \beta \text{ are comparable}\}$.

The class of l-groups G for which the set $\overline{\Gamma(G)}$ satisfies the descending chain condition (DCC) is denoted $\overline{D}$, and $\overline{D}$ is a *torsion class*, i.e., it is closed under convex l-subgroups, l-homomorphic images, and joins of convex l-subgroups [**5**].

1. Abelian a-closed lattice-ordered groups. It was shown in [**7**] that if the principal l-ideal $G(g)$ is a-closed for each $g > 0$ in an l-group G, then G is a-closed. In this section we will show that the converse of the above statement is true for abelian l-groups.

Theorem 1.1. *Let G be an abelian l-group. If G is a-closed, then $G(g)$ is a-closed for each $g \in G^+$.*

Proof. 1. Let H be a proper a-extension of $G(g_0)$ where $0 < g_0 \in G$. We claim that $H \nsubseteq G$.

For if $H \subseteq G$, then we will have $G(g_0) \subseteq H \subseteq G$. So, for each $h \in H$, $h \in G$ and h is a-equivalent to some $g \in G(g_0)$. Thus $h \in G(g_0)$ and hence $H \subseteq G(g_0)$. This contradicts the fact that H is a proper a-extension of $G(g_0)$.

2. Since G is abelian, we may assume that $G(g_0) \subseteq G \subseteq V(\Gamma(G), R)$, by Conrad-Harvey-Holland embedding theorem for abelian lattice-ordered groups.

By lifting the identity map $i : G(g_0) \to V(\Gamma(G), R)$, to a map from

H to $V(\Gamma(G), R)$, we may assume that $H \subseteq V(\Gamma(G), R)$. Now let $K = \langle G, H \rangle$ be the l-subgroup generated by H and G in $V(\Gamma(G), R)$.

3. K is an a-extension of G.

We first show that every positive element in the group generated by G and H is a-equivalent to some $g \in G^+$. Consider $g + h > 0$, where $g \in G$ and $h \in H$. We have $g = (g \wedge g_0) + (g - g \wedge g_0)$, hence

$$g + h = (h + g \wedge g_0) + (g - g \wedge g_0),$$

where $h + g \wedge g_0 \in H$. Moreover, $h + g \wedge g_0$ is positive. For $h+g\wedge g_0 = (h+g)\wedge(h+g_0)$, where $h+g > 0$. We can make $h+g_0 > 0$ by replacing g_0 with some $ng_0 > -h$. Thus $h+g\wedge g_0 = (h+g)\wedge(h+g_0)$ is a positive element of H, but H is an a-extension of $G(g_0)$; hence, there exists $\overline{g} \in G(g_0)^+$ such that $h + g \wedge g_0$ is a-equivalent to $\overline{g}$. Now we have that $g + h = (h + g \wedge g_0) + (g - g \wedge g_0)$ is a-equivalent to $\overline{g} + g - g \wedge g_0 \in G^+$, because $\overline{g}$ and $g - g \wedge g_0$ are both positive elements of G, and there are no cancellations in their maximal components.

We hence have shown that every positive element in the subgroup of $V(\Gamma(G), R)$ generated by G and H is a-equivalent to some $g \in G^+$.

To show that every positive element in the l-subgroup generated by G and H is a-equivalent to some $g \in G^+$, we first observe that if $(g_1+h_1)\wedge(g_2+h_2) > 0$ with $g_1, g_2 \in G$ and $h_1, h_2 \in H$, then $g_1+h_1 > 0$ and $g_2 + h_2 > 0$.

From the above argument, $g_1 + h_1$ is a-equivalent to some $\overline{g}_1 \in G^+$, and $g_2 + h_2$ is a-equivalent to some $\overline{g}_2 \in G^+$. Therefore $(g_1 + h_1) \wedge (g_2 + h_2)$ is a-equivalent to $\overline{g}_1 \wedge \overline{g}_2 \in G^+$.

We now show $(g_1+h_1)\vee(g_2+h_2) > 0$ is a-equivalent to some positive element in G. Although $(g_1 + h_1) \vee (g_2 + h_2) > 0$ does not imply that $g_1 + h_1 > 0$ and $g_2 + h_2 > 0$, we observe that $(g_1 + h_1) \vee (g_2 + h_2)$ is equal to $(g_1+h_1)^+ \vee (g_2+h_2)^+$; hence, it suffices to show that $(g+h)^+$ is a-equivalent to some $\overline{g} \in G^+$ where $g \in G$ and $h \in H$.

Now $(g+h)^+ = (g+h) \vee 0 = h + (g \vee (-h)) = h + (g \vee (-h)) \wedge g_0 + (g \vee (-h)) - ((g \vee (-h)) \wedge g_0) = f1 + f_2$, where $f_1 = h + (g \vee (-h)) \wedge g_0$, and $f_2 = (g \vee (-h)) - ((g \vee (-h)) \wedge g_0)$. We will show that f_1 and f_2 are a-equivalent to some positive elements in G.

We observe that $f_1 = h + (g \vee (-h)) \wedge g_0 = ((g+h) \vee 0) \wedge (h+g_0) \geq 0$, for we can always make $h + g_0 \geq 0$ by replacing g_0 with ng_0, where n is a positive integer with $ng_0 \geq -h$. Also $f_1 = h + (g \wedge g_0) \vee (g_0 - h) \in H$, hence f_1 is a-equivalent to some $\overline{g} \in G(g_0)^+$.

Now consider $f_2 = (g \vee (-h)) - ((g \vee (-h)) \wedge g_0) \geq 0$.

Let $A = \{\gamma \in \Gamma(G) \mid \gamma$ is a maximal component of f_2 and a maximal component of $g \vee (-h)\}$, and let $B = \{\gamma \in \Gamma(G) \mid \gamma$ is a maximal component of f_2 but not a maximal component of $g \vee (-h)\}$.

There are no cancellations at maximal components that belong to A when we subtract $(g \vee (-h)) \wedge g_0$ from $g \vee (-h)$ to get f_2 because each maximal component $\gamma \in A$ of $g \vee (-h)$ lies above some maximal component of g_0. Since we can always replace g_0 with $ng_0 > -h$, we may assume that $f_2 = (g \vee (-h)) - ((g \vee (-h)) \wedge g_0)$ and $g - g \wedge g_0$ take the same value on $\gamma \in A$.

However, there are cancellations at maximal components lying above some element in B when we subtract $(g \vee (-h)) \wedge g_0$ from $g \vee (-h)$ to get the value of f_2 on $\gamma \in B$. We observe that $f_2 = (g \vee (-h)) - ((g \vee (-h)) \wedge g_0) > g - g \wedge g_0$ on $\gamma \in B$ and that elements in B lie below some maximal component of g_0. Hence there exists some $h \in H$ such that $nf_2 > h$ and $nh > f_2$ on $\gamma \in B$ for some $n \in Z^+$. Therefore there exists some $\overline{g_0} \in G(g_0)^+$ such that $nf_2 > \overline{g_0}$ and $n\overline{g_0} > f_2$ on $\gamma \in B$ for some $n \in Z^+$ and $\overline{g_0} \ll f_2$ on $\gamma \in A$. Hence we have $nf_2 > g - g \wedge g_0 + \overline{g_0}$ and $n(g - g \wedge g_0 + \overline{g_0}) > f_2$. We now have that f_2 is a-equivalent to some element in G^+.

Now we have shown that $(g + h)^+ = f_1 + f_2$ is a-equivalent to $\overline{g} + g - g \wedge g_0 + \overline{g}_0 \in G^+$ which means that $K = \langle H, G \rangle$ is an a-extension of G. This contradicts the fact that G is a-closed. Thus we must have $H \subseteq G$ which implies that $H = G(g_0)$, hence $G(g_0)$ is a-closed.

Corollary 1.2. *Let G be an abelian l-group. If G has a unique a-closure, then $G(g)$ has a unique a-closure for each $g \in G$.*

Proof. Assume that G has a unique a-closure K. We then have that $K(g)$ is an a-extension of $G(g)$ for each $g \in G^+$ and $K(g)$ is a-closed. If H is another a-closure of $G(g)$ which is not isomorphic to $K(g)$, then $H \not\subseteq K$. Now from the proof of Theorem 1.1, the l-subgroup $\langle G, H \rangle$ of $V(\Gamma(G), R)$ generated by G and H is an a-extension of G which is not contained in K. This contradicts the fact that K is the a-closure of G.

Let G be an l-group and $\Gamma(G)$ the set of regular subgroups of G. For α, β in Γ, we define $\alpha \sim \beta$ if α and β lie on the same roots of Γ. This is an equivalence relation and we shall denote the equivalence class that contains α by $\overline{\alpha}$ and the set of all equivalence classes by $\overline{\Gamma(G)}$. Define $\overline{\beta} > \overline{\alpha}$ if $\overline{\beta} \neq \overline{\alpha}$ and $\beta > \alpha$, or equivalently if $\beta > \alpha$ and $\beta > \gamma$, with $\alpha \parallel \beta$. Then $\overline{\Gamma(G)}$ is also a root system, and the map $\alpha \to \overline{\alpha}$ is an o-homomorphism of $\Gamma(G)$ onto $\overline{\Gamma(G)}$ with $a \parallel \beta$ if and only if $\overline{\alpha} \parallel \overline{\beta}$. Since $\overline{\beta} > \overline{\alpha}$ implies that $\overline{\beta} > \overline{\gamma}$ for some $\overline{\gamma}$ with $\overline{\alpha} \parallel \overline{\gamma}$, it follows that $\overline{\Gamma(G)}$ is a root system of "branch points." For each $\overline{\gamma} \in \overline{\Gamma(G)}$, we define $G^{\overline{\gamma}} = \cup_{\alpha \in \overline{\gamma}} G^{\alpha}$ and $G_{\overline{\gamma}} = \cap_{\alpha \in \overline{\gamma}} G_{\alpha}$ [**5**].

If an l-group G satisfies property F, then each g in G has only a finite number of values in $\overline{\Gamma(G)}$; hence each $G_{\overline{\gamma}}$ is special, and therefore $G_{\overline{\gamma}} \triangleleft G^{\overline{\gamma}}$ for all $\overline{\gamma}$ in $\overline{\Gamma(G)}$.

Theorem 1.3. *Let G be a lattice-ordered group. If G has property F and $\overline{\Gamma}$ satisfies DCC and $G^{\overline{\gamma}}/G_{\overline{\gamma}} \cong R$ for each $\overline{\gamma} \in \overline{\Gamma(G)}$, then G is a-closed.*

Proof. Let H be an a-extension of G. First we assume that G has a finite basis of n elements and that $\overline{\Gamma}$ satisfies DCC and $G^{\overline{\gamma}}/G_{\overline{\gamma}} \cong R$ for each $\overline{\gamma} \in \overline{\Gamma(G)}$. Then $\Gamma(G)$ has exactly n roots [**13**] and we use induction on n.

If $n = 1$, then $\Gamma(G)$ is totally-ordered, and $\overline{\Gamma}$ contains a single element. Thus $G = G^{\overline{\gamma}}$, $H = H^{\overline{\gamma}}$, $G_{\overline{\gamma}} = H_{\overline{\gamma}} = 0$. If $H \neq G$, then $G^{\overline{\gamma}}/G_{\overline{\gamma}} = G^{\overline{\gamma}} \subset H^{\overline{\gamma}} = H^{\overline{\gamma}}/H_{\overline{\gamma}}$. This contradicts the fact that $G^{\overline{\gamma}}/G_{\overline{\gamma}}$ is a-closed.

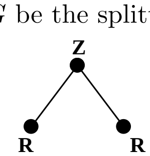
If $n > 1$, then from the structure theorem for an l-group with a finite basis [**15**], it follows that $G = lex(A \oplus B)$ where A and B are nonzero convex l-subgroups of G, and A and B have bases of fewer than n elements. By induction hypothesis, A and B are a-closed; thus, $A \oplus B$ is a-closed [**7**]. $A \oplus B$ is the subgroup of G generated by non-units, hence it is also the subgroup of H generated by non-units. Thus $A \oplus B \triangleleft H$ [**11**]. Therefore $H/(A \oplus B)$ is an a-extension of $G/(A \oplus B)$, but since $G/(A \oplus B)$ is a-closed o-group by the argument used for $n = 1$, it follows that $H = G$. Hence G is a-closed.

To prove the theorem, it suffices to show that $G(g)$ is a-closed for each $0 < g \in G$ [**7**]. Since G has property F, each $G(g)$ has a finite basis. Thus, in order to complete the proof, it suffices to show that $\overline{\Gamma(G(g))}$ satisfies DCC and each $G(g)^{\overline{\gamma}}/G(g)_{\overline{\gamma}} \cong R$.

By Theorem 3.7 in [**6**], $G(g) = G(g_1) \oplus \cdots \oplus G(g_k)$, where each $G(g_i)$ is a lexico-extension of a proper l-ideal. Clearly it suffices to show that each $\overline{\Gamma(G(g_i))}$ satisfies DCC. Thus, without loss of generality, we may assume that $G(g)$ is a lexico-extension of a proper l-ideal. Then $(G(g) + G(g)')^+$ consists of all elements in G^+ that do not exceed $G(g)$ [**11**]. Let $x \in G(g)$, and let M be a convex l-subgroup of $G(g)$ that is maximal without x. Then we claim that $M \oplus G(g)'$ is a maximal convex l-subgroup of G without x. Then we claim that $M \oplus G(g)'$ is a maximal convex l-subgroup of G without x. For suppose $x \notin N$, where N is a convex l-subgroup of G and $N \supset M \oplus G(g)'$. Pick $0 < z \in N \setminus (M \oplus G(g)')$ and, $N \cap G(g) \supseteq M$. If $N \cap G(g) \supset M$, then $x \in N$, this contradicts that $x \notin N$. Thus, if $z = z_1 + z_2 \in G(g) + G(g)'$, then $z - z_2 = z_1 \in N \cap G(g) = M$ and hence $z \in M + G(g)'$, a contradiction. Therefore, $M \oplus G(g)' \in \Gamma(G)$ is maximal without containing x. Thus since $\overline{\Gamma(G)}$ satisfies DCC it follows that $\overline{\Gamma(G(g))}$ satisfies DCC.

Now we show that $G(g)^{\overline{\gamma}}/G(g)_{\overline{\gamma}} \cong R$. Since $G(g)$ has a finite basis, each $G(g)_{\overline{\gamma}}$ is special, hence $G(g)_{\overline{\gamma}} \lhd G(g)^{\overline{\gamma}}$. As above, we can assume that $G(g)$ is a lexico-extension of a proper l-ideal. Then $G(g)_{\overline{\gamma}} \oplus G(g)'$ is a special convex l-subgroup of G and $\dfrac{G(g)^{\overline{\gamma}}}{G(g)_{\overline{\gamma}}} \cong \dfrac{G(g)^{\overline{\gamma}} \oplus G(g)'}{G(g)_{\overline{\gamma}} \oplus G(g)'} \cong R$.

We observe that if H is a-closed, then it does not follow that $H^{\overline{\gamma}}/H_{\overline{\gamma}}$ is a-closed. For example, let G be the splitting extension on



i.e., G is the splitting extension of $R \oplus R$ by Z determined by $\alpha : Z \to \mathrm{Auto}\,(R \oplus R)$ such that $(r_1, r_2)\alpha(z) = (r_1, r_2)$ if z is even; $(r_1, r_2)\alpha(z) = (r_2, r_1)$, if z is odd. Define $(z(r_1, r_2))$ to be positive if $z > 0$ or $z = 0$ and $r_1, r_2 \geq 0$. If H is an a-closure of G, then H is an extension of $R \oplus R$ by an l-group K, but K cannot be divisible and $H^{\overline{\gamma}}/H_{\overline{\gamma}}$ is not a-closed.

However, if G is an abelian l-group, and G has property F, then G is a-closed if and only if $G \cong F(\Gamma, R)$ for some root system Γ. But we also know that $G \cong F(\overline{\Gamma}, G^{\overline{\gamma}}/G_{\overline{\gamma}})$ [**5**], and there is a natural l-isomorphism from $F(\Gamma, R)$ to $F(\overline{\Gamma}, V(\overline{\gamma}, R))$. Hence it is easy to see $G^{\overline{\gamma}}/G_{\overline{\gamma}} \cong V(\overline{\gamma}, R)$. Therefore $G^{\overline{\gamma}}/G_{\overline{\gamma}}$ is a-closed.

2. The class of abelian a-closed lattice-ordered groups. We recall that a nonempty class $\Re$ of l-groups is a radical class if it is closed with respect to convex l-subgroups and joins of convex l-subgroups.

Lemma 2.1. *Let H be an a-extension of an l-group G, and let C be an a-closed convex l-subgroup of G. Then $C = H\langle C\rangle$, the convex l-subgroup of H generated by C.*

Proof. Let $0 < h \in H\langle C\rangle$. Then there exists $c \in C$ such that $0 < h \le c$. And there exists a positive element $g \in G$ such that h and g are a-equivalent. Since $g \le nh \le nc$ for some positive integer n, we have $g \in C$. So $H\langle C\rangle$ is an a-extension of C, and thus $C = H\langle C\rangle$. $\square$

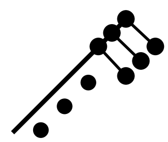
Lemma 2.2. *Let A and B be abelian a-closed convex l-subgroups of an l-group G. Then $A \vee B = A + B$ is a-closed.*

Proof. Suppose $A+B$ is not a-closed. Let H be a proper a-extension of $A + B$, and let $0 < h \in H \setminus (A + B)$. There exist $0 < a \in A$ and $0 < b \in B$ such that $h \le a + b$. Thus, there exist $0 \le c_1 \le a$ and $0 \le c_2 \le b$ such that $h = c_1 + c_2$. Since $c_1 \in H\langle A\rangle = A$, and $c_2 \in H\langle B\rangle = B$, we have that $h \in A + B$. $\square$

Theorem 2.3. *The class of abelian a-closed lattice-ordered groups is a radical class.*

Proof. Let $\mathcal{M} = \{C_\lambda\}$ be the set of all abelian a-closed convex l-subgroups of an l-group G. Let $\mathcal{C}$ be a chain in $\{C_\lambda\}$. Then, for each $g \in \cup\mathcal{C}$, $G(g)$ is a-closed and so $\cup\mathcal{C}$ is a-closed. So $\mathcal{M}$ has maximal element. But by the above lemma, there must be a unique maximal convex l-subgroup of G in $\mathcal{M}$. For, if A and B are maximal elements, then so is $A \vee B$. $\square$

3. Conditions for $F(\Delta, R)$ to be the unique a-closure of $\sum(\Delta, R)$. Let Δ be a root system. It is shown in [**7**] that $F(\Delta, R)$ is an abelian a-closure of $\sum(\Delta, R)$. It is also shown in [**3**] that $F(\Delta, R)$ is the unique a-closure of $\sum(\Delta, R)$ if and only if Δ does not contain a copy of



We will provide more conditions equivalent to $F(\Delta, R)$ being the unique a-closure of $\Sigma(\Delta, R)$.

Lemma 3.1. *Let Δ be a root system and G an l-subgroup of $V(\Delta, R)$ and $\Lambda = \{\lambda \in \Delta \mid g_\lambda$ is a maximal component of some $g \in G\}$. Let $V_\Lambda = \{v \in V(\Delta, R) \mid$ support of v is contained in $\Lambda\}$.*

(1) *Λ is a root system.*

(2) *V_Λ is an l-subgroup of $V(\Delta, R)$.*

(3) $V_\Lambda \cong V(\Lambda, R)$.

(4) *The projection ρ of $V(\Delta, R)$ onto V_Λ induces an l-isomorphism of G into V_Λ.*

Proof. 1. Λ is a subset of a root system, hence it is a root system.

2. If the support of v is contained in Λ, then so is that of $v \vee 0$.

3. There is a natural isomorphism of V_Λ into $V(\Lambda, R)$. This isomorphism is also onto, for if $v \in V(\Lambda, R)$, then it is the image of $v \in V(\Delta, R)$ with support in Λ.

4. The projection ρ induces an isomorphism of G into V_Λ, for g and $g\rho$ have the same maximal components. Actually, it is an l-isomorphism. For if x is a special negative component of g, then $x\rho$ is a special negative component of $g\rho$, hence $(g \vee 0)\rho = g\rho \vee 0$.

Let G be a vector lattice, and let Δ be a plenary subset of $\Gamma(G)$.

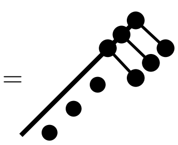
Then there exists a linear v-isomoprhism τ of G into $V(\Delta, R)$. We say that G has the *projective property* if there exists an embedding τ of G into $V(\Delta, R)$ so that the projection of $G\tau$ onto each dual ideal of Δ belongs to $G\tau$.

Let F_v be the torsion class of finite-valued lattice-ordered groups and $\overline{D}$ the torsion class of l-groups whose root system of "branch points" of regular subgroups satisfies the descending chain condition [**5**].

Theorem 3.2 [**14**]. *If G belongs to the torsion class $F_v \cap \overline{D}$, then G has the projective property.*

Theorem 3.3. *For a root system Δ, the following are equivalent.*

(1) $\sum(\Delta, R)$ *belongs to torsion class* $\overline{D}$.

(2) Δ *contains no copies of* $\Lambda =$ 

(3) $\overline{\Delta}$ *satisfies the descending chain condition.*

(4) $F(\Delta, R)$ *is the unique abelian a-closure of* $\sum(\Delta, R)$.

(5) *If K is a finite-valued l-subgroup of $V(\Delta, R)$ such that $\sum(\Delta, R) \subseteq K \subseteq V(\Delta, R)$, then $K \subseteq F(\Delta, R)$.*

Proof. 1, 2 and 3 are clearly equivalent since Δ can be identified with the regular subgroups of $\sum(\Delta, R)$.

(2$\rightarrow$4). Suppose H is an a-closure of $\sum(\Delta, R)$. Without loss of generality, $H \subseteq V(\Delta, R)$. If H is not isomorphic to $F(\Delta, R)$, then there exists $h \in H$ such that the support of h is not contained in finitely many roots in Δ. h is finite-valued since it is a-equivalent to some $g \in \sum(\Delta, R)$. Hence h can be represented as $h = \vee_{i=1}^{n} h_i$ where h_i are disjoint and special, and there is at least one h_i with support not contained in finitely many roots of Δ. So we may assume that h is special and the support of h is not contained in finitely many roots of Δ.

Since Δ contains no copy of Λ, each bounded root in Δ has only finitely many branch points.

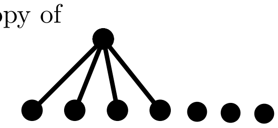
Let δ be the value of h and $h_{\overline{\delta}}$ is the projection of h onto $\overline{\delta}$ where $\overline{\delta} = \{\alpha \in$ support of $h \mid \alpha < \delta$, α belongs to the equivalence class in $\overline{\Gamma}$ containing $\delta\}$. Then $h_{\overline{\delta}} \in H$, since each $H \in F_v \cap \overline{D}$ has the projective property. Thus $h_1 = h - h_{\overline{\delta}} \in H$ is finite-valued. Now let $\Gamma = \{\delta \in \Delta \mid \delta$ is a maximal component of $h_1\}$, and let $h_2 = h_1 - h_{\overline{\Gamma}}$, where $h_{\overline{\Gamma}} =$ (projection of h_1 onto the union of $\overline{\delta}$ with $\delta \in \Gamma$). Then $h_{\overline{\Gamma}} \in H$, and $h_2 \in H$ is finite-valued. Since each bounded root has only a finite number of branch points, we may keep projecting onto $\overline{\delta}$ with δ a maximal component until there is no branch point in the support, and this only takes finitely many steps. Thus the support of h is contained in finitely many roots. Hence $h \in F(\Delta, R)$. This contradicts the fact that $h \in H \setminus F(\Delta, R)$.

(4$\to$2). Suppose that Δ contains a copy of Λ. By the results in [**4**], $\sum(\Lambda, R)$ has uncountably many non-isomorphic a-closures. Let A be an a-closure of $\sum(\Lambda, R)$ in $V(\Lambda, R)$ that is not isomorphic to $F(\Lambda, R)$, and let $B = \sum(\Delta \setminus \Lambda, R)$. There exist natural embeddings of A and B into $V(\Delta, R)$ so we may assume that A and B are l-subgroups of $V(\Delta, R)$. The support of A is contained in Λ, and the support of B is contained in $\Delta \setminus \Lambda$. Also $\sum(\Delta, R) \subseteq A \oplus B$ is a subgroup of $V(\Delta, R)$.

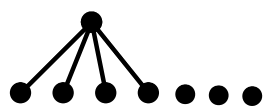
Actually, $A \oplus B$ is a finite-valued l-subgroup of $V(\Delta, R)$, and hence an a-extension of $\sum(\Delta, R)$. We need to show that $a + b \in A \oplus B$ implies that $(a + b) \vee 0 \in A \oplus B$. Suppose $a = a_1 + a_2 + \cdots + a_n$, and $b = b_1 + b_2 + \cdots + b_m$, where a_i and b_j are special, $|a_i| \wedge |a_j| = 0$ and $|b_i| \wedge |b_j| = 0$, if $i \neq j$. Supports of a and b are disjoint, hence $a + b$ can be written as $c_1 + c_2 + \cdots + c_l$, where each c_k is special, and $|c_k| \wedge |c_j| = 0$. Each c_k is either in the form of $c_k = a_i + \sum b_j$, where the values of b_j are less than that of a_is or in the form of $c_k = b_i + \sum a_j$, where the values of a_j are less than that of b_is. Therefore, $(a + b) \vee 0 = c_1 + c_2 + \cdots + c_n$, where $c_k > 0$ are all the positive components from the representation of $a + b$, hence $(a + b) \vee 0 \in A \oplus B$.

Let C be an abelian a-closure of $A \oplus B$ in $V(\Delta, R)$. C is an abelian a-closure of $\sum(\Delta, R)$. By assumption, C is isomorphic to $F(\Delta, R)$. Thus there exists an l-isomorphism σ such that $A\sigma \subseteq C\sigma = F(\Delta, R) \subseteq V(\Delta, R)$. Since A is finite-valued, we have $\Lambda \cong \Gamma(A) \cong \Gamma(A\sigma)$ where $A\sigma$ is finite-valued and $\Gamma(A\sigma)$ corresponds to the maximal components of $A\sigma$. Thus, without loss of generality, we assume $\Gamma(A\sigma) = \Lambda \subseteq \Delta$. Now the projection ρ of $V(\Delta, R)$ onto V_Λ induces an l-isomorphism of $A\sigma$ to V_Λ, and since $A\sigma \subseteq F(\Delta, R)$, $A\sigma\rho \subseteq F_\Lambda(\Delta, R) \cong F(\Lambda, R)$ which is an a-extension of $\sum(\Lambda, R)$. This contradicts our choice of A as an a-closure of $\sum(\Lambda, R)$ that is not isomorphic to $F(\Lambda, R)$.

(2→5). Let K be finite-valued. We have that $\sum(\Delta, R) \subseteq K \subseteq V(\Delta, R)$. We claim that $K \subseteq F(\Delta, R)$. Suppose not, then there exists some $k \in K$, such that the support of k is not contained in a finite number of roots of Δ. Since the support of k does not contain a copy of Λ it must contain a copy of



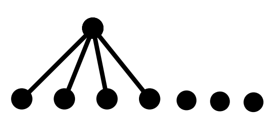
Without loss of generality, assume the support of

$k =$ 

Let δ be the maximal component of the support of k. The characteristic function χ_δ on δ belongs to $\sum(\Delta, R) \subseteq K$. It follows that $k - k(\delta)\chi_\delta \in K$, where $k(\delta)$ is the value of k at δ. But now $k - k(\delta)_{\chi_\delta}$ is not finite-valued. This contradicts the fact that $k - k(\delta)_{\chi_\delta} \in K$.

(5→4). Let K be an abelian a-closure of $\sum(\Delta, R)$. By the Conrad-Harvey-Holland embedding theorem, we can extend the identity map on $\sum(\Delta, R)$ to an l-isomorphism of K into $V(\Delta, R)$ so, without loss of generality, $\sum(\Delta, R) \subseteq K \subseteq V(\Delta, R)$. By (5), $K \subseteq F(\Delta, R)$, and since $F(\Delta, R)$ is an a-extension of $\sum(\Delta, R)$, we have $K = F(\Delta, R)$.

We observe that the assumption that K is a finite-valued l-group of $V(\Delta, R)$ does not imply that $K \subseteq F(\Delta, R)$. For example, let

$\Delta =$ 

and χ be the characteristic function on Δ. Then $K = R\chi + \sum_{i=1}^{\infty} R_i$ is a finite-valued l-group, but $K \nsubseteq F(\Delta, R)$.

Theorem 3.4. *For a root system Δ, the following are equivalent.*

(1) *Each finite-valued vector lattice G with $\Gamma(G) = \Delta$ has a unique abelian a-closure.*

(2) *$\overline{\Delta}$ satisfies the descending chain condition.*

If this is the case, then $F(\Delta, R)$ is the abelian a-closure of G.

Proof. $(2 \rightarrow 1)$. Each finite-valued vector lattice G with $\Gamma(G) = \Delta$ can be embedded into $V(\Delta, R)$ such that $\sum(\Delta, R) \subseteq G \subseteq V(\Delta, R)$ and G is an a-extension of $\sum(\Delta, R)$. But we know $F(\Delta, R)$ is the unique a-closure of $\sum(\Delta, R)$ and hence the unique a-closure of G.

$(1 \rightarrow 2)$ is the result of Theorem 3.3.

Corollary 3.5. *The above theorem holds for an abelian l-group G.*

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