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SPECTRAL PROPERTIES OF THE NONHOMOGENEOUS KLEIN-GORDON s-WAVE EQUATIONS

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ABSTRACT. In this article we investigate the eigenvalues and the spectral singularities of the boundary value problem

$$y'' + [\lambda - p(x)]^2 y = f(x), \quad x \in \mathbf{R}_+ = [0, \infty),$$
$$\alpha y'(0) - \beta y(0) = 0,$$

in the space $L_2(\mathbf{R}_+)$ where p and f are complex-valued functions and $\alpha, \beta \in \mathbf{C}$, with $|\alpha| + |\beta| \neq 0$.

1. Introduction. Let L denote the operator generated in $L_2(\mathbf{R}_+)$ by the differential expression

$$l(y) = -y'' + q(x)y, \quad x \in \mathbf{R}_+ = [0, \infty),$$

with the boundary condition $y'(0) - hy(0) = 0$, where q is a complex-valued function and $h \in \mathbf{C}$. The study of the spectral analysis of L was begun by Naimark [**18**]. He proved that the spectrum of L consists of the eigenvalues, the continuous spectrum and the spectral singularities. The spectral singularities are poles of the kernel of the resolvent and are also imbedded in the continuous spectrum, but they are not eigenvalues.

Lyance [**16**] showed that the spectral singularities play an important role in the spectral analysis of L. He also investigated the effect of the spectral singularities in the spectral expansion. The spectral singularities of dissipative Schrödinger operators with rapidly decreasing potential were considered by Hruscev [**9**]. In [**1**] and [**15**], by means of the uniqueness theorems of analytic functions, the dependence of the structure of the spectral singularities of Quadratic Pencil of Schrödinger Operators (QPSO) was studied. A two-fold spectral expansion in terms of the principal functions of QPSO with spectral singularities has been derived in [**2**]. In that article the effect of the spectral singularities in the spectral expansion of QPSO has also been investigated via the

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regularization of divergent integrals. Using the analytical properties of the generalized spectral function in the sense of Marchenko [**17**], the role of the spectral singularities in the spectral expansion of discrete Dirac operators have been studied in [**3**].

Let us consider the boundary value problem (BVP)

$$y'' + [\lambda - p(x)]^2 y = 0, \quad x \in \mathbf{R}_+, \tag{1.1}$$

$$y(0) = 0. \tag{1.2}$$

In relativistic quantum mechanics the equation (1.1) is called the Klein-Gordon s-wave equation for a particle of zero mass with static potential p [**8**]. The spectrum of the BVP (1.1)–(1.2) for a real-valued function p and for a complex-valued function p were studied in [**5, 6, 10, 11**] and in [**4**], respectively.

In the space $L_2(\mathbf{R}_+)$ we consider the following nonhomogeneous BVP

$$y'' + [\lambda - p(x)]^2 y = f(x), \quad x \in \mathbf{R}_+, \tag{1.3}$$

$$\alpha y'(0) - \beta y(0) = 0, \tag{1.4}$$

where p and f are complex-valued functions, p is continuously differentiable on $\mathbf{R}_+$ and $\alpha, \beta \in \mathbf{C}$, with $|\alpha| + |\beta| \neq 0$.

In this paper we discuss the eigenvalues and the spectral singularities of the BVP (1.3)–(1.4) and prove that this BVP has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity under the conditions

$$\lim_{x\to\infty} p(x) = 0, \quad \sup_{x\in\mathbf{R}_+} [|p'(x)| e^{\varepsilon\sqrt{x}}] < \infty,$$

and

$$\sup_{x\in\mathbf{R}_+} [|f(x)| e^{\varepsilon x^{1+\delta}}] < \infty, \quad \varepsilon > 0, \quad \delta > 0.$$

Some results on the study of the spectral analysis of the nonhomogeneous Schrödinger operators may be found in [**12–14**].

2. Jost solutions of homogeneous equation (1.1). Let us suppose that

$$\lim_{x\to\infty} p(x) = 0, \quad \int_0^\infty x|p'(x)|\,dx < \infty. \tag{2.1}$$

We will denote the solutions of (1.1) satisfying

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1 \quad \text{for } \lambda \in \overline{\mathbf{C}}_+ := \{\lambda : \lambda \in \mathbf{C},\ \ \operatorname{Im}\lambda \geq 0\},$$

and

$$\lim_{x\to\infty} y(x,\lambda)e^{i\lambda x} = 1 \quad \text{for } \lambda \in \overline{\mathbf{C}}_- := \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im}\lambda \leq 0\},$$

by $e^+(x,\lambda)$ and $e^-(x,\lambda)$, respectively. The solutions $e^+(x,\lambda)$ and $e^-(x,\lambda)$ are called the Jost solutions of (1.1). Under the condition (2.1) the Jost solutions have the representations [**10**],

$$e^+(x,\lambda) = e^{iw(x)+i\lambda x} + \int_x^\infty K^+(x,t)e^{i\lambda t}\,dt, \tag{2.2}$$

and

$$e^-(x,\lambda) = e^{-iw(x)-i\lambda x} + \int_x^\infty K^-(x,t)e^{-i\lambda t}\,dt, \tag{2.3}$$

for $\lambda \in \overline{\mathbf{C}}_+$ and $\lambda \in \overline{\mathbf{C}}_-$, respectively, where $w(x) = \int_x^\infty p(t)\,dt$, and the kernels $K^\pm(x,t)$ may be expressed in terms of $p \cdot K^\pm(x,t)$ are continuously differentiable with respect to their arguments and

$$|K^\pm(x,t)| \leq C\int_{(x+t)/2}^\infty \theta(s)\,ds, \tag{2.4}$$

$$|K^\pm_{x_i}(x_1,x_2)| \leq C\bigg\{\int_{(x_1+x_2)/2}^\infty \theta(t)\,dt + \theta\bigg(\frac{x_1+x_2}{2}\bigg)\bigg\}, \quad i=1,2, \tag{2.5}$$

where $C>0$ is a constant and

$$\theta(x) = |p(x)|^2 + |p'(x)|.$$

Therefore $e^+(x,\lambda)$ and $e^-(x,\lambda)$ are analytic with respect to λ in $\mathbf{C}_+ := \{\lambda : \lambda \in \mathbf{C},\ \operatorname{Im}\lambda > 0\}$ and $\mathbf{C}_- := \{\lambda : \lambda \in \mathbf{C},\ \operatorname{Im}\lambda < 0\}$, respectively, and continuous on the real axis. $e^\pm(x,\lambda)$ also satisfy

$$e^\pm(x,\lambda) = e^{\pm i[w(x)+\lambda x]} + O\bigg(\frac{e^{\mp x\ \operatorname{Im}\lambda}}{|\lambda|}\bigg), \quad \lambda \in \overline{\mathbf{C}}_\pm|\lambda| \longrightarrow \infty.$$

According to (2.2)–(2.5), the Wronskian of $e^+(x,\lambda)$ and $e^-(x,\lambda)$ is

$$W[e^+(x,\lambda), e^-(x,\lambda)] = -2i\lambda,$$

for $\lambda \in \mathbf{R} = (-\infty, \infty)$.

Let $\varphi^+(x,\lambda)$ and $\varphi^-(x,\lambda)$ denote the unbounded solutions of (1.1) subject to the conditions $\lim_{x\to\infty} e^{i\lambda x}\varphi^+(x,\lambda) = 1$ for $\lambda \in \overline{\mathbf{C}}_+$ and $\lim_{x\to\infty} e^{-i\lambda x}\varphi^-(x,\lambda) = 1$ for $\lambda \in \overline{\mathbf{C}}_-$, respectively. Then

$$W[e^{\pm}(x,\lambda), \varphi^{\pm}(x,\lambda)] = \mp 2i\lambda, \quad \lambda \in \overline{\mathbf{C}}_{\pm}.$$

3. Discrete spectrum of (1.3)–(1.4). Let us define the following functions

$$\begin{aligned} A^{\pm}(\lambda) &= \int_0^\infty f(x)e^{\pm}(x,\lambda)\,dx + \alpha e_x^{\pm}(0,\lambda) - \beta e^{\pm}(0,\lambda), \\ \beta^{\pm}(\lambda) &= \int_0^\infty f(x)\varphi^{\pm}(x,\lambda)\,dx + \alpha \varphi_x^{\pm}(0,\lambda) - \beta \varphi^{\pm}(0,\lambda). \end{aligned} \tag{3.1}$$

It is obvious that the functions

$$\begin{aligned} E^{\pm}(x,\lambda) = \pm\frac{1}{2i\lambda}\Big\{ &\varphi^{\pm}(x,\lambda)\int_x^\infty f(t)e^{\pm}(t,\lambda)\,dt \\ &- e^{\pm}(x,\lambda)\int_x^\infty f(t)\varphi^{\pm}(t,\lambda)\,dt \\ &+ B^{\pm}(\lambda)e^{\pm}(x,\lambda) - A^{\pm}(\lambda)\varphi^{\pm}(x,\lambda)\Big\}, \end{aligned} \tag{3.2}$$

are the solutions of the BVP (1.3)–(1.4).

Let us denote the eigenvalues and the spectral singularities of (1.3)–(1.4) by σ_p and σ_{ss}, respectively.

Lemma 3.1. *If* (2.1) *and*

$$\sup_{x\in\mathbf{R}_+}[|f(x)|e^{\varepsilon x^{1+\delta}}] < \infty, \quad \varepsilon > 0, \quad \delta > 0, \tag{3.3}$$

hold, then

$$\sigma_p = \{\lambda : \lambda \in \mathbf{C}_+, A^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbf{C}_-, A^-(\lambda) = 0\}. \tag{3.4}$$

Proof. Let $\lambda_0 \in \mathbf{C}_+$. From (2.2) and (2.4), we get that $e^+(x,\lambda_0) \in L_2(\mathbf{R}_+)$ and $\varphi^+(x,\lambda_0) \notin L_2(\mathbf{R}_+)$. Since

$$\varphi^+(x,\lambda)\int_x^\infty f(t)e^+(t,\lambda)\,dt = O(e^{-(\varepsilon/2)x^{1+\delta}}), \quad x\to\infty,$$

and

$$e^+(x,\lambda)\int_x^\infty f(t)\varphi^+(t,\lambda)\,dt = O(e^{-(\varepsilon/2)x^{1+\delta}}), \quad x\to\infty,$$

it follows from (3.2) that $E^+(x,\lambda_0)$ belongs to $L_2(\mathbf{R}_+)$ if and only if $A^+(\lambda_0)=0$. Let $\lambda_0 \in \mathbf{C}_-$. Similarly $E^-(x,\lambda_0)$ belongs to $L_2(\mathbf{R}_+)$ if and only if $A^-(\lambda_0)=0$. □

Analogously to the homogeneous Schrödinger and Klein-Gordon equations we have

$$\sigma_{ss} = \{\lambda : \lambda \in \mathbf{R}^*, A^+(\lambda)=0\} \cup \{\lambda : \lambda\in\mathbf{R}^*, A^-(\lambda)=0\}, \tag{3.5}$$

where $\mathbf{R}^* = \mathbf{R}\setminus\{0\}$, [**1, 4**].

From (3.4) and (3.5) we see that in order to investigate the structure of the eigenvalues and the spectral singularities of the BVP (1.3)–(1.4) we need to discuss the structure of the zeros of A^+ and A^- in $\overline{\mathbf{C}}_+$ and $\overline{\mathbf{C}}_-$, respectively. For the sake of simplicity we will consider only the zeros of A^+ in $\overline{\mathbf{C}}_+$.

Let us define

$$P_1^\pm = \{\lambda : \lambda\in\mathbf{C}_\pm, A^\pm(\lambda)=0\},\ P_2^\pm = \{\lambda : \lambda\in\mathbf{R}, A^\pm(\lambda)=0\}.$$

It follows from (3.4) and (3.5) that

$$\sigma_p = P_1^+ \cup P_1^-, \sigma_{ss} = \{P_2^+\cup P_2^-\}\setminus\{0\}. \tag{3.6}$$

Lemma 3.2. *If* (2.1) *and* (3.3) *hold, then*

(i) *The set P_1^+ is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.*

(ii) *The set P_2^+ is compact and its linear Lebesgue measure is zero.*

Proof. Equations (2.2) and (3.1) yield that A^+ is analytic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$, and has the form

$$A^+(\lambda) = i\lambda\alpha e^{iw(0)} + \eta + \int_0^\infty g(t)e^{i\lambda t}\,dt, \tag{3.7}$$

where

$$\eta = -\alpha[ip(0)e^{iw(0)} + K^+(0,0)] - \beta e^{iw(0)},$$

$$g(t) = f(t)e^{iw(t)} + \int_0^t f(x)K^+(x,t)\,dx + \alpha K_x^+(0,t) - \beta K^+(0,t). \tag{3.8}$$

By (3.8), we have $g \in L_1(\mathbf{R}_+)$, hence (3.7) implies that

$$A^+(\lambda) = i\lambda\alpha e^{iw(0)} + \eta + 0(1), \quad \lambda \in \overline{\mathbf{C}}_+, \quad |\lambda| \longrightarrow \infty, \tag{3.9}$$

which shows that the boundedness of the set P_1^+ and P_2^+. Since the function A^+ is analytic in $\mathbf{C}_+$, we get that P_1^+ has at most a countable number of elements and its limit points can lie only in a bounded subinterval of the real axis. By the boundary value uniqueness theorem of analytic function we obtain that the set P_2^+ is closed and its linear Lebesgue measure is zero [**7**]. □

From (3.6) and Lemma 3.2 we immediately get the following.

Theorem 3.3. *Under the conditions* (2.1) *and* (3.3)*, we have*

(i) *The set of eigenvalues of the* BVP (1.3)–(1.4) *is bounded, is no more than countable and its limit points can lie only in a bounded subinterval of the real axis.*

(ii) *The set of spectral singularities of the* BVP (1.3)–(1.4) *is bounded and its linear Lebesgue measure is zero.*

Defintion 3.4. The multiplicity of zero A^+, or A^-, in $\overline{\mathbf{C}}_+$, or $\overline{\mathbf{C}}_-$, is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.3)–(1.4).

Theorem 3.5. *If* (3.3) *and*

$$\lim_{x\to\infty} p(x) = 0, \quad \sup_{x\in\mathbf{R}_+} [e^{\varepsilon x}|p'(x)|] < \infty, \quad \varepsilon > 0, \tag{3.10}$$

hold, then the BVP (1.3)–(1.4) *has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity.*

Proof. From (2.4), (2.5), (3.3), (3.8) and (3.10), we find that

$$|g(t)| \le Ce^{-(\varepsilon/4)t}, \tag{3.11}$$

where $C > 0$ is a constant. Equations (3.7) and (3.11) show that the function A^+ has an analytic continuation from the real axis to the half-plane $\operatorname{Im}\lambda > -\varepsilon/4$. So the limit points of the set P_1^+ and P_2^+ cannot lie in $\mathbf{R}$, i.e., the bounded sets P_1^+ and P_2^+ have no limit points (see Lemma 3.2). Therefore, we have the finiteness of the zeros of A^+ in $\overline{\mathbf{C}}_+$. Moreover, all zeros of A^+ in $\overline{\mathbf{C}}_+$ has a finite multiplicity. Similarly, we get that the function A^- has a finite number of zeros with a finite multiplicity in $\overline{\mathbf{C}}_-$. $\square$

It is seen that the conditions (3.3) and (3.10) guarantee the analytic continuation of A^+ and A^- from the real axis to lower and upper half-planes, respectively. So the finiteness of eigenvalues and spectral singularities of the BVP (1.3)–(1.4) are obtained as a result of this analytic continuation.

Now let us suppose that

(3.12)
$$\lim_{x\to\infty} p(x) = 0, \quad \sup_{x\in\mathbf{R}_+} \{e^{\varepsilon x^\gamma}|p'(x)|\} < \infty, \quad \varepsilon > 0, \quad \frac{1}{2} \le \gamma < 1,$$

hold, which is weaker than (3.10). It is evident that, under the conditions (3.3) and (3.12), the function A^+ is analytic in $\mathbf{C}_+$ and infinitely differentiable on the real axis. But A^+ does not have an analytic continuation from the real axis to the lower half-plane. Similarly A^- does not have an analytic continuation from the real axis to the upper half-plane. Therefore, under the conditions (3.3) and (3.12), the finiteness of eigenvalues and spectral singularities of the BVP (1.3)–(1.4) cannot be proved by the same technique used in Theorem 3.5.

Let us denote the set of all limit points of P_1^+ and P_2^+ by P_3^+ and P_4^+, respectively, and the set of all zeros of A^+ with infinite multiplicity in $\overline{\mathbf{C}}_+$ by P_5^+.

It is obvious that

$$P_1^+ \cap P_5^+ = \phi,\, P_3^+ \subset P_2^+,\, P_4^+ \subset P_2^+,\, P_5^+ \subset P_2^+$$

and the linear Lebesgue measures of P_3^+, P_4^+ and P_5^+ are zero. Using the continuity of all derivatives of A^+ on the real axis, we get

$$P_3^+ \subset P_5^+,\, p_4^+ \subset P_5^+. \tag{3.13}$$

To prove the next result, we will use the following uniqueness theorem for the analytic functions on the upper half-plane.

Theorem 3.6 [**1**]. *Let us assume that the function u is analytic in $\mathbf{C}_+$, all of is derivatives are continuous up to the real axis and there exist $T > 0$ such that*

$$|u^{(n)}(z)| \le C_n, \quad n = 0, 1, \dots, \quad z \in \overline{C}_+, \quad |z| < 2T, \tag{3.14}$$

and

$$\left| \int_{-\infty}^{-T} \frac{\ln |u(x)|}{1+x^2}\, dx \right| < \infty, \left| \int_{T}^{\infty} \frac{\ln |u(x)|}{1+x^2}\, dx \right| < \infty. \tag{3.15}$$

If the set Q with linear Lebesgue measure zero is the set of all zeros of the function u with infinite multiplicity, and if

$$\int_0^h \ln F(s)\, d\mu(Q_s) = -\infty,$$

where $F(s) = \inf_n (C_n s^n / n!)$, $n = 0, 1, 2, \dots, \mu(Q_s)$ is the linear Lebesgue measure of s-neighborhood of Q and h is an arbitrary positive constant, then $u(z) = 0$.

Lemma 3.7. *If* (3.3) *and* (3.12) *hold, then $P_5^+ = \phi$.*

Proof. It is easy to see from Lemma 3.2 and (3.7), (3.8) that A^+ satisfies (3.14) and (3.15). Since the function A^+ is not equal to zero identically, then by Theorem 3.6, P_5^+ satisfies

$$\int_0^h \ln F(s)\, d\mu(P_{5,s}^+) > -\infty, \tag{3.16}$$

where $F(s) = \inf_n (C_n s^n/n!)$, $\mu(P_{5,s}^+)$ is the linear Lebesgue measure of s-neighborhood of P_5^+ and $h > 0$ is a constant.

Using (2.4), (2.5), (3.7), (3.8) and (3.12), we obtain

$$\left| \frac{d^n}{d\lambda^n} A^+(\lambda) \right| \le C_n = 2^n C \int_0^\infty x^n e^{-\varepsilon x^\gamma}\, dx \le D d^n n! n^{n(1-\gamma/\gamma)}, \tag{3.17}$$

where D and d are constants depending on $\varepsilon, \delta, \gamma$ and C. Substituting (3.17) in the definition of $F(s)$ we arrive at

$$F(s) = \inf_n \frac{C_n s^n}{n!} \le D \exp\left\{ -\frac{1-\gamma}{\gamma} e^{-1/(1-\gamma)} d^{-\gamma/(1-\gamma)} s^{-\gamma/(1-\gamma)} \right\},$$

or

$$\int_0^h s^{-\gamma/(1-\gamma)}\, d\mu(P_{5,s}^+) < \infty, \tag{3.18}$$

by (3.16). Since $\gamma/(1-\gamma) \ge 1$, consequently (3.18) holds for arbitrary s if and only if $\mu(P_{5,s}^+) = 0$ or $P_5^+ = \phi$. □

Theorem 3.8. *Under the conditions* (3.3) *and* (3.12), *the* (BVP) (1.3)–(1.4) *has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the functions A^+ and A^- have a finite number of zeros with a finite multiplicities in $\overline{\mathbf{C}}_+$ and $\overline{\mathbf{C}}_-$, respectively. We will prove it only for A^+.

From (3.13) and Lemma 3.7, we find that $P_3^+ = P_4^+ = \phi$. So the bonded sets P_1^+ and P_2^+ have no limit points, i.e., the function A^+ has only a finite number of zeros in $\overline{\mathbf{C}}_+$. Since $P_5^+ = \phi$ these zeros are of finite multiplicity. $\square$

It follows from Theorem 3.8 that the weakest conditions which guarantee the finiteness of eigenvalues and spectral singularities of the BVP (1.3)–(1.4) are

$$\lim_{x\to\infty} p(x) = 0, \quad \sup_{x\in\mathbf{R}_+} [e^{\varepsilon\sqrt{x}}|p'(x)|] < \infty,$$
$$\sup_{x\in\mathbf{R}_+} [e^{\varepsilon x^{1+\delta}}|f(x)|] < \infty, \quad \varepsilon > 0,\ \delta > 0.$$

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