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ON INFINITE TENSOR PRODUCTS OF PROJECTIVE UNITARY REPRESENTATIONS

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ABSTRACT. We initiate a study of infinite tensor products of projective unitary representations of a discrete group G. Special attention is given to regular representations twisted by 2-cocycles and to projective representations associated with CCR-representations of bilinear maps. Detailed computations are presented in the case where G is a finitely generated free abelian group. We also discuss an extension problem about product type actions of G, where the projective representation theory of G plays a central role.

1. Introduction. The theory of infinite tensor products of Hilbert spaces started with the seminal paper by von Neumann [**22**]. Later on, Guichardet [**11, 12**] approached this matter from a slightly different point of view and developed a unified framework for treating several related concepts involving operators, algebras and functionals. The notion of infinite tensor product has been mainly used in this form in operator algebras and quantum field theory over the last three decades, see, e.g., [**10**] for a recent overview.

The existence of some infinite tensor product of representations of a group has been established and used in some recent works. For example, it was shown in [**1**] that a locally compact group is σ-compact and amenable if and only if there exists an infinite tensor power of its regular representation. Such an infinite tensor power construction was then a useful tool for studying covariance of certain (induced) product-type representations of generalized Cuntz algebras with respect to natural product-type actions. This circle of ideas has been generalized and thoroughly investigated in [**4**]. In another direction, the infinite

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tensor product of certain unitary representations of some group of diffeomorphisms was shown to exist under suitable assumptions in [**13**].

In this paper we initiate a study of infinite tensor products of *projective* unitary representations of a discrete group. It is in fact not obvious that such infinite tensor products exist at all. Indeed it is quite easy to realize that it is impossible to form the infinite tensor power of a single projective unitary representation unless the associated 2-cocycle vanishes. Besides its intrinsic interest, this new generality has the potential advantage to allow for extensions of the analysis given in [**1, 4**] to a broader class of product-type actions on the 0^{th}-degree part of extended Cuntz algebras. It is also relevant when studying extensions of product-type actions from the algebraic to the von Neumann algebra level. Finally it provides a way to represent faithfully on infinite tensor product spaces some familiar C^*-algebras like noncommutative tori. To avoid technicalities, we stick to the case of a discrete group, although it could be of interest in the future to consider a locally compact, or even just a topological, group and strongly continuous projective unitary representations of such a group.

The paper is organized as follows. Section 2 is devoted to some preliminaries on projective unitary representations, product sequences of 2-cocycles and infinite tensor products. Section 3 contains our main existence results for infinite tensor products of projective unitary representations. We especially display some sufficient conditions for countable amenable groups in the case of projective regular representations and in the case of projective representations associated with CCR-representations of bilinear maps. To illustrate our work we specialize in Section 4 to the case of finitely generated free abelian groups. The final section deals with infinite tensor products of actions of a discrete group G on von Neumann algebras. We concentrate our attention to the existence problem of such product actions in the case of unitarily implemented actions. One of our results exhibits an obstruction for extending some algebraic tensor power action of G to the weak closure that lies in the second cohomology group $H^2(G, \mathbf{T})$. In another result, the obstruction lies in the non-amenability of G.

2. Preliminaries. Throughout this note G denotes a non-trivial discrete group, with neutral element e.

A 2-*cocycle*, or multiplier, on G with values in the circle group $\mathbf{T}$ is a map $u : G \times G \to \mathbf{T}$ such that

$$u(x, y)u(xy, z) = u(y, z)u(x, yz), \quad x, y, z \in G,$$

see, e.g., [**5**, Chapter IV]. We will consider only *normalized* 2-cocycles, satisfying

$$u(x, e) = u(e, x) = 1, \quad x \in G.$$

The set of all such 2-cocycles, which is denoted by $Z^2(G, \mathbf{T})$, becomes an abelian group under pointwise product. We equip $Z^2(G, \mathbf{T})$ with the topology of pointwise convergence.

A 2-cocycle v on G is called a *coboundary* whenever $v(x, y) = \rho(x)\rho(y)\overline{\rho(xy)}$, $(x, y \in G)$, for some $\rho : G \to \mathbf{T}$, $\rho(e) = 1$, in which case we write $v = \mathrm{d}\rho$ (such a ρ is uniquely determined up to multiplication by a character). The set of all coboundaries, which is denoted by $B^2(G, \mathbf{T})$, is a subgroup of $Z^2(G, \mathbf{T})$, which is easily seen to be closed. (Indeed, assume that $(\mathrm{d}\rho_\alpha)$ is a net in $B^2(G, \mathbf{T})$ converging to $v \in Z^2(G, \mathbf{T})$. Due to Tychonov's theorem, we may, by passing to a subnet if necessary, assume that ρ_α converges pointwise to ρ, for some $\rho : G \to \mathbf{T}$, $\rho(e) = 1$. Then we have $v = \mathrm{d}\rho$.)

The quotient group $H^2(G, \mathbf{T}) := Z^2(G, \mathbf{T})/B^2(G, \mathbf{T})$ is called the *second cohomology group* of G with values in $\mathbf{T}$. We denote elements in $H^2(G, \mathbf{T})$ by $[u]$ and write $v \sim u$ when $[v] = [u] \quad (u, v \in Z^2(G, \mathbf{T}))$. We also write $v \sim_\rho u$ when we have $v = (\mathrm{d}\rho)u$ for some coboundary $\mathrm{d}\rho$.

We recall a few facts concerning infinite products of complex numbers, see [**17**]. Let (z_i) denote a sequence of complex numbers. We say that the infinite product $\prod_i z_i$ exists, or converges, if the limit of the net $(\prod_{i \in J} z_i)_{J \in \mathcal{F}}$ exists, where $\mathcal{F}$ denotes the family of nonempty finite subsets of $\mathbf{N}$ ordered by inclusion. We then also use $\prod_i z_i$ to denote this limit. We will need the following result:

Assume that $\sum_i |1 - z_i| < \infty$. Then $\prod_i z_i$ exists, and $\prod_i z_i \neq 0$ if all z_i's are nonzero. Conversely, assume that $\prod_i z_i$ converges to a nonzero element. Then $\sum_i |1 - z_i| < \infty$.

We shall be interested in *product* sequences in $Z^2(G, \mathbf{T})$: we call a sequence (u_i) in $Z^2(G, \mathbf{T})$ a *product* sequence whenever the (pointwise) infinite product $u = \prod_i u_i$ exists on $G \times G$ (u being then obviously a 2-cocycle itself).

A cohomological problem concerning product sequences is that perturbing a product sequence (by a coboundary in each component) does not necessarily lead to a product sequence, as may be illustrated by taking all u_i's to be 1 and perturbing by the same coboundary $v \neq 1$ in each component. The following lemma somewhat clarifies this problem.

Lemma 2.1. *Let (u_i) and (v_i) be two sequences in $Z^2(G, \mathbf{T})$ satisfying $v_i \sim_{\rho_i} u_i$ for every i.*

i) *Assume that $\rho := \prod_i \rho_i$ exists. Then (v_i) is a product sequence if and only if (u_i) is a product sequence, in which case we have $\prod_i v_i \sim_\rho \prod_i u_i$.*

ii) *Assume that (u_i) and (v_i) are both product sequences. Then $\prod_i v_i \sim \prod_i u_i$, even if $\prod_i \rho_i$ does not necessarily exist.*

Proof. As i) is straightforward, we only show ii). So we assume that $u = \prod_i u_i$ and $v = \prod_i v_i$ both exist. Then $w := \prod_i \mathrm{d}\rho_i = \prod_i \overline{u_i} v_i$ also exists and is the limit of a net of 2-coboundaries. As $B^2(G, \mathbf{T})$ is closed, this implies that $w \in B^2(G, \mathbf{T})$. Since $v = wu$, this shows that $v \sim u$, as asserted.

(To see that $\prod_i \rho_i$ does not necessarily exist, assume that G possesses a nontrivial character γ. Set $u_i = v_i = 1$ and $\rho_i = \gamma$ for all i. Then clearly $v_i \sim_{\rho_i} u_i$ while $\prod_i \rho_i$ does not exist.) □

A *projective unitary representation* U of G on a Hilbert space $\mathcal{H}$ associated with some $u \in Z^2(G, \mathbf{T})$ is a map from G into the group of unitaries on $\mathcal{H}$ such that

$$U(x)U(y) = u(x, y)U(xy), \quad x, y \in G.$$

If we pick a $\rho : G \to \mathbf{T}$ satisfying $\rho(e) = 1$ and set $V = \rho U$, then V is also a projective unitary representation of G on $\mathcal{H}$ associated with a 2-cocycle v satisfying $v \sim_\rho u$. Such a V is called a *perturbation* of U.

To each $u \in Z^2(G, \mathbf{T})$ one may associate the left u-regular projective unitary representation λ_u of G on $l^2(G)$ defined by

$$(\lambda_u(x)f)(y) = u(y^{-1}, x)f(x^{-1}y), \quad f \in l^2(G),\ x, y \in G.$$

Choosing $u = 1$ gives the left regular representation of G which we will just denote by λ. It is well known, and easy to see, that if $v \sim_\rho u$, then λ_v is unitarily equivalent to $\rho\lambda_u$.

For $i = 1, 2$, let U_i be a projective unitary representation of G on a Hilbert space $\mathcal{H}_i$ associated with $u_i \in Z^2(G, \mathbf{T})$. Then the naturally defined tensor product representation $U_1 \otimes U_2$ is easily seen to be a projective unitary representation of G on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ associated with the product cocycle $u_1 u_2$. In the case of ordinary unitary representations of a group, it is a classical result of Fell (cf. [**8**]) that the left regular representation acts in an absorbing way with respect to tensoring (up to multiplicity and equivalence). In the projective case we have the following analogue.

Proposition 2.2. *Let u, v be elements in $Z^2(G, \mathbf{T})$, and let V be any projective unitary representation of G on a Hilbert space $\mathcal{H}$ associated with v. Then the tensor product representation $\lambda_u \otimes V$ is unitarily equivalent to $\lambda_{uv} \otimes id_{\mathcal{H}}$, i.e., to $(dim\ V) \cdot \lambda_{uv}$.*

Proof. We leave to the reader to check that the same unitary operator W as in the nonprojective case, which is determined on $l^2(G) \otimes \mathcal{H}$ ($\cong l^2(G, \mathcal{H})$) by $(W(f \otimes \psi))(x) = f(x)V(x^{-1})\psi$, implements the asserted equivalence. □

We conclude this section with a short review on infinite tensor products of Hilbert spaces and operators. (See [**11, 12**] for more information).

Let $\mathcal{H} = \{\mathcal{H}_i\}$ denote a sequence of Hilbert spaces and $\phi = \{\phi_i\}$ be a sequence of unit vectors where $\phi_i \in \mathcal{H}_i$ for each $i \geq 1$. We denote by $\mathcal{H}^\phi$ or by $\bigotimes_i^\phi \mathcal{H}_i$ the associated infinite tensor product Hilbert space of the $\mathcal{H}_i$'s along the sequence ϕ.

For any sequence $\psi_i \in \mathcal{H}_i$ such that

$$\sum_i |\, 1 - \|\psi_i\| \,| < \infty \quad \text{and} \quad \sum_i |\, 1 - (\psi_i, \phi_i) \,| < \infty,$$

there corresponds a so-called decomposable vector in $\mathcal{H}^\phi$ denoted by

$\otimes_i\psi_i$. If $\otimes_i\xi_i$ is another decomposable vector in $\mathcal{H}^\phi$, then

$$(\otimes_i\psi_i, \otimes_i\xi_i) = \prod_i (\psi_i, \xi_i).$$

A decomposable vector of the form $\psi_1 \otimes \cdots \otimes \psi_k \otimes \phi_{k+1} \otimes \phi_{k+2} \otimes \cdots$ is called elementary. The set of all elementary decomposable vectors is total in $\mathcal{H}^\phi$.

Let $T_1, T_2, \ldots$ be a sequence of bounded linear operators where each T_i acts on $\mathcal{H}_i$. For each fixed $n \in \mathbf{N}$ there exists a unique bounded linear operator $\tilde{T}_n$ acting on $\mathcal{H}^\phi$ which is determined by

$$\tilde{T}_n(\otimes_i\psi_i) = T_1\psi_1 \otimes \cdots \otimes T_n\psi_n \otimes \psi_{n+1} \otimes \psi_{n+2} \otimes \cdots$$

for each decomposable vector $\otimes_i\psi_i$. Similarly, one may define $\tilde{T}_J$ for each (nonempty) finite $J \subset \mathbf{N}$. Under certain assumptions, the net $\{\tilde{T}_J\}$ converges in the strong operator topology to a bounded linear operator on $\mathcal{H}^\phi$ which is then denoted by $\otimes_i T_i$.

By [**12**, Proposition 6], a sufficient condition for $\otimes_i T_i$ to exist is that

$$\prod_i \|T_i\| \quad \text{exists,} \quad \sum_i |1 - \|T_i\phi_i\|\,| < \infty \quad \text{and} \quad \sum_i |1 - (T_i\phi_i, \phi_i)\,| < \infty,$$

in which case we have $(\otimes_i T_i)\,(\otimes_i\psi_i) = \otimes_i T_i\psi_i$ for all elementary decomposable vectors $\otimes_i\psi_i$.

When all T_i's are unitaries (which is the case of interest in this paper) we have the following result, which will be used several times in the sequel.

Proposition 2.3. *Let (T_i) be a sequence of unitaries where each T_i acts on $\mathcal{H}_i$. Then $\otimes_i T_i$ exists on $\mathcal{H}^\phi$ if and only if*

$$(*) \qquad \sum_i |1 - (T_i\phi_i, \phi_i)| < \infty,$$

in which case $\otimes_i T_i$ is a unitary on $\mathcal{H}^\phi$ satisfying $(\otimes_i T_i)^* = \otimes_i T_i^*$.

Proof. Assume first that $(*)$ holds. It is then quite elementary to deduce from Guichardet's result mentioned above that $\otimes_i T_i$ and $\otimes_i T_i^*$

both exist. Moreover, these two operators are then isometries, being the strong limit of a net of unitaries, and they are easily seen to be the inverse of each other. So both are unitaries satisfying $(\otimes_i T_i)^* = \otimes_i T_i^*$.

Assume now that $T := \otimes_i T_i$ exists on $\mathcal{H}^\phi$. Then T is nonzero, being an isometry, so there are elementary decomposable vectors $\otimes_i \psi_i$ and $\otimes_i \xi_i$ such that

$$0 \neq c := (T \otimes_i \psi_i, \otimes_i \xi_i).$$

Let J be any finite subset of $\mathbf{N}$ large enough so that $\psi_i = \xi_i = \phi_i$ for all $i \notin J$. Then we have

$$(\tilde{T}_J \otimes_i \psi_i, \otimes_i \xi_i) = \prod_{i \in J} (T_i \, \psi_i, \xi_i).$$

Since $T = \lim_J \tilde{T}_J$, we get $c = \lim_J \prod_{i \in J} (T_i \, \psi_i, \xi_i)$, i.e., $\prod_{i \in \mathbf{N}} (T_i \, \psi_i, \xi_i)$ converges to a nonzero value.

Thus we get $\sum_i |1 - (T_i \, \psi_i, \xi_i)| < \infty$. Therefore $\sum_i |1 - (T_i \, \phi_i, \phi_i)| < \infty$ since $\psi_i = \xi_i = \phi_i$ for all but finitely many i's. $\square$

3. Infinite tensor products of projective unitary representations. Before attacking the main problem whether it is possible to form an infinite tensor product of a sequence of projective unitary representations, at least in some cases, we first show that this construction, when possible, produces a new projective unitary representation of G, and also make some general observations.

Theorem 3.1. *Let U_i be a sequence of projective unitary representations of G acting respectively on a Hilbert space $\mathcal{H}_i$ and with associated $u_i \in Z^2(G, \mathbf{T})$. Let $\phi = (\phi_i)$ be a sequence of unit vectors where each $\phi_i \in \mathcal{H}_i$. Assume that $\otimes_i U_i(x)$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \mathcal{H}_i$ for each $x \in G$. Then we have*

i) *(u_i) is a product sequence in $Z^2(G, \mathbf{T})$.*

ii) *The map $x \to U^\phi(x) := \otimes_i U_i(x)$ is a projective unitary representation of G on $\mathcal{H}^\phi$ with $u = \prod_i u_i$ as its associated 2-cocycle.*

iii) *If there exists one k such that U_k is unitarily equivalent to λ_{u_k}, then U^ϕ is unitarily equivalent to $\lambda_u \otimes id_{\mathcal{H}}$, for some Hilbert space $\mathcal{H}$.*

iv) *$\lambda \otimes U^\phi$ is unitarily equivalent to $\lambda_u \otimes id_{\mathcal{H}^\phi}$.*

Proof. Notice first that Proposition 2.3 implies that each $U^\phi(x) := \otimes_i U_i(x)$ is a unitary.

i) Let $g, h \in G$. We must show that $\prod_i u_i(g,h)$ converges. Now

$$\otimes_i U_i(gh)$$

and

$$(\otimes_i U_i(g))(\otimes_i U_i(h)) = \otimes_i U_i(g) U_i(h) = \otimes_i u_i(g,h)\, U_i(gh)$$

are both unitaries. Putting $a_i = (\, U_i(gh)\phi_i\,,\, \phi_i\,)$, we deduce from Proposition 2.3 that

$$\sum_i |\, 1 - a_i\,| < \infty \quad \text{and} \quad \sum_i |\, 1 - u_i(g,h) a_i\,| < \infty.$$

This implies that $\sum_i |\, 1 - u_i(g,h)\,| < \infty$, and therefore that $\prod_i u_i(g,h)$ converges, as desired. (We use here implicitly that whenever $z \in \mathbf{T}$ and $a \in \mathbf{C}$, then $|1 - z| = |1 - \bar{z}| \le |1 - a| + |a - \bar{z}| = |1 - a| + |za - 1|$).

ii) Using i) we get

$$\begin{aligned} U^\phi(x)\, U^\phi(y) &= \otimes_i u_i(x,y)\, U_i(xy) \\ &= \Big(\prod_i u_i(x,y)\Big) \otimes_i U_i(xy) \\ &= u(x,y)\, U^\phi(xy) \end{aligned}$$

for all $x, y \in G$, as asserted.

iii) and iv) follow easily from Proposition 2.2. □

An obvious, but noteworthy consequence of part i) of this theorem is that it is impossible to form the infinite tensor power of a single projective unitary representation unless the associated 2-cocycle vanishes. In another direction, the case where infinitely many of the U_i's are projective regular representations of G cannot occur in this theorem when G is uncountable or nonamenable, as easily follows from our next theorem. (We refer to [**17**] or [**18**] for information on amenability).

Theorem 3.2. *Let (u_i) be a sequence in $Z^2(G, \mathbf{T})$ and set $U_i = \lambda_{u_i}$ for every i. Let $\phi = (\phi_i)$ be a sequence of unit vectors in $l^2(G)$. Assume* *that* $\otimes_i U_i(x)$ *exists on* $\mathcal{H}^\phi = \otimes_i^\phi l^2(G)$ *for each* $x \in G$. *Then* G *is countable and amenable.*

Proof. Using Proposition 2.3, it follows that $\sum_i |1 - (U_i(x)\phi_i, \phi_i)| < \infty$ for every $x \in G$. Notice that

$$|(U_i(x)\phi_i, \phi_i)| \leq (\lambda(x)|\phi_i|, |\phi_i|) \leq 1.$$

Hence we get

$$(\lambda(x)|\phi_i|\,,\,|\phi_i|) \to 1 \quad x \in G.$$

This means that the trivial one-dimensional representation of G is weakly contained in λ and the amenability of G follows.

Moreover, setting $f_i(x) := (\lambda(x)|\phi_i|, |\phi_i|) \geq 0$ we have $0 \leq f_i \leq 1$, $f_i \in C_0(G)$ (cf. [**8**]) and $f_i \to 1$ pointwise. Then $f_i^{-1}([1/2, 1]) =: H_i$ is finite and $G = \cup_i H_i$, so G is countable. □

In view of this theorem, it is quite natural to wonder whether some converse holds. We shall provide a partial answer in Corollary 3.4. To ease our exposition, we introduce some terminology. A sequence (F_i) of nonempty, finite subsets of G will be called an F-sequence (respectively σF-sequence) for G whenever

$$\lim_i \frac{\#(F_i \cap xF_i)}{\#F_i} = 1 \quad \text{for all } x \in G$$
$$\Big(\text{resp.} \;\; \sum_i |1 - \frac{\#(F_i \cap xF_i)}{\#F_i}| < \infty \quad \text{for all } x \in G\Big).$$

An F-sequence (F_i) for G is often called a *Fölner* sequence in the literature. We remark that the definition is usually phrased in a slightly different, but equivalent, way (involving the symmetric difference of sets) and that some authors also require that $F_i \subseteq F_{i+1}$ for every i. Anyhow, thanks to Fölner (see [**17, 18**]) we know that G is countable and amenable if and only if G has an F-sequence. Now, obviously, a σF-sequence for G is also an F-sequence. Moreover, any F-sequence has some subsequence which is a σF-sequence, as is easily checked. Hence we can also conclude that G is countable and amenable if and only if G has a σF-sequence.

When F is a subset of G, we denote by χ_F its characteristic function.

Theorem 3.3. *Let (u_i) be a sequence in $Z^2(G,\mathbf{T})$. Assume that G is countable and amenable, and has a σF-sequence (F_i) which satisfies*

$$(*) \qquad \sum_i \frac{1}{\#F_i} \sum_{y\in F_i} |1-u_i(y^{-1},x)| < \infty \quad \text{for all } x \in G.$$

Set $U_i = \lambda_{u_i}$ and $\phi_i := \chi_{F_i}/(\#F_i)^{1/2}$ for every i. Then $\phi = (\phi_i)$ is a sequence of unit vectors in $l^2(G)$ such that $\otimes_i U_i$ exists on $\mathcal{H}^\phi = \otimes_i^\phi l^2(G)$.

Proof. We first record some easy calculations. Let F be a finite nonempty subset of G, and set $\phi_F := \chi_F/(\#F)^{1/2}$. Let $u \in Z^2(G,\mathbf{T})$. Then we have

$$(\lambda(x)\phi_F, \phi_F) = \frac{1}{\#F}\,\#(F\cap xF)$$

for every $x \in G$. More generally we have

$$(\lambda_u(x)\phi_F,\phi_F) = \frac{1}{\#F}\sum_{y\in F\cap xF} u(y^{-1},x)$$

and therefore

$$(\,(\lambda(x)-\lambda_u(x))\phi_F,\phi_F\,) = \frac{1}{\#F}\sum_{y\in F\cap xF}(1-u(y^{-1},x))$$

for all $x\in G$.

Using the triangle inequality and the above computations, we get

$$\begin{aligned}
&\sum_i |1-(U_i(x)\phi_i,\phi_i)| \\
&\quad \le \sum_i |1-(\lambda(x)\phi_i,\phi_i)| + \sum_i |(\,(\lambda(x)-U_i(x))\phi_i,\phi_i\,)| \\
&\quad = \sum_i \left|1-\frac{\#(F_i\cap xF_i)}{\#F_i}\right| + \sum_i \frac{1}{\#F_i}\left|\sum_{y\in F_i\cap xF_i}(1-u_i(y^{-1},x))\right| \\
&\quad \le \sum_i \left|1-\frac{\#(F_i\cap xF_i)}{\#F_i}\right| + \sum_i \frac{1}{\#F_i}\sum_{y\in F_i}|1-u_i(y^{-1},x)|
\end{aligned}$$

for all $x \in G$. Since (F_i) is a σF-sequence for G satisfying $(*)$, both sums above converge for all $x \in G$. Hence, $\sum_i |1 - (U_i(x)\phi_i, \phi_i)| < \infty$ for all $x \in G$ and the assertion follows from Proposition 2.3. $\square$

Clearly, if $u_i = 1$ for all but finitely many i's, any σF-sequence (F_i) for G trivially satisfies $(*)$. In this case, the above theorem could also have been deduced from [**7**].

Corollary 3.4. *Let G be countable and amenable, and let (v_j) be a product sequence in $Z^2(G, \mathbf{T})$. Then there exist a subsequence (u_i) of (v_j) and a sequence $\phi = (\phi_i)$ of unit vectors in $l^2(G)$ such that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi l^2(G)$.*

Proof. First we pick a σF-sequence (F_i) for G and a growing sequence (H_i) of nonempty finite subsets of G satisfying $\cup_i H_i = G$. Since the (pointwise) product $\prod_j v_j$ exists, we can choose a subsequence (u_i) of (v_j) satisfying

$$|1 - u_i(y^{-1}, x)| \leq 1/i^2 \quad \text{for all } x \in H_i,\ y \in F_i,\ i \in \mathbf{N}.$$

Let $x \in G$ and choose $N \in \mathbf{N}$ such that $x \in H_N$. Then we get

$$\begin{aligned}\sum_i \frac{1}{\#F_i} \sum_{y \in F_i} &|1 - u_i(y^{-1}, x)| \\ &\leq \sum_{i<N} 2 \ + \sum_{i \geq N} \frac{1}{\#F_i} \sum_{y \in F_i} 1/i^2 = 2(N-1) \ + \sum_{i \geq N} 1/i^2 < \infty.\end{aligned}$$

This shows that (F_i) satisfies $(*)$ in Theorem 3.3, from which the result then clearly follows. $\square$

Corollary 3.5. *Let G be countable and amenable. Then there always exist some product sequence (u_i) in $Z^2(G, \mathbf{T})$ satisfying $u_i \neq 1$ for all i and some sequence $\phi = (\phi_i)$ of unit vectors in $l^2(G)$ such that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi l^2(G)$. If $H^2(G, \mathbf{T})$ is nontrivial and $1 \neq [u] \in H^2(G, \mathbf{T})$, then the sequence (u_i) above may be chosen so that $u = \prod_i u_i$.*

Proof. We call a product sequence (u_i) in $Z^2(G,\mathbf{T})$ 1-free if $u_i \neq 1$ for all i. It is easy to see that 1-free product sequences do exist in $B^2(G,\mathbf{T})$. As 1-freeness is clearly preserved when passing to subsequences, the first assertion follows from the previous corollary. The 1-free product sequence (u_i) is then in $B^2(G,\mathbf{T})$. Therefore (by closedness) $\prod_i u_i \in B^2(G,\mathbf{T})$, so we may write it as $\mathrm{d}\rho$ for some normalized $\rho : G \to \mathbf{T}$. Assume now $H^2(G,\mathbf{T})$ is non-trivial and $1 \neq [u] \in H^2(G,\mathbf{T})$. Set $v_1 = \overline{\mathrm{d}\rho}\, u$ and $v_i = u_{i-1}, i > 1$. Then (v_i) is a 1-free product sequence satisfying $u = \prod_i v_i$. Further we can define a new sequence $\psi = (\psi_i)$ of unit vectors in $l^2(G)$, by setting $\psi_1 = \delta_e$ and $\psi_i = \psi_{i-1}$, $i > 1$. It is then obvious that $\otimes_i \lambda_{v_i}$ exists on $\mathcal{H}^\psi$, which proves the second assertion. □

Remarks. 1) It follows from Theorem 3.1 iii) that representations obtained as the infinite tensor product of projective regular representations are never irreducible.

2) Let G be countable and amenable, and let (u_i) and (v_i) be two sequences in $Z^2(G,\mathbf{T})$ satisfying $v_i \sim_{\rho_i} u_i$ for every i. Assume that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi l^2(G)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $l^2(G)$. As $\prod_i v_i$ does not necessarily exist, it may happen that $\otimes_i \lambda_{v_i}$ cannot be formed at all, cf. Theorem 3.1. However, it is quite clear that $\rho_1 \lambda_{v_1} \otimes \rho_2 \lambda_{v_2} \otimes \cdots$ exists on $\otimes^{\psi_i} l^2(G)$, where ψ_i is defined by $\psi_i(x) = \overline{\rho_i(x^{-1})}\phi_i(x)$, and this may be considered as a problem of gauge fixing. On the other hand, let us also assume that $\otimes_i \lambda_{v_i}$ exists on $\mathcal{H}^\psi = \otimes_i^\psi l^2(G)$ for some sequence $\psi = (\psi_i)$ of unit vectors in $l^2(G)$. Then we may conclude that $\otimes_i \lambda_{v_i}$ is, up to unitary equivalence, just a perturbation of $\otimes_i \lambda_{u_i}$.

(To prove this, we first appeal to Theorem 3.1 and obtain that both $u = \prod_i u_i$ and $v = \prod_i v_i$ exist. Using Lemma 2.1 we may then write $v = \mathrm{d}\rho\, u$ for some normalized $\rho : G \to \mathbf{T}$. Now, using that $\lambda_v \simeq \rho\lambda_u$ and Theorem 3.1, we get

$$\otimes_i \lambda_{v_i} \simeq \lambda_v \otimes id \simeq \rho(\lambda_u \otimes id) \simeq \rho(\otimes_i \lambda_{u_i}),$$

where id denotes the identity representation of G on any infinite separable Hilbert space.)

3) To produce examples of infinite tensor product of projective unitary representations of not necessarily amenable groups, one can pro-

ceed as follows. Let G be any countable group possessing a nontrivial amenable factor group K (one can here for instance let G be any non-perfect, non-amenable group, e.g., any non-abelian countable free group, since the abelianized group $G/[G,G]$ is then nontrivial and abelian) and let (v_i) be a sequence in $Z^2(K,\mathbf{T})$ such that $\otimes_i\lambda_{v_i}$ exists on $\otimes_i^\phi l^2(K)$. Using the canonical homomorphism $\pi : G \to K$, we may lift each v_i to a $u_i \in Z^2(G,\mathbf{T})$ in the obvious way. Set $U_i(x) := \lambda_{v_i}(\pi(x)), x \in G$, for each i. It is then a simple matter to check that each U_i is a projective unitary representation of G on $l^2(K)$ associated to u_i, and that $\otimes_i U_i$ exists on $\otimes_i^\phi l^2(K)$.

We now turn to another class of examples which is in spirit related to the setting of the Stone-Mackey-von Neumann theorem, i.e., with so-called CCR-representations of a locally compact abelian group and its dual, cf. [**19**].

Let A and B be two discrete groups and $\sigma : A \times B \to \mathbf{T}$ be a bilinear map. We call a triple $\{V, W, \mathcal{H}\}$ for a CCR-*representation* of σ whenever V and W are unitary representations of respectively A and B on $\mathcal{H}$ which satisfy the CCR-relation

$$V(a)W(b) = \sigma(a,b)\ W(b)V(a)$$

for all $a \in A, b \in B$.

We now set $G = A \times B$ and define $u_\sigma : G \times G \to \mathbf{T}$ by

$$u_\sigma(\,(a_1,b_1),(a_2,b_2)\,) = \overline{\sigma(a_2,b_1)}.$$

It is an easy exercise to check that u_σ is a 2-cocycle on G (in fact a bicharacter, i.e., a bilinear map on $G \times G$ into $\mathbf{T}$). When both A and B are abelian, then $[u_\sigma] \neq 1$ in $H^2(G,\mathbf{T})$ whenever σ is nontrivial, as follows from [**16**] since u_σ is then clearly nonsymmetric. Note that there is a one-to-one correspondence between CCR-representations of σ and projective unitary representations of G associated with u_σ (being given by setting $U(a,b) = V(a)W(b)$ whenever $\{V, W, \mathcal{H}\}$ is a CCR-representation of σ).

There is a canonical way to produce a CCR-representation of σ on $l^2(B)$, to which we may associate a projective unitary representation U_σ of G on $l^2(B)$ associated with u_σ. We recall this construction (and remark that a similar construction can be done on $l^2(A)$ in an analogous way):

For each $a \in A, b \in B$ we set $\sigma_a(b) = \sigma(a,b)$, so the map $(a \mapsto \sigma_a)$ belongs to $\mathrm{Hom}\,(A, \hat{B})$ where $\hat{B} := \mathrm{Hom}\,(B, \mathbf{T})$. Let then $V_\sigma(a)$ denote the multiplication operator by the function σ_a on $l^2(B)$ and λ_B be the left regular representation of B on $l^2(B)$. By computation we have

$$V_\sigma(a)\lambda_B(b) = \sigma(a,b)\ \lambda_B(b)V_\sigma(a)$$

for all $a \in A, b \in B$. Hence, the triple $\{V_\sigma, \lambda_B, l^2(B)\}$ is a CCR-representation of σ and we can put $U_\sigma(a,b) := V_\sigma(a)\lambda_B(b)$ for all $(a,b) \in G$.

Assume now that (σ_i) is a sequence of bilinear maps from $A \times B$ into $\mathbf{T}$. The question whether is it possible to form $\otimes_i U_{\sigma_i}$ on $\otimes_i^\phi l^2(B)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $l^2(B)$ is then clearly equivalent to whether it is possible to form the infinite tensor product of the CCR-representations associated with the σ_i's. In the case of a positive answer, the product $\prod_i u_{\sigma_i}$ will exist (as a consequence of Theorem 3.1) so $\prod_i \sigma_i$ will then exist too and the infinite tensor product of the CCR-representations associated with the σ_i's will be a CCR-representation of this product map.

Quite similarly to Theorem 3.2 and Theorem 3.3 we have:

Theorem 3.6. *Let (σ_i) be a sequence of bilinear maps from $G = A \times B$ into $\mathbf{T}$. Set $U_i := U_{\sigma_i}$.*

i) *Assume that $\otimes_i U_i$ exists on $\otimes_i^\phi l^2(B)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $l^2(B)$. Then B is countable and amenable.*

ii) *Assume that B is countable and amenable, and that (F_i) be a σF-sequence for B satisfying*

$$\sum_i \frac{1}{\#(F_i)} \sum_{b \in F_i} |1 - \sigma_i(a,b)| < \infty$$

for every $a \in A$. Set $\phi = (\phi_i)$ where $\phi_i := \chi_{F_i}/\#(F_i)^{1/2}$.

Then $\otimes_i U_i$ exists on $\otimes_i^\phi l^2(B)$.

Proof. i) Since $U_i(e,b) = \lambda_B(b)$, this follows from [**1**] (or Theorem 3.2).

ii) Let B be countable and amenable, and (F_i) be as in ii). Since (F_i) is a σF-sequence for B, it follows from [**7**] (or Theorem 3.3) that $\otimes_i U_i(e,b) = \otimes_i \lambda_B(b)$ exists on $\otimes_i^\phi l^2(B)$ for every $b \in B$. The existence of $\otimes_i U_i$ on $\otimes_i^\phi l^2(B)$ reduces then to whether $\otimes_i V_{\sigma_i}$ exists on $\otimes_i^\phi l^2(B)$, i.e., whether

$$\sum_i |1 - (V_{\sigma_i}(a)\phi_i, \phi_i)| = \sum_i |1 - ((\sigma_i)_a\, \phi_i, \phi_i)| < \infty$$

holds for every $a \in A$. As we have

$$|1-((\sigma_i)_a\, \phi_i, \phi_i)| = \frac{1}{\#(F_i)} \Big| \sum_{b\in F_i} (1-\sigma_i(a,b)) \Big| \leq \frac{1}{\#(F_i)} \sum_{b\in F_i} |1-\sigma_i(a,b)|$$

for every $a \in A$, this follows from the assumption on (F_i). □

We leave it to the reader to deduce from this theorem the analogous versions of Corollary 3.4 and Corollary 3.5 in this setting.

4. The case of free abelian groups. The purpose of this section is to exemplify the results of the previous section in the concrete case where G is a finitely generated free abelian group.

We let $N \in \mathbf{N}$ and set $G = \mathbf{Z}^N$.

When $x = (x_1, \ldots, x_N) \in G$, we set $|x|_1 = \sum_{j=1}^N |x_j|$.

When $m \in \mathbf{N}$, we define $K_m \subset G$ by

$$K_m = \{x \in G \mid 0 \leq x_i \leq m,\ i = 1 \ldots N\} \quad (= \{0, 1, \ldots, m\}^N).$$

To each $N \times N$ real matrix A, one may associate $u_A \in Z^2(G, \mathbf{T})$ by

$$u_A(x,y) = e^{ix\cdot(Ay)}.$$

In fact, every element in $H^2(G, \mathbf{T})$ may be written as $[u_A]$ for some skew-symmetric A, see [**2, 3**]. Without loss of generality, we can assume that $A \in M_N((-\pi, \pi])$, i.e., all of A's coefficients belong to $(-\pi, \pi]$. We set

$$|A|_\infty = \max\{|a_{ij}|,\ 1 \leq i,j \leq N\}.$$

We first record a technical lemma.

Lemma 4.1. *Let* $A \in M_N((-\pi,\pi])$, $x, y \in G$ *and* $m \in \mathbf{N}$. *Then*

(1) $|1 - u_A(x,y)| \leq |A|_\infty |x|_1 |y|_1$

(2) $\displaystyle\sum_{x \in K_m} |x|_1 = \frac{Nm(m+1)^N}{2}$

(3) $\displaystyle 1 - \frac{\#((x+K_m) \cap K_m)}{\#K_m} \leq \frac{|x|_1}{m+1}$.

Proof. 1) follows from $|1 - e^{ix\cdot(Ay)}| \leq |x \cdot (Ay)| \leq |A|_\infty |x|_1 |y|_1$.

2) $\displaystyle\sum_{x \in K_m} |x|_1 = \sum_{j=1}^{N} \sum_{x \in K_m} |x_j| = N(m+1)^{N-1}\Big(\sum_{k=0}^{m} k\Big) = \frac{Nm(m+1)^N}{2}$.

3) $$1 - \frac{\#((x+K_m) \cap K_m)}{\#K_m} = \frac{\#(K_m \backslash (x+K_m))}{\#K_m} \leq \frac{(m+1)^{N-1}}{(m+1)^N}|x|_1 = \frac{|x|_1}{m+1}. \qquad \square$$

Proposition 4.2. *Let* (A_i) *be a sequence in* $M_N((-\pi,\pi])$ *and* (m_i) *be a sequence in* $\mathbf{N}$. *For each* $i \in \mathbf{N}$, *we set*

$$F_i = K_{m_i} \subset G,$$
$$\phi_i = \frac{1}{(\#F_i)^{1/2}}\, \chi_{F_i} \in l^2(G),$$
$$u_i = u_{A_i} \in Z^2(G, \mathbf{T}).$$

Then we have:

(1) (F_i) *is an* F*-sequence for* G *if and only if* $m_i \to +\infty$.

(2) (F_i) *is a* σF*-sequence for* G *if and only if* $\sum_{i=1}^{\infty} \frac{1}{m_i} < \infty$.

(3) $\prod_i u_i$ *exists* $\Leftrightarrow \sum_i |A_i|_\infty < \infty$.

(4) *The projective unitary representation* $\otimes_i \lambda_{u_i}$ *of* G *exists on* $\otimes_i^{\phi_i} l^2(G)$ *whenever*

$$\sum_{i=1}^{\infty} \frac{1}{m_i} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} m_i |A_i|_\infty < \infty$$

(*and* $\prod_i u_i$ *is then the associated* 2*-cocycle*).

Proof. The nontrivial parts of (1) and (2) are consequences of Lemma 4.1, part (3). Assertion (3) relies on the inequality $2|\theta|/\pi \le |1-e^{i\theta}| \le |\theta|$ which holds when $|\theta| \le \pi$. Concerning (4) let $x, y \in G$. Then we have

$$\begin{aligned}
&\frac{1}{\#F_i}\sum_{y\in F_i}|1-u_i(-y,x)| \\
&\quad \le \frac{1}{(m_i+1)^N}\Big(\sum_{y\in F_i}|A_i|_\infty|x|_1|y|_1\Big) \qquad \text{(by Lemma 4.1, (1))} \\
&\quad = \frac{|x|_1|A_i|_\infty}{(m_i+1)^N}\sum_{y\in F_i}|y|_1 \\
&\quad = \frac{|x|_1|A_i|_\infty}{(m_i+1)^N}\,\frac{Nm_i(m_i+1)^N}{2} \qquad \text{(by Lemma 4.1, (2))} \\
&\quad = \frac{N|x|_1}{2}\,m_i|A_i|_\infty
\end{aligned}$$

for every $i \in \mathbf{N}$. Hence we have

$$\sum_i \frac{1}{\#F_i}\sum_{y\in F_i}|1-u_i(-y,x)| \le \frac{N|x|_1}{2}\sum_i m_i|A_i|_\infty .$$

Now if we assume that $\sum_{i=1}^\infty 1/m_i < \infty$ and $\sum_{i=1}^\infty m_i|A_i|_\infty < \infty$, then $\{F_i\}$ is a σF-sequence for G (by (2)) and

$$\sum_i \frac{1}{\#F_i}\sum_{y\in F_i}|1-u_i(-y,x)| < \infty$$

for all $x \in G$, and the conclusion follows from Theorem 3.3. □

Example. Let $A \in M_N((-\pi,\pi])$. Set $A_i = 2^{-i}A$ and $u_i = u_{A_i}$ ($i \in \mathbf{N}$). Then clearly $u_A = \prod_i u_i$. Further, if we let $m_i = i^2$, then

$$\sum_i \frac{1}{m_i} < \infty \quad \text{and} \quad \sum_i m_i|A_i|_\infty = |A|_\infty \sum_i \frac{i^2}{2^i} < \infty$$

so (4) in the above proposition applies. Theorem 3.1 then gives

$$\lambda_{u_A} \otimes id \cong \otimes_i \lambda_{u_i},$$

thus producing an infinite tensor product decomposition of the amplification of λ_{u_A}. It is well known that the C*-algebra $C^*(\lambda_{u_A})$ generated by λ_{u_A} on $l^2(G)$ is a so-called noncommutative N-torus. Using this decomposition result, we can clearly obtain a faithful representation of $C^*(\lambda_{u_A})$ onto the C^*-algebra generated by $\otimes_i \lambda_{u_i}$ on $\otimes_i^\phi l^2(G)$ for some suitably chosen sequence ϕ of unit vectors in $l^2(G)$.

We shall now exhibit projective unitary representations arising from CCR-representations of bilinear maps on some direct product decomposition of G.

We assume from now on that $N \geq 2$ and write $G = \mathbf{Z}^N \simeq \mathbf{Z}^P \times \mathbf{Z}^Q$ where $1 \leq P$, $Q < N$ and $P + Q = N$.

To each $P \times Q$ matrix D with coefficients in $(-\pi, \pi]$, we associate a bilinear map $\sigma_D : \mathbf{Z}^P \times \mathbf{Z}^Q \to \mathbf{T}$ by

$$\sigma_D(a, b) = e^{ia\cdot(Db)}.$$

Using the construction described at the end of the previous section, we then obtain a CCR-representation of σ_D on $l^2(\mathbf{Z}^Q)$, or, equivalently, a projective unitary representation U_D of $G = \mathbf{Z}^N$ with associated 2-cocycle u^D. This cocycle is easy to describe: a simple computation gives

$$u^D(x, y) = e^{ix\cdot(\tilde{D}y)} \quad (x, y \in G)$$

where $\tilde{D}$ is the $N \times N$ matrix given by

$$\tilde{D} = \begin{pmatrix} 0 & 0 \\ -D^t & 0 \end{pmatrix}.$$

Notice that $u^D = u_{\tilde{D}}$ and $[u^D]$ is nontrivial whenever $D \neq 0$.

Proposition 4.3. *Let (D_i) be a sequence of $P \times Q$ matrices with coefficients in $(-\pi, \pi]$, and let $(U_i) = (U_{D_i})$ be the associated sequence of projective unitary representations of G on $l^2(\mathbf{Z}^Q)$. Let (n_i) be a sequence in $\mathbf{N}$.*

Set $H_i = \{b \in \mathbf{Z}^Q \mid 0 \leq b_i \leq n_i,\ i = 1 \dots Q\}$. *Further, let* $\psi_i = 1/(\#H_i)^{1/2}\chi_{H_i}$, $i \in \mathbf{N}$.

Then $\otimes_i U_i$ *exists on* $\otimes_i^{\psi_i} l^2(\mathbf{Z}^Q)$ *whenever*

$$\sum_i 1/n_i < \infty \quad \textit{and} \quad \sum_i n_i |D_i|_\infty < \infty.$$

Proof. This follows from Theorem 3.6. As the details are quite similar to the proof of the previous proposition, we leave these to the reader. □

Example. We take $P = Q = 1$ so that $G = \mathbf{Z} \times \mathbf{Z} = \mathbf{Z}^2$, and let $(D_j) = (\theta_j)$ be a sequence in $(-\pi, \pi]$. This gives rise to the sequence (U_j) of representations of $\mathbf{Z}^2$ on $l^2(\mathbf{Z})$ with associated 2-cocycles

$$u_j(x, y) = e^{-i\theta_j x_1 y_2}\ (x, y \in \mathbf{Z}^2).$$

By Proposition 4.3 we can then form the infinite tensor representation $\otimes_j U_j$ whenever we can choose a sequence (n_j) in $\mathbf{N}$ such that $\sum_j 1/n_j < \infty$ and $\sum_j n_j|\theta_j| < \infty$, e.g., $n_j = j^2$ will do if $(j^4|\theta_j|)$ is bounded.

By a more careful analysis of this example involving the familiar Dirichlet sums, one can deduce that $\otimes_j U_j$ will exist whenever we can choose (n_j) such that

$$\sum_j \frac{1}{n_j} < \infty \quad \text{and} \quad \sum_j \Big| 1 - \frac{1}{2n_j + 1} \frac{\sin((2n_j+1)\theta_j/2)}{\sin(\theta_j/2)} \Big| < \infty.$$

Assuming that $\sum_j |\theta_j| < \infty$ (so $\prod_j u_j$ exists), it would be interesting to know whether such a choice of (n_j) can always be made.

5. Infinite products of actions. For each $i \in \mathbf{N}$ let $\mathcal{H}_i$ be a Hilbert space, $\phi_i \in \mathcal{H}_i$ be a unit vector, $\mathcal{M}_i \subset \mathcal{B}(\mathcal{H}_i)$ be a von Neumann algebra and $\alpha_i : G \to \mathrm{Aut}\,(\mathcal{M}_i)$ be an action of G on $\mathcal{M}_i$. We denote by I_i the identity operator on $\mathcal{H}_i$. We then form the $*$-algebra $\odot_i \mathcal{M}_i$, respectively von Neumann algebra $\otimes_i(\mathcal{M}_i, \phi_i)$, acting on $\otimes_i^{(\phi_i)}\mathcal{H}_i$ generated by operators of the form $\otimes_i T_i$ where $T_i \in \mathcal{M}_i$ and $T_i = I_i$ for all but finitely many i's. At the $*$-algebraic level we define an action $\odot_i\alpha_i$ of G on $\odot_i\mathcal{M}_i$ such that for every finite $J \subset \mathbf{N}$ we have

$$\odot_i\alpha_i((\otimes_{i\in J}T_i)\otimes(\otimes_{i\notin J}I_i)) = (\otimes_{i\in J}\alpha_i(T_i))\otimes(\otimes_{i\notin J}I_i).$$

One natural question is whether $\odot_i\alpha_i$ may be extended to an action of G on the von Neumann algebra $\otimes_i(\mathcal{M}_i,\phi_i)$. As we shall see, the answer may be negative in some situations, regardless of the choice of unit vectors ϕ_i.

We restrict ourselves to the case where each α_i is unitarily implemented, i.e., we assume that for every i and g there exists a unitary $U_i(g)$ on $\mathcal{H}_i$ such that $\alpha_{i,g} = \mathrm{Ad}\,(U_i(g))$. This assumption is automatically satisfied for many classes of von Neumann algebras, see [**20**]. Note that if $U_i(g) \in \mathcal{M}_i$ for all $g \in G$ and $\mathcal{M}_i$ is a factor, especially if $\mathcal{M}_i = \mathcal{B}(\mathcal{H}_i)$, then $g \to U_i(g)$ is a projective unitary representation of G on $\mathcal{H}_i$.

We consider the following condition:

$$(*) \qquad \sum_i (1 - |(U_i(g)\phi_i,\phi_i)|) < \infty, \quad \text{for all } g \in G.$$

Proposition 5.1. *Condition $(*)$ is equivalent to the following condition:*

$$(**) \quad \forall\, i \in \mathbf{N},\ \exists\, \rho_i : G \to \mathbf{T}, \rho_i(e) = 1, \quad \text{such that } \otimes_i \rho_i U_i \text{ exists on } \otimes_i^{\phi_i}\mathcal{H}_i.$$

When $(*)$ holds, then $\odot_i\alpha_i$ extends to a unitarily implemented action α on $\otimes_i(\mathcal{M}_i,\phi_i)$, which is inner whenever $U_i(g) \in \mathcal{M}_i$ for every i and $g \in G$.

Proof. The first assertion follows from Proposition 2.3, using [**11**]. When $(*)$ holds, then $\alpha_g = \mathrm{Ad}\,(U(g))$ where $U(g) = \otimes_i\rho_i(g)U_i(g)$ is well defined on $\otimes_i^{\phi_i}\mathcal{H}_i$. Clearly $U(g) \in \otimes_i(\mathcal{M}_i,\phi_i)$ whenever $U_i(g) \in \mathcal{M}_i$ for every i and $g \in G$, and α_g is then inner for every $g \in G$. ∎

We now treat the case where every $\mathcal{M}_i$ is a type I factor. We use the well-known fact that every automorphism of a type I factor is inner and also that $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i) = \mathcal{B}(\otimes_i^{\phi_i}\mathcal{H}_i)$, [**11**, Proposition 1.6].

Theorem 5.2. *Assume that* $\mathcal{M}_i = \mathcal{B}(H_i)$ *for all* i. *Then* $\odot_i\alpha_i$ *extends* (*uniquely*) *to an action* $\alpha = \otimes\alpha_i$ *on* $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i)$ *if and only if condition* $(*)$ *holds.*

Proof. Assume that an extension α of $\odot_i\alpha_i$ exists on $\mathcal{M}^\phi = \otimes_i(\mathcal{M}_i, \phi_i)$. Using the facts recalled above, we have $\alpha_g = \mathrm{Ad}\,(U(g))$ for some $U(g) \in \mathcal{U}(\otimes_i^{\phi_i}\mathcal{H}_i)$ for every $g \in G$.

Let J be a nonempty finite subset of $\mathbf{N}$.

We identify $\mathcal{M}^\phi$ with $(\otimes_{i\in J}\mathcal{M}_i) \otimes {}_J\mathcal{M}$ where ${}_J\mathcal{M} := \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$, and consider ${}_J\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{M}^\phi$ in the obvious way. It is easy to see that α restricts to an action ${}_J\alpha$ of G on ${}_J\mathcal{M}$ such that $\alpha = (\otimes_{i\in J}\alpha_i) \otimes {}_J\alpha$. Since${}_J\mathcal{M}$ is a also type I factor, we can write ${}_J\alpha_g = \mathrm{Ad}\,({}_JU(g))$ for some ${}_JU(g) \in \mathcal{U}(\otimes_{i\notin J}^{\phi_i}\mathcal{H}_i)$ for each $g \in G$.

Set now $U_J(g) = \otimes_{i\in J}U_i(g)$ for each $g \in G$. Clearly, we then have $\alpha_g = \mathrm{Ad}\,(U_J(g) \otimes {}_JU(g))$. Therefore, for each $g \in G$, there exists some $z_J(g) \in \mathbf{T}$ such that $U(g) = z_J(g)U_J(g) \otimes {}_JU(g)$.

Let $g \in G$. Since $U(g) \neq 0$ we can pick two elementary decomposable vectors $\otimes_i\psi_i$ and $\otimes_i\xi_i$ in $\otimes_i^{\phi_i}\mathcal{H}_i$, which do not depend on J, satisfying

$$\begin{aligned} 0 \neq c(g) &:= |(U(g) \otimes_i \psi_i, \otimes_i\xi_i)| \\ &= \prod_{i\in J} |(U_i(g)\psi_i, \xi_i)|\ |({}_JU(g) \otimes_{i\notin J} \psi_i, \otimes_{i\notin J}\xi_i)| \end{aligned}$$

Since $|({}_JU(g) \otimes_{i\notin J} \psi_i, \otimes_{i\notin J}\xi_i)| \leq 1$ we get

$$0 < c(g) \leq \prod_{i\in J} |(U_i(g)\psi_i, \xi_i)|.$$

As this holds for every J, one easily deduces that $\prod_{i\in\mathbf{N}} |(U_i(g)\psi_i, \xi_i)|$ converges to a non-zero number. Since $\psi_i = \xi_i = \phi_i$ for all but finitely many i's, this implies that $(*)$ holds. Hence, we have shown the only if part of the assertion. The converse part follows from Proposition 5.1. $\square$

The proof of the above result is reminiscent of the proof of a lemma in [**21**], see also [**9**]. In the same line of ideas, we have the following result, which is related to [**6**, Lemma 1.3.8].

Theorem 5.3. *Assume that all $\mathcal{M}_i$'s are factors and that $\odot_i \alpha_i$ extends to an action α on $\mathcal{M}^\phi = \otimes_i(\mathcal{M}_i, \phi_i)$. Then α is inner if and only if there exists for each $g \in G$ and each i a unitary $v_i(g) \in \mathcal{M}_i$ implementing $\alpha_{i,g}$ such that the following condition holds:*

$$\sum_i (1 - |(v_i(g)\phi_i, \phi_i)|) < \infty \quad \text{for all } g \in G. \tag{1}$$

On the other hand, α is outer if and only if, for each $g \in G$, $g \neq e$, at least one of the $\alpha_{i,g}$ is outer or there exists for each i a unitary $v_i(g) \in \mathcal{M}_i$ implementing $\alpha_{i,g}$ such that

$$\sum_i (1 - |(v_i(g)\phi_i, \phi_i)|) = \infty. \tag{2}$$

Proof. Assume first that α is inner. So we have $\alpha_g = \mathrm{Ad}\,(U(g))$ for some unitary $U(g) \in \mathcal{M}^\phi$ for every $g \in G$. Recall from [**11**] that $\mathcal{M}^\phi$ is a factor. Using [**14**, Corollary 1.14], it follows easily that each α_i is inner. Hence, there exists for each $g \in G$ and each i a unitary $v_i(g) \in \mathcal{M}_i$ implementing $\alpha_{i,g}$.

Let J be a nonempty finite subset of $\mathbf{N}$. As in the previous proof, we identify $\mathcal{M}^\phi$ with $(\otimes_{i\in J}\mathcal{M}_i) \otimes {}_J\mathcal{M}$ where ${}_J\mathcal{M} := \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$. We set $V_J(g) = \otimes_{i\in J} v_i(g)$ and $W_J(g) = (\, V_J(g) \otimes (\otimes_{i\notin J} I_i)\,)^* U(g)$ for each $g \in G$. Then, using that we may write $\alpha = (\otimes_{i\in J}\alpha_i) \otimes {}_J\alpha$, we get

$$W_J(g) \in (\otimes_i(\mathcal{M}_i, \phi_i)) \cap ((\otimes_{i\in J}\mathcal{M}_i) \otimes (\otimes_{i\notin J}\mathbf{C}I_i))'.$$

Using that all $\mathcal{M}_i$ are factors, it is a simple exercise to deduce that $W_J(g) \in (\otimes_{i\in J}\mathbf{C}I_i) \otimes (\otimes_{i\notin J}(\mathcal{M}_i, \phi_i))$. We may therefore write $W_J(g) = (\otimes_{i\in J} I_i) \otimes {}_J V(g)$ for some unitary ${}_J V(g) \in \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$. This gives $U(g) = V_J(g) \otimes {}_J V(g)$ and we can clearly proceed further in the same way as in the previous proof to show that (1) holds, thereby proving the only if part of the first assertion. The converse part of this assertion follows from Proposition 5.1. The second assertion follows from a similar argument. □

The following corollary may be seen as generalization of [**10**, Theorem 6.7].

Corollary 5.4. *Assume for each* $i \in \mathbf{N}$ *that* β_i *is an action of* G *on some von Neumann algebra* $\mathcal{N}_i$ *and that there exists a normal* β_i*-invariant state* τ_i *on* $\mathcal{N}_i$. *Denote the GNS-triple of* τ_i *by* $(\pi_i, \mathcal{H}_i, \xi_i)$ *and set* $\mathcal{M}_i = \pi_i(\mathcal{N}_i)$. *Let* α_i *be the action of* G *on* $\mathcal{M}_i$ *induced by* β_i. *Then* $\odot_i \alpha_i$ *extends to an action* α *of* G *on* $\otimes_i(\mathcal{M}_i, \xi_i)$.

Assume further that all $\mathcal{N}_i$*'s are factors and all* π_i*'s are faithful. Then* α *is inner if and only if there exists for each* $g \in G$ *and each* i *a unitary* $v_i(g) \in \mathcal{N}_i$ *implementing* $\beta_{i,g}$ *such that the following condition holds:*

$$\sum_i (1 - |\tau_i(v_i(g))|) < \infty \quad \text{for all } g \in G. \tag{1}$$

On the other hand, α *is outer if and only if, for each* $g \in G, g \neq e$, *at least one of the* $\beta_{i,g}$ *is outer or there exists each* i *a unitary* $v_i(g) \in \mathcal{N}_i$ *implementing* $\beta_{i,g}$ *such that*

$$\sum_i (1 - |\tau_i(v_i(g))|) = \infty. \tag{2}$$

Proof. We first recall that there exists for each i a unitary representation $V_i : G \to \mathcal{B}(\mathcal{H}_i)$ such that

$$\pi_i(\beta_{i,g}(x)) = V_i(g)\pi_i(x)V_i(g)^* \quad \text{and} \quad V_i(g)\pi_i(x)\xi_i = \pi_i(\beta_{i,g}(x))\xi_i$$

for all $g \in G, x \in \mathcal{N}_i$, see [**8**]. The induced action α_i on $\mathcal{M}_i$ is then defined by $\alpha_{i,g}(\pi_i(x)) = \pi_i(\beta_{i,g}(x))$. As $V_i(g)\xi_i = \xi_i$ for all $g \in G$, the first assertion follows obviously from Proposition 3.1. The second assertion is then easily deduced from Theorem 5.3. □

Example. Let u_i be a sequence in $Z^2(G, \mathbf{T})$. Set $\mathcal{N}_i = \lambda_{u_i}(G)'' \subset \mathcal{B}(l^2(G))$ and let $\beta_{i,g}$ be the inner automorphism of $\mathcal{N}_i$ implemented by $\lambda_{u_i}(g)$ for all $g \in G, i \in \mathbf{N}$. Let τ_i denote the canonical normal faithful tracial state of $\mathcal{N}_i$ (determined by $\tau_i(\lambda_{u_i}(g)) = 1$ if $g = e$ and 0 otherwise), which is trivially β_i-invariant. If ξ denotes the normalized delta-function at e, then $\tau_i = \omega_\xi|_{\mathcal{N}_i}$. So we may identify the GNS-triple of τ_i with $(id_i, l^2(G), \xi_i)$, where id_i denotes the identity representation of $\mathcal{N}_i$ and $\xi_i = \xi$, i.e., we may take $\mathcal{M}_i = \mathcal{N}_i$ and $\alpha_i = \beta_i$ in the notation of Corollary 5.4. Hence, $\odot\alpha_i = \odot\beta_i$ extends to an action α on $\otimes_i(\lambda_{u_i}(G)^{''}, \xi_i)$.

Further, if all $\lambda_{u_i}(G)^{''}$ are factors, then α is outer, as

$$\sum_i (1 - |\tau_i(\lambda_{u_i}(g))|) = \sum_i 1 = \infty \quad \text{for all } g \neq e.$$

A necessary and sufficient condition for a twisted group von Neumann algebra $\lambda_u(G)''$ to be a factor may be found in [**15**].

If we replace each $\mathcal{N}_i$ with $\mathcal{B}(l^2(G))$ in this example, the extended product action may be formed in many cases under the assumption that G is countable and amenable, as follows from Theorem 3.3 and Proposition 5.1. This requires a suitable choice of unit vectors ϕ_i in $l^2(G)$. This product action restricts then to an action on $\otimes_i(\lambda_{u_i}(G)'', \phi_i)$ which is inner, in contrast to the factor case above. When G is either uncountable or nonamenable, we have the following:

Theorem 5.5. *Let u_i be a sequence in $Z^2(G, \mathbf{T})$ and $\alpha_i = \mathrm{Ad}\,\lambda_{u_i}$ be the associated sequence of actions of G on $\mathcal{B}(l^2(G))$. If G is either uncountable or non-amenable, then $\odot_i\alpha_i$ does not extend to an action of G on $\otimes_i(\mathcal{B}(l^2(G)), \phi_i)$, regardless of the choice of vectors ϕ_i.*

Proof. According to Proposition 5.1 and Theorem 5.2, the existence of such an extension $\otimes_i(\mathcal{B}(l^2(G)), \phi_i)$ would imply the existence of $\otimes_i\rho_i\lambda_{u_i}$ on $\otimes_i^{\phi_i} l^2(G)$ for some choice of functions $\rho_i : G \to \mathbf{T}$ with $\rho_i(e) = 1$. It is straightforward to see that this amounts to the existence of $\otimes_i\lambda_{v_i}$ on $\otimes_i^{\psi_i} l^2(G)$ for some $v_i \in Z^2(G, \mathbf{T})$ with $v_i \sim u_i$ and some sequence ψ_i of unit vectors in $l^2(G)$. This is impossible if G is either uncountable or nonamenable, as follows from Theorem 3.2. □

Another type of possible obstruction for extending a product action from the $*$-algebraic level to the von Neumann algebra level is of cohomological nature, as we now illustrate:

Theorem 5.5. *Let α_i be a sequence of actions of G on $\mathcal{B}(\mathcal{H}_i)$ and write each α_i as $\operatorname{Ad} U_i(g)$ where U_i is a projective representation of G with associated 2-cocycle u_i. Assume that $[u_i] = [u]$ for every i and $[u] \neq [1]$ in $H^2(G, \mathbf{T})$. Then $\odot_i \alpha_i$ does not extend to an action of G on $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i)$, regardless of the choice of vectors ϕ_i.*

Proof. Assume that such an extension exists $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i)$. Using Proposition 5.1 and Theorem 5.2, we deduce that $\otimes \rho_i U_i$ exists on $\otimes_i^{\phi_i} \mathcal{H}_i$ for some choice of functions $\rho_i : G \to \mathbf{T}$ with $\rho_i(e) = 1$. It follows then from Theorem 3.1 that $\prod_i (\mathrm{d}\rho_i) u_i$ exists. Hence $\mathrm{d}\rho_i u_i \to 1$ (in the pointwise topology). As each $u_i = (\mathrm{d}\rho_i')u$ for some ρ_i', we get that u is a limit of 2-coboundaries. Since $B^2(G, \mathbf{T})$ is closed, this means that u is itself a coboundary, i.e., $[u] = 1$, which gives a contradiction. □

Example. The simplest case where the above situation occurs is when $G = \mathbf{Z}_2 \times \mathbf{Z}_2$. Indeed, let

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A projective unitary representation of $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ on $\mathbf{C}^2$ is then obtained by setting $U((a, b)) = V^a W^b$ $(a, b \in \mathbf{Z}_2)$. Since $V^a W^b = \sigma(a, b) W^b U^a$ where $\sigma(a, b) = -1$ if $a = b = 1$ and 1 otherwise, the associated cocycle u is easily computed to be $u((a_1, b_1), (a_2, b_2)) = (-1)^{a_2 b_1}$. It is not difficult to check that $[u] \neq 1$. Remark that U is nothing but the projective representation associated to the CCR representation of σ on $\mathbf{C}^2 = l^2(\mathbf{Z}_2)$ determined by V and W.

For each $i \in \mathbf{N}$ consider the action α_i of G on $M_2(\mathbf{C})$ given by $\alpha_{i,(a,b)} = \operatorname{Ad}(U((a, b)))$. Then, according to Theorem 5.6, the infinite tensor product of the α_i's does never make sense as an action on $\otimes_i(M_2(\mathbf{C}), \phi_i)$.

On the other hand, the canonical tracial state of $M_2(\mathbf{C})$ is trivially α_i-invariant. Therefore we may use Corollary 5.4 to form the infinite tensor product action after passing to the GNS-representation with respect to this tracial state for each i. As another application of Corollary 5.4, the resulting product action is easily seen to be outer.

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