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WEIL REPRESENTATIONS OF SYMPLECTIC GROUPS OVER NON-PRINCIPAL RINGS

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ABSTRACT. Let $W : \mathrm{Sp}(2n, R) \to \mathrm{GL}(X)$ be a Weil representation of the symplectic group of rank $2n$ over a finite commutative ring R of odd characteristic. This is a complex representation of degree $|R|^n$ defined in terms of the action of $\mathrm{Sp}(2n, R)$ on a two-step nilpotent group called Heisenberg group. We address the problem of decomposing the $\mathrm{Sp}(2n, R)$-module X into irreducible constituents. The problem can easily be reduced to the case when R is local and quasi-Frobenius. Further, the case when R is a principal ring has already been solved. This was achieved by means of the following recursive property of the Weil representation: precisely two irreducible constituents of X do not admit trivial action by any congruence subgroup of $\mathrm{Sp}(2n, R)$; the remaining irreducible constituents lie inside an $\mathrm{Sp}(2n, R)$-submodule Y of X that affords a Weil representation for a quotient symplectic group $\mathrm{Sp}(2n, T)$. We show here that this recursive property of Y holds only when R is principal, failing in all other cases. This failure opens the following Pandora box: given any finite commutative quasi-Frobenius local ring R_0 of odd characteristic, we can choose R so that R_0 is quotient of R and *every* complex irreducible character of $\mathrm{Sp}(2n, R_0)$ enters Y when inflated to $\mathrm{Sp}(2n, R)$. Thus, the problem of decomposing the Weil module X into irreducible constituents is, in general, as difficult as the problem of finding all complex irreducible characters of all symplectic groups $\mathrm{Sp}(2n, R_0)$. In spite of this, we manage to identify submodules of X that do admit either a Weil representation or the tensor product of various Weil representations for a quotient symplectic group.

1. Introduction. Let R be a finite commutative local ring of odd characteristic. Let V be a free R-module of rank $2n$ endowed with a non-degenerate R-bilinear form $\langle\ ,\ \rangle$. Denote by $\mathrm{Sp}(2n, R)$ the symplectic group of rank $2n$ over R, namely the subgroup of $\mathrm{GL}(V)$ that preserves $\langle\ ,\ \rangle$. Let $W : \mathrm{Sp}(2n, R) \to \mathrm{GL}(X)$ be the complex

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representation of degree $|R|^n$ referred to as Weil representation in [**3**]. A natural problem in this context is to decompose the $\mathrm{Sp}\,(2n,R)$-module X into irreducible constituents.

The simplest case occurs when $R=F_q$ is a field. In this case, one has the decomposition into irreducible constituents $X=X^+\oplus X^-$, where $X^\pm$ is the ± 1-eigenspace of -1_V acting on X. Details may be found in [**5**]. Various properties of W have been investigated in the classical case $R=F_q$. For instance, the character values for W were computed in [**10**], the character field and Schur index of both X^+ and X^- were determined in [**6**], the restriction of W to the unitary group $\mathrm{U}(2n,q^2)$ was analyzed in [**5, 12**], lattices associated the Weil representation were studied in [**4, 6, 9**], etc.

The next case in relative difficulty takes place when R is a principal ring. Under this hypothesis all irreducible constituents of X where determined in [**3**]. Indeed, let us denote by $\mathfrak{m}$ the maximal ideal of R and by l the nilpotency degree of $\mathfrak{m}$. As the field case has already been considered, we may assume that $l>1$. Denote by $\mathfrak{a}$ the conductor of $\mathfrak{m}$ into the minimal ideal $\mathfrak{m}^{l-1}$ of R. Write T for the quotient ring $R/\mathfrak{a}$. There is a canonical epimorphism $B:\mathrm{Sp}\,(2n,R)\to\mathrm{Sp}\,(2n,T)$ whose kernel is the congruence subgroup $\Gamma(\mathfrak{a})=\{g\in\mathrm{Sp}\,(2n,R)\mid gv\equiv v \mod \mathfrak{a}V\}$. Let Y denote the fixed points of $\Gamma(\mathfrak{a})$ in X. Then Y is an $\mathrm{Sp}\,(2n,R)$-submodule of X affording a representation $\overline{W}$ of $\mathrm{Sp}\,(2n,T)$ given by $\overline{W}(B(g))=W(g)|_Y$ for $g\in\mathrm{Sp}\,(2n,R)$. It is shown in [**2**] that Y is non-zero and properly contained in X.

The decomposition of X thus falls naturally into two cases: the study of the quotient $\mathrm{Sp}\,(2n,R)$-module $Z=X/Y$, and the investigation of Y as a module for the quotient symplectic group $\mathrm{Sp}\,(2n,T)$.

The $\mathrm{Sp}\,(2n,R)$-module Z is shown in [**3**] to have exactly two irreducible constituents, namely Z^+ and Z^-, the ± 1-eigenspaces of -1_V acting on Z. Further, it is shown in [**2**] that Z^+ and Z^- truly pertain to $\mathrm{Sp}\,(2n,R)$ in the sense that no congruence subgroup acts trivially on them. In fact, the kernels of the representations afforded by Z^+ and Z^- are as small as possible: the kernel of Z^+ is $\{1_V,-1_V\}$, while the kernel of Z^- is trivial. Further, the Clifford theory of Z^+ and Z^- is explicitly elucidated in [**2**].

In regards to Y, it is natural to inquire about the nature of this $\mathrm{Sp}\,(2n,T)$-module. There is a priori no reason to suspect that Y will be again a Weil module. However, [**3**] proves that this is indeed the case. Thus the irreducible constituents of X are $Z^{\pm}$ along with the irreducible constituents of Y, viewed as a Weil module for $\mathrm{Sp}\,(2n, T)$. The Weil module X has $l+1$ irreducible constituents, all inequivalent to each other.

Let us now remove the hypothesis that R is principal. As shown below, one may assume without loss of generality that R is a quasi-Frobenius ring. In this case, the representation $\overline{W}$ of $\mathrm{Sp}\,(2n, T)$ afforded by Y need no longer be Weil. In fact, we show that $\overline{W}$ is *never* a Weil representation if R is not principal. Thus the fact that $\overline{W}$ is a Weil representation depends exclusively on whether R is a principal ring or not. Oddly enough, Z^{+} and Z^{-} are shown in [**2**] to remain irreducible, regardless of the structure of R.

Not only does Y fail to be a Weil module, we prove that its irreducible constituents can be quite arbitrary. In fact, let R_0 be any finite commutative quasi-Frobenius local ring of odd characteristic, and let ϕ be an arbitrary complex character of $\mathrm{Sp}\,(2n, R_0)$. Then we can choose R so that R_0 is a quotient of R, and the inflation of ϕ to $\mathrm{Sp}\,(2n, R)$ is equal to the character afforded by some $\mathrm{Sp}\,(2n, R)$-submodule of Y. Thus, the problem of decomposing the Weil module X into irreducible constituents is, in general, as difficult as the problem of finding all complex irreducible characters for all symplectic groups $\mathrm{Sp}\,(2n, R_0)$.

In spite of the above, we show that, under certain hypotheses, the $\mathrm{Sp}\,(2n, T)$-module Y is similar to the tensor product of Weil modules. The number and type of factors in this product is explicitly described in terms of certain quadratic spaces naturally related to R.

2. Preliminaries. Let R be a finite commutative local ring of odd characteristic. Let V be a free R-module of rank $2n$ endowed with non-degenerate alternating R-bilinear form $\langle\ ,\ \rangle$. We associate two groups to these data: the symplectic group $\mathrm{Sp}\,(V)$, which is the group of all $g \in \mathrm{GL}\,(V)$ satisfying

$$\langle gv, gw\rangle = \langle v, w\rangle, \quad v, w \in V,$$

and the Heisenberg group $H(V)$, whose underlying set is $R \times V$, with multiplication $(r_1, v_1)(r_2, v_2) = (r_1 + r_2 + \langle v_1, v_2\rangle, v_1 + v_2)$. The symplectic group $\mathrm{Sp}\,(V)$ acts on $H(V)$ by means of ${}^{g}(r, v) = (r, gv)$.

For $r \in R$ and $v \in V$ we have the symplectic transvection $\rho_{r,v} : V \to V$, defined by

$$\rho_{r,v}(x) = x + r\langle v, x\rangle v, \quad x \in V.$$

This is R-linear with inverse $\rho_{-r,v}$ and preserves $\langle \ , \ \rangle$. Thus $\rho_{r,v} \in \mathrm{Sp}\,(V)$. A distinguished element of $\mathrm{Sp}\,(V)$ is the central involution ι, defined by $\iota(v) = -v$ for $v \in V$.

Let $\mathfrak{a}$ be an arbitrary ideal of R. Consider the $R/\mathfrak{a}$-module $V/\mathfrak{a}V$. This is a free $R/\mathfrak{a}$-module, whose rank is $2n$ if $\mathfrak{a}$ is properly contained and 0 otherwise. Moreover, $V/\mathfrak{a}V$ is endowed with the non-degenerate alternating $R/\mathfrak{a}$-bilinear form $\ll \ , \ \gg$, defined by

$$\ll v + \mathfrak{a}V, w + \mathfrak{a}V \gg = \langle v, w\rangle, \quad v, w \in V.$$

We have the group homomorphisms $A : H(V) \to H(V/\mathfrak{a}V)$ and $B : \mathrm{Sp}\,(V) \to \mathrm{Sp}\,(V/\mathfrak{a}V)$, given by $A(r, v) = (r + \mathfrak{a}, v + \mathfrak{a}V)$ and $B(g)(v + \mathfrak{a}V) = gv + \mathfrak{a}V$. The map A is surjective with kernel $(\mathfrak{a}, \mathfrak{a}V)$. The kernel of B is the congruence subgroup associated to $\mathfrak{a}$

$$\Gamma(\mathfrak{a}) = \{g \in \mathrm{Sp}\,(V) \mid gv \equiv v \mod \mathfrak{a}V \text{ for all } v \in V\}.$$

Moreover, B is also surjective. Indeed, it is known that symplectic groups are generated by symplectic transvections, cf. Theorem 2 of [**8**]. Since $B(\rho_{r,v}) = \rho_{r+\mathfrak{a}, v+\mathfrak{a}V}$, the result follows. Thus $H(V/\mathfrak{a}V)$ is canonically isomorphic to $H(V)/(\mathfrak{a}, \mathfrak{a}V)$ and $\mathrm{Sp}\,(V/\mathfrak{a}V)$ is canonically isomorphic to $\mathrm{Sp}\,(V)/\Gamma(\mathfrak{a})$. We also observe that the epimorphisms A and B are compatible with the actions of $\mathrm{Sp}\,(V)$ on $H(V)$ and $\mathrm{Sp}\,(V/\mathfrak{a}V)$ on $H(V/\mathfrak{a}V)$, in the sense that

$$(1) \qquad {}^{B(g)}A(h) = A({}^{g}h), \quad g \in \mathrm{Sp}\,(V),\ h \in H(V).$$

Let $\lambda : R \to \mathbf{C}^*$ be an additive linear character of R. We think of λ as a linear character of the center of $H(V)$, via the canonical isomorphism $Z(H(V)) = (R, 0) \cong R^+$. Let $S : H(V) \to \mathrm{GL}\,(X)$ be a complex irreducible representation that is $\mathrm{Sp}\,(V)$-invariant and lies over λ. By a Weil representation of $\mathrm{Sp}\,(V)$ of type λ, we understand a complex representation $W : \mathrm{Sp}\,(V) \to \mathrm{GL}\,(X)$ that satisfies

$$(2) \qquad W(g)S(h)W(g)^{-1} = S\left({}^{g}h\right), \quad h \in H,\ g \in \mathrm{Sp}\,(V).$$

We recall from [**3**] the construction of S and W, considering first the case when λ is primitive. By this we mean that (0) is the only ideal of R contained in the kernel of λ. Fix a basis $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ of V, which is symplectic in the sense that

$$\langle u_i, v_j\rangle = \delta_{ij}, \quad \langle u_i, u_j\rangle = 0, \quad \langle v_i, v_j\rangle = 0.$$

The existence of such a basis can be established much as in the case when R is a field, cf. Section 1 of [**8**]. Setting $M = Ru_1 \oplus \cdots \oplus Ru_n$ we observe that $\langle M, M\rangle = (0)$. Further, M is a maximal submodule of V relative to this property. Consider the normal subgroup $H(M) = (R, M)$ of $H(V)$. We define a one-dimensional representation of $H(M)$ afforded by $Y = \mathbf{C}y$ as follows:

$$(r, u)y = \lambda(r)y.$$

An elementary calculation that makes use of the maximality of M and the primitivity of λ reveals that the inertia group of Y in $H(V)$ is $H(M)$ itself. It follows that the induced module

$$X = \mathrm{ind}_{H(M)}^{H(V)} Y = \mathbf{C}H(V) \otimes_{\mathbf{C}H(M)} Y$$

is irreducible. Let S be the representation of $H(V)$ afforded by X and denote its character by χ. We claim that χ is the only irreducible character of $H(V)$ that lies over λ. To substantiate this claim we make use of the following well-known result.

2.1 Lemma. *Let G be a finite group with normal subgroup N. Let β be a complex irreducible character of N, and let α be a complex irreducible character of G that lies over β. Suppose furthermore that $\alpha|_N = e\beta$ for some positive integer e satisfying $e^2 = [G : N]$. Then α is the only complex irreducible character of G lying over β.*

Proof. By Frobenius reciprocity $[\mathrm{ind}_N^G \beta, \alpha] = [\beta, \alpha|_N] = e$. As

$$\deg \mathrm{ind}_N^G \beta = [G : N] \deg \beta = e^2 \deg \beta = e \deg \alpha,$$

we infer $\mathrm{ind}_N^G \beta = e\alpha$. If γ is a complex irreducible character of G that lies over β, then Frobenius reciprocity ensures that γ enters $\mathrm{ind}_N^G \beta$. Since $\mathrm{ind}_N^G \beta = e\alpha$, the result follows. □

We use Lemma 2.1 with $G = H(V)$, $N = Z(H(V))$, $\alpha = \chi$, $\beta = \lambda$ and $e = \sqrt{|V|}$. Since $\chi|_{Z(H(V))} = \sqrt{|V|}\lambda$ and $[H(V) : Z(H(V))] = |V|$, the claim follows. We refer to χ as the Schrödinger character of $H(V)$ of type λ.

For $g \in \mathrm{Sp}(V)$ we consider the conjugate character χ^g, defined by $\chi^g(h) = \chi({}^g h)$. As $\mathrm{Sp}(V)$ acts trivially on $Z(H(V))$, the above claim implies $\chi^g = \chi$ for all $g \in \mathrm{Sp}(V)$. Thus, to each $g \in \mathrm{Sp}(V)$ there corresponds an operator $P(g) \in \mathrm{GL}(X)$ such that

$$P(g)S(h)P(g)^{-1} = S({}^g h), \quad h \in H(V). \tag{3}$$

Since S is irreducible, Schur's lemma ensures that each operator $P(g)$ is unique up to multiplication by a non-zero constant $c(g)$. It is shown in Section 3 of [**3**] that these scalars can be chosen so that $W(g) = c(g)P(g)$ defines representation of $\mathrm{Sp}(V)$, namely a Weil representation of type λ.

A second application of Schur's lemma yields that W is unique up to multiplication by a linear character of $\mathrm{Sp}(V)$. It is known that $\mathrm{Sp}(V)$ is a perfect group unless $n = 1$ and the residue class field F_q of R has three elements, cf. Section 3 of [**8**] if $q > 3$ and Section 2.4 of [**11**] if $q = 3$. For ease of exposition it will be assumed henceforth that $(n, q) \neq (1, 3)$. Thus the Weil representation of type λ is unique up to similarity. Its degree is equal to the degree of S, namely $\sqrt{|V|}$. Hence,

$$\deg W = |R|^n. \tag{4}$$

Suppose now that $\lambda : R \to \mathbf{C}^*$ is an arbitrary additive linear character. Let $\mathfrak{i}_\lambda$ be the conductor of λ, that is the sum of all ideals of R contained in the kernel of λ. Consider the additive linear character $\overline{\lambda} : R/\mathfrak{i}_\lambda \to \mathbf{C}^*$, defined by $\overline{\lambda}(r + \mathfrak{i}_\lambda) = \lambda(r)$ for $r \in R$. The definition of $\mathfrak{i}_\lambda$ guarantees the primitivity of $\overline{\lambda}$. Let $\overline{S}$ be a Schrödinger representation of $H(V/\mathfrak{i}_\lambda V)$ of type $\overline{\lambda}$ and let $\overline{W}$ be the associated Weil representation of $\mathrm{Sp}(V/\mathfrak{i}_\lambda V)$ of type $\overline{\lambda}$. Let S be the inflation of $\overline{S}$ to $H(V)$ via A and let W be the inflation of $\overline{W}$ to $\mathrm{Sp}(V)$ via B. The compatibility condition (1) ensures that S and W satisfy (2). In particular, S is $\mathrm{Sp}(V)$-invariant. Moreover, as A is surjective, S is also irreducible. Further, S lies over λ. All in all, W is a Weil representation of type λ.

Suppose next that T is an arbitrary irreducible representation of $H(V)$ that is $\mathrm{Sp}\,(V)$-invariant and lies over λ. Given $(r,v) \in (\mathfrak{i}_\lambda, \mathfrak{i}_\lambda V)$ and $(s,w) \in H(V)$ we have

$$\begin{aligned} T(r,v)T(s,w)T(r,v)^{-1} &= T(s+2\langle v,w\rangle, u) = \lambda(2\langle v,w\rangle)T(s,w) \\ &= T(s,w) \end{aligned}$$

since $Z(H(V))$ acts via multiplication by λ and $\mathfrak{i}_\lambda$ is contained in the kernel of λ. We infer that each $T(r,v)$ with $(r,v) \in (\mathfrak{i}_\lambda, \mathfrak{i}_\lambda V)$ is in the commuting ring of T. As T is irreducible, we deduce that $(\mathfrak{i}_\lambda, \mathfrak{i}_\lambda V)$ acts under T via multiplication by a linear character μ. Since T is $\mathrm{Sp}\,(V)$-invariant and $(\mathfrak{i}_\lambda, \mathfrak{i}_\lambda V)$ is preserved by $\mathrm{Sp}\,(V)$, we see that μ is $\mathrm{Sp}\,(V)$-invariant. In particular, $\mu(0,v) = \mu(0,\iota v)$ for $v \in \mathfrak{i}_\lambda V$, whence $(0, 2\mathfrak{i}_\lambda V)$ is in the kernel of μ. Since 2 is invertible, it follows that $(0, \mathfrak{i}_\lambda V) \subseteq \ker \mu$. But we also have $\mu(r,0) = \lambda(r) = 1$ for all $r \in \mathfrak{i}_\lambda$. Hence μ is the trivial character. Thus T is the inflation via A of an irreducible representation $\overline{T}$ of $H(V/\mathfrak{i}_\lambda V)$ that lies over $\overline{\lambda}$. Since $\overline{\lambda}$ is primitive, it follows by uniqueness that $\overline{T}$ is a Schrödinger representation of $H(V/\mathfrak{i}_\lambda V)$ of type $\overline{\lambda}$. All in all, T is similar to the representation S constructed above.

As a result, the Weil representation of type λ is uniquely determined up to similarity. In particular, the Weil representation of $\mathrm{Sp}\,(V)$ of type λ is similar to the inflation via B of the Weil representation of $\mathrm{Sp}\,(V/\mathfrak{i}_\lambda V)$ of type $\overline{\lambda}$. It infer that the kernel of the Weil representation of type λ contains the kernel of B, namely the congruence subgroup $\Gamma(\mathfrak{i}_\lambda)$. Furthermore, since B is surjective, for the purpose of studying the irreducible constituents of the Weil representation, it suffices to assume that λ itself is primitive. We shall henceforth make this assumption.

We resume now the construction of W under the assumption that λ is primitive. Set $N = Rv_1 \oplus \cdots \oplus Rv_n$. Note that $(0,N)$ is a transversal for $H(M)$ in $H(V)$. Thus, if $e_v = (0,v) \otimes y \in X$ then $(e_v)_{v\in N}$ is a basis for X over $\mathbf{C}$. If $u \in M$, $v,w \in N$ and $r \in R$, then the definition of X yields

(5)

$$\begin{gathered} S(0,w)e_v = (0,w)(0,v) \otimes y = (0,w+v) \otimes y = e_{v+w}, \\ S(0,u)e_v = (0,u)(0,v) \otimes y = (0,v)(0,u) \otimes (2\langle u,v\rangle, 0)y = \lambda(2\langle u,v\rangle)e_v, \\ S(r,0)e_v = (r,0)(0,v) \otimes y = (0,v) \otimes (r,0)y = \lambda(r)e_v. \end{gathered}$$

Let $G(M)$ denote the subgroup of $\mathrm{Sp}(V)$ that fixes every point of M. Given $g \in G(M)$ consider the operator $P(g)$ of $\mathrm{GL}(X)$ defined by

$$P(g)e_v = \lambda(\langle gv, v\rangle)e_v, \quad v \in N.$$

One verifies by direct computation that $P(g)$ satisfies (3). One also checks that (3) is satisfied by the operator $P(\iota) \in \mathrm{GL}(X)$, defined by

$$P(\iota)e_v = e_{-v}.$$

Let $X_\pm$ denote the ± 1-eigenspace of $P(\iota)$ acting on X. Then Theorem 3.1 of [**3**] states that $X_\pm$ is $P(g)$-invariant for each $P(g) \in \mathrm{GL}(X)$ satisfying (3), and moreover,

$$W(g) = (\det P(g)|_{X_+})^{-1}(\det P(g)|_{X_-})P(g), \quad g \in \mathrm{Sp}(V).$$

Let $\mathcal{T}$ be a set of representatives of $N \setminus \{0\}$ relative to the action of ι. Then $(e_v - e_{-v})_{v\in\mathcal{T}}$ is basis of X_-, and $(e_v + e_{-v})_{v\in\mathcal{T}}$ along with e_0 form a basis of X_+. As

$$P(g)(e_v \pm e_{-v}) = \lambda(\langle gv, v\rangle)(e_v \pm e_{-v}), \quad v \in \mathcal{T},\ g \in G(M)$$

it follows that $(\det P(g)|_{X_+})^{-1} \det P(g)|_{X_-} = \lambda(\langle g0, 0\rangle)^{-1} = 1$ for $g \in G(M)$, whence

$$W(g)e_v = \lambda(\langle gv, v\rangle)e_v, \quad v \in N,\ g \in G(M). \tag{6}$$

3. $\mathrm{Sp}(V)$**-submodules of** X **and congruence subgroups of** $\mathrm{Sp}(V)$**.** Let $\mathfrak{i}$ be an ideal of R of square (0). Let $\mathfrak{j}$ be the annihilator of $\mathfrak{i}$ in R, and let $\mathfrak{k}$ be the conductor of $\mathfrak{j}$ into $\mathfrak{i}$, that is, $(\mathfrak{i} : \mathfrak{j}) = \{r \in R \mid r\mathfrak{j} \subseteq \mathfrak{i}\}$. Observe that $\mathfrak{i} \subseteq \mathfrak{j}$ for $\mathfrak{i}^2 = (0)$. Further, remark

$$\Gamma(\mathfrak{k}) = \{g \in \mathrm{Sp}(V) \mid gv \equiv v \bmod \mathfrak{i}V \text{ for all } v \in \mathfrak{j}V\}. \tag{7}$$

Denote by $X(\mathfrak{i})$ the set of all points in X fixed by the subgroup $(0, \mathfrak{i}V)$ of H. Since $(0, \mathfrak{i}V)$ is normalized by $\mathrm{Sp}(V)$, we see that $X(\mathfrak{i})$ is an $\mathrm{Sp}(V)$-submodule of X. Further, since the subgroup $(R, \mathfrak{j}V)$ centralizes $(0, \mathfrak{i}V)$, it follows that $X(\mathfrak{i})$ is also $(R, \mathfrak{j}V)$-invariant. In fact, Proposition 4.1 and Lemma 4.2 of [**3**] yield the following result.

3.1 Proposition. *$X(\mathfrak{i})$ is an irreducible $(\mathfrak{j}^2, \mathfrak{j}V)$-module of degree $\sqrt{|\mathfrak{j}V/\mathfrak{i}V|}$.*

For future reference we prove the following generalization of Theorem 4.5 of [**3**].

3.2 Theorem. *The $\mathrm{Sp}\,(V)$-module $\mathrm{End}_{\mathbf{C}}(X(\mathfrak{i}))$ is canonically isomorphic to the permutation $\mathrm{Sp}\,(V)$-module $\mathbf{C}(\mathfrak{j}V/\mathfrak{i}V)$.*

Proof. Let $(f_{v+\mathfrak{i}V})_{v+\mathfrak{i}V\in\mathfrak{j}V/\mathfrak{i}V}$ be a complex basis of $\mathbf{C}(\mathfrak{j}V/\mathfrak{i}V)$ that is permuted by $\mathrm{Sp}\,(V)$ according to the rule: ${}^g(f_{v+\mathfrak{i}V}) = f_{gv+\mathfrak{i}V}$.

Consider the linear map $T : \mathbf{C}(\mathfrak{j}V/\mathfrak{i}V) \to \mathrm{End}_{\mathbf{C}}(X(\mathfrak{i}))$, defined on the above basis by $T(f_{v+\mathfrak{i}V}) = S(0,v)|_{X(\mathfrak{i})}$. This is well defined since $S(0,v) = 1_{X(\mathfrak{i})}$ for all $v \in \mathfrak{i}V$ and $(0, \mathfrak{j}V)$ preserves $X(\mathfrak{i})$. To see that T is a homomorphism of $\mathrm{Sp}\,(V)$-modules note that $\mathrm{Sp}\,(V)$ acts on $\mathrm{End}_{\mathbf{C}}(X(\mathfrak{i}))$ via: ${}^gE = W(g)|_{X(\mathfrak{i})}E\,W(g)|_{X(\mathfrak{i})}^{-1}$. Thus, given $g \in \mathrm{Sp}\,(V)$ we have

$$\begin{aligned} T({}^gf_{v+\mathfrak{i}V}) &= T(f_{gv+\mathfrak{i}V}) = S(0,gv)|_{X(\mathfrak{i})} \\ &= S({}^g(0,v))|_{X(\mathfrak{i})} = (W(g)S(0,v)W(g)^{-1})|_{X(\mathfrak{i})} \\ &= W(g)|_{X(\mathfrak{i})}S(0,v)|_{X(\mathfrak{i})}W(g)|_{X(\mathfrak{i})}^{-1} \\ &= {}^gS(0,v)|_{X(\mathfrak{i})} = {}^gT(f_{v+\mathfrak{i}V}), \end{aligned}$$

as required. Since the representation of $(R, \mathfrak{j}V)$ afforded by $X(\mathfrak{i})$ is irreducible, a well-known theorem of Burnside ensures $X(\mathfrak{i}) = \mathrm{span}\,\{S(r,v) \mid r \in R, v \in \mathfrak{j}V\}$. But $S(r,v) = \lambda(r)S(0,v)$, hence $X(\mathfrak{i}) = \mathrm{span}\,\{T(f_{v+\mathfrak{i}V}) \mid v \in \mathfrak{j}V\} = \mathrm{im}\,T$. Thus T is surjective. From Proposition 3.1 we have $\dim \mathrm{End}_{\mathbf{C}}(X(\mathfrak{i})) = |\mathfrak{j}V/\mathfrak{i}V|$, which is also equal to $\dim \mathbf{C}(\mathfrak{j}V/\mathfrak{i}V)$. We conclude that T is injective, and hence an isomorphism. □

3.3 Theorem. *The kernel of the representation of $\mathrm{Sp}\,(V)$ afforded by $X(\mathfrak{i})$ is the congruence subgroup $\Gamma(\mathfrak{k})$.*

Proof. Let $h(\mathfrak{i}) \in \mathrm{End}_{\mathbf{C}}(X)$ be the linear operator defined by

$$h(\mathfrak{i}) = \frac{1}{|(0, \mathfrak{i}V)|} \sum_{v \in \mathfrak{i}V} S(0, v).$$

By construction $X(\mathfrak{i}) = h(\mathfrak{i})X$. Also, since $(0, \mathfrak{i}V)$ is preserved by $\mathrm{Sp}\,(V)$, we see that $h(\mathfrak{i}) \in \mathrm{End}_{\mathbf{C}\mathrm{Sp}\,(V)}(X)$. To compute with $h(\mathfrak{i})$ we make use of (5). Given $u \in \mathfrak{i}M$, $w \in \mathfrak{i}N$ and $v \in N$, we have

$$\begin{aligned} S(0, u + w) &= S(0, u)S(0, w)\lambda(\langle w, u\rangle)e_v \\ &= S(0, u)\lambda(\langle w, u\rangle)e_{v+w} \\ &= \lambda(2\langle u, v\rangle)\lambda(\langle u, w\rangle)e_{v+w}. \end{aligned}$$

As $\langle \mathfrak{i}M, \mathfrak{i}N\rangle = \mathfrak{i}^2\langle M, N\rangle = (0)$ and $\mathfrak{i}V = \mathfrak{i}M \oplus \mathfrak{i}N$, we obtain

$$h(\mathfrak{i})e_v = \frac{1}{|(0, \mathfrak{i}V|)} \bigg(\sum_{u \in \mathfrak{i}M} \lambda(2\langle u, v\rangle) \bigg) \bigg(\sum_{w \in \mathfrak{i}N} e_{v+w} \bigg), \quad v \in N.$$

For $v \in N$ the map $u \mapsto 2\langle u, v\rangle$ is a linear character of $\mathfrak{i}M$, which is trivial if and only if $v \in \mathfrak{j}N$. In particular, $h(\mathfrak{i})e_v = 0$ for $v \in N \setminus \mathfrak{j}N$. We infer that $X(\mathfrak{i})$ is generated by $(h(\mathfrak{i})e_v)_{v \in \mathfrak{j}N}$.

For $r \in \mathfrak{k}$ and $u \in M \setminus \mathfrak{m}M$, the symplectic transvection $g = \rho_{r,u}$ belongs to both $\Gamma(\mathfrak{k})$ and $G(M)$. Since $g \in G(M)$, the formula (6) yields

$$W(g)h(\mathfrak{i})e_v = h(\mathfrak{i})W(g)e_v = h(\mathfrak{i})\lambda(\langle gv, v\rangle)e_v, \quad v \in N.$$

But, from $g \in \Gamma(\mathfrak{k})$ and (7), we infer

$$gv \equiv v \bmod \mathfrak{i}V, \quad v \in \mathfrak{j}N.$$

As $\langle \mathfrak{i}V, \mathfrak{j}V\rangle = (0)$, we deduce

$$\lambda(\langle gv, v\rangle) = \lambda(\langle gv - v, v\rangle) = 1, \quad v \in \mathfrak{j}N,$$

whence

$$W(g)h(\mathfrak{i})e_v = h(\mathfrak{i})e_v, \quad v \in \mathfrak{j}N.$$

This proves

$$W(g) = 1_{X(\mathfrak{i})}. \tag{8}$$

Now, for $f \in \mathrm{Sp}(V)$, one has

$$fgf^{-1} = f\rho_{r,u}f^{-1} = \rho_{r,fu}. \tag{9}$$

Moreover, as any vector in $V \setminus \mathfrak{m}V$ belongs to a symplectic basis of V, cf. Section 1 of [**8**], it follows that $\mathrm{Sp}(V)$ acts transitively on $V \setminus \mathfrak{m}V$. We deduce from (8) and (9) that $W(\rho_{r,w}) = 1_{X(\mathfrak{i})}$ for all $r \in \mathfrak{k}$ and $w \in V \setminus \mathfrak{m}V$. We now appeal to Theorem 2 of [**8**], which asserts that the set of all these $\rho_{r,w}$ generates $\Gamma(\mathfrak{k})$. This proves that $\Gamma(\mathfrak{k})$ acts trivially on $X(\mathfrak{i})$.

Suppose conversely that $f \in \mathrm{Sp}(V)$ acts trivially on $X(\mathfrak{i})$. Then f acts trivially on $\mathrm{End}_{\mathbf{C}}(X(\mathfrak{i}))$. By virtue of Theorem 3.2 we see that f acts trivially on $\mathbf{C}(\mathfrak{j}V/\mathfrak{i}V)$, whence $f \in \Gamma(\mathfrak{k})$ by (7). This completes the proof of the theorem. □

3.4 Corollary. *A Weil representation of primitive type is faithful.*

Proof. Apply Theorem 3.3 to the ideal $\mathfrak{i} = (0)$. □

4. $\mathrm{Sp}(V)$**-submodules of** X **as tensor product of Weil modules.** For the remainder of the paper we denote by $\mathfrak{m}$ the unique maximal ideal of R. Recall that R is a quasi-Frobenius ring if $\operatorname{ann}\operatorname{ann}\mathfrak{a} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of R. For future reference we record the following result.

4.1 Lemma. *The following conditions on the ring R are equivalent*:

(a) *R possesses a primitive linear character λ.*

(b) *R is a quasi-Frobenius ring.*

(c) *R has a unique minimal.*

Proof. (a) ⇒ (b). Given an ideal $\mathfrak{a}$ of R, let $\widehat{\mathfrak{a}}$ denote the group of linear characters of $\mathfrak{a}$, and let $\mathfrak{a}^0$ denote the R-module of linear functionals of $\mathfrak{a}$. Let $\ell : R \to R^0$ be the left-multiplication map. Consider the homomorphisms $R \to \mathfrak{a}^0$ and $\mathfrak{a}^0 \to \widehat{\mathfrak{a}}$, given by $r \mapsto \ell_r|_{\mathfrak{a}}$ and $\phi \mapsto \lambda \circ \phi$. The latter is injective, by the primitivity of λ, while the former has kernel $\operatorname{ann}\mathfrak{a}$. Applying this to the case when $\mathfrak{a} = R$, we obtain that $|R| \leq |R^0| \leq |\widehat{R}| = |R|$, whence both maps are bijective when $\mathfrak{a} = R$.

For an arbitrary ideal $\mathfrak{a}$ and $\phi \in \widehat{\mathfrak{a}}$, let $\varphi \in \widehat{R}$ be an extension of ϕ to R (which exists because the abelian group $\mathbf{C}^*$ is divisible). The above ensures that φ is of the form $\lambda \circ \ell_r$, hence ϕ is of the form $\lambda \circ \ell_r|_{\mathfrak{a}}$ for some $r \in R$. It follows that the composite map $R \to \widehat{\mathfrak{a}}$, given by $r \mapsto \lambda \circ \ell_r|_{\mathfrak{a}}$ is a surjection with kernel $\operatorname{ann} \mathfrak{a}$, whence $|R| = |\operatorname{ann} \mathfrak{a}||\widehat{\mathfrak{a}}| = |\operatorname{ann} \mathfrak{a}||\mathfrak{a}|$. Applying this formula to $\operatorname{ann} \mathfrak{a}$ yields $|R| = |\operatorname{ann} \operatorname{ann} \mathfrak{a}||\operatorname{ann} \mathfrak{a}|$. Since $\mathfrak{a} \subseteq \operatorname{ann} \operatorname{ann} \mathfrak{a}$, we conclude that $\mathfrak{a} = \operatorname{ann} \operatorname{ann} \mathfrak{a}$.

(b) $\Rightarrow$ (c). As R is finite, it has minimal, say $\mathfrak{s}$. As such, $\operatorname{ann} s = \mathfrak{m}$. But by hypothesis $\mathfrak{s} = \operatorname{ann} \operatorname{ann} \mathfrak{s} = \operatorname{ann} \mathfrak{m}$, whence the only minimal ideal of R is $\operatorname{ann} \mathfrak{m}$.

(c) $\Rightarrow$ (a). Let $\mathfrak{s}$ be the only minimal ideal of R. There is a canonical bijective correspondence between the set of non-primitive linear characters of R and the set of all linear characters of $R/\mathfrak{s}$. Hence the former set has $|R/\mathfrak{s}|$ elements. It follows that $|R| - |R/\mathfrak{s}| > 0$ linear characters of R are primitive. □

Let $\mathfrak{i}$ be an ideal of R of square (0). Further, let $\mathfrak{j} = \operatorname{ann} \mathfrak{i}$ and $\mathfrak{k} = (\mathfrak{i} : \mathfrak{j})$. We claim that $\mathfrak{k} = \operatorname{ann} \mathfrak{j}^2$. Indeed, $r \in \operatorname{ann} \mathfrak{j}^2$ if and only if $r\mathfrak{j} \subseteq \operatorname{ann} \mathfrak{j}$. As R is a quasi-Frobenius ring, cf. Lemma 4.1, we have $\operatorname{ann} \mathfrak{j} = \mathfrak{i}$, whence the claim follows.

Recall the canonical epimorphism $B : \operatorname{Sp}(V) \to \operatorname{Sp}(V/\mathfrak{k}V)$ of Section 2. The kernel of B is the congruence subgroup $\Gamma(\mathfrak{k})$. By Theorem 3.3 we know that $\Gamma(\mathfrak{k})$ acts trivially on $X(\mathfrak{i})$. Thus we obtain a representation $\overline{W} : \operatorname{Sp}(V/\mathfrak{k}V) \to \operatorname{GL}(X(\mathfrak{i}))$, defined by

$$\overline{W}(B(g)) = W(g)|_{X(\mathfrak{i})}, \quad g \in \operatorname{Sp}(V).$$

We intend to describe $\overline{W}$ under the assumption that $\mathfrak{j}^2$ is a principal ideal and $\mathfrak{j}/\mathfrak{i}$ is a free $R/\mathfrak{k}$-module.

Assume that $\mathfrak{j}^2$ is indeed a principal ideal. We fix a generator t of $\mathfrak{j}^2$ and consider the map $f : R/k \to \mathfrak{j}^2$, given by

$$f(r + \mathfrak{k}) = rt, \quad r \in R.$$

Since $\mathfrak{k} = \operatorname{ann} \mathfrak{j}^2 = \operatorname{ann} t$, we see that f is an isomorphism of $R/\mathfrak{k}$-modules.

Consider the multiplication map $\{\ ,\ \} : \mathfrak{j}/\mathfrak{i} \times \mathfrak{j}/\mathfrak{i} \to \mathfrak{j}^2$, given by

$$\{x + \mathfrak{i}, y + \mathfrak{i}\} = xy, \quad x, y \in \mathfrak{j}.$$

This is a well-defined symmetric $R/\mathfrak{k}$-bilinear map. Moreover, as $\mathfrak{i} = \operatorname{ann}\mathfrak{j}$ we infer that $\{\ ,\ \}$ is non-degenerate. Assume that $\mathfrak{j}/\mathfrak{i}$ is a free $R/\mathfrak{k}$-module of rank $m > 0$. It follows that $\mathfrak{j}/\mathfrak{i}$ endowed with the form $(\ ,\ ) = f^{-1} \circ \{\ ,\ \}$ is a quadratic space of rank m over $R/\mathfrak{k}$. As such it has basis relative to which the Gram matrix of $(\ ,\ )$ is diagonal, with every diagonal entry being a unit (this can be shown much as in the case when $R = F_q$ is a field). Let $\{x_1 + \mathfrak{k}, \dots, x_m + \mathfrak{k}\}$ be an $R/\mathfrak{k}$-basis of $\mathfrak{j}/\mathfrak{i}$ relative to which the Gram matrix of $(\ ,\ )$ is equal to $\operatorname{diag}(d_1 + \mathfrak{k}, ..., d_m + \mathfrak{k})$, where $d_1, \dots, d_m$ are units of R.

The isomorphism f may also be used to render $\mathfrak{j}V/\mathfrak{i}V$ into a symplectic space over $R/\mathfrak{k}$, as follows. Consider the map $\ll\ ,\ \gg : \mathfrak{j}V/\mathfrak{i}V \to \mathfrak{j}^2$, given by

$$\ll v + \mathfrak{i}V, w + \mathfrak{i}V \gg = \langle v, w\rangle, \quad v, w \in \mathfrak{j}V.$$

Then $\ll\ ,\ \gg$ is a well-defined alternating $R/\mathfrak{k}$-bilinear map. Moreover, as $\mathfrak{i} = \operatorname{ann}\mathfrak{j}$ we see that $\ll\ ,\ \gg$ is non-degenerate. It follows that $\mathfrak{j}V/\mathfrak{i}V$ endowed with the form $[\ ,\ ] = f^{-1}\circ \ll\ ,\ \gg$ is a symplectic space of rank $2nm$ over $R/\mathfrak{k}$.

Consider finally the map $\lambda' : R/\mathfrak{k} \to \mathbf{C}^*$, defined by $\lambda' = \lambda \circ f$, that is,

$$\lambda'(r + \mathfrak{k}) = \lambda(rt), \quad r \in R.$$

As λ is primitive and $\mathfrak{k} = \operatorname{ann} t$ we see that λ' is also primitive. For d a unit of R we let $\lambda'[d + \mathfrak{k}]$ denote the primitive linear character of $R/\mathfrak{k}$ defined by

$$\lambda'[d + \mathfrak{k}](r + \mathfrak{k}) = \lambda'((d + k)(r + \mathfrak{k})) = \lambda(drt), \quad r \in R.$$

With this notation we may state the following result.

4.2 Theorem. *Suppose that* $\mathfrak{j}^2$ *is a principal ideal and* $\mathfrak{j}/\mathfrak{i}$ *is a free* $R/\mathfrak{k}$*-module of rank* $m > 0$. *Then the representation* $\overline{W}$ *of* $\operatorname{Sp}(V/\mathfrak{k}V)$ *afforded by* $X(\mathfrak{i})$ *is similar to the tensor product of* m *Weil representations of primitive types* $\lambda'[d_1 + \mathfrak{k}], \dots, \lambda'[d_m + \mathfrak{k}]$.

Proof. Denote by $H(V/\mathfrak{k}V)^m$ and $\operatorname{Sp}(V/\mathfrak{k}V)^m$ the direct product of m copies of the groups $H(V/\mathfrak{k}V)$ and $\operatorname{Sp}(V/\mathfrak{k}V)$, respectively. The action of $\operatorname{Sp}(V/\mathfrak{k}V)$ on $H(V/\mathfrak{k}V)$ yields an action of $\operatorname{Sp}(V/\mathfrak{k}V)^m$ on

$H(V/\mathfrak{k}V)^m$. This in turn gives an action of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ on $H(V/\mathfrak{k}V)^m$, by means of the diagonal embedding $\mathrm{Sp}\,(V/\mathfrak{k}V) \to \mathrm{Sp}\,(V/\mathfrak{k}V)^m$.

Consider the additive linear character $\mu : (R/\mathfrak{k})^m \to \mathbf{C}^*$, defined by

$$\mu(r_1 + \mathfrak{k}, \dots , r_m + \mathfrak{k}) = \lambda((d_1 r_1 + \cdots + d_m r_m)t), \quad r_i \in R.$$

Let Z denote the center of $H(V/\mathfrak{k}V)^m$. Since $Z = Z(H(V/\mathfrak{k}V))^m \cong (R/\mathfrak{k})^m$, we may identify Z with $(R/\mathfrak{k})^m$ and think of μ as a linear character of Z.

We claim there exists a representation $\overline{S} : H(V/\mathfrak{k}V)^m \to \mathrm{GL}\,(X(\mathfrak{i}))$ satisfying:

(a) $\overline{S}$ is irreducible.

(b) $\overline{S}$ lies over the linear character μ of Z.

(c) $\overline{W}(g)\overline{S}(h)\overline{W}(g)^{-1} = \overline{S}({}^g h)$ for all $h \in H(V/\mathfrak{k}V)^m$ and $g \in \mathrm{Sp}\,(V/\mathfrak{k}V)$.

Assume such a representation exists. Let $S_1, \dots , S_m$ be Schrödinger representations of $H(V/\mathfrak{k}V)$ of types $\lambda'[d_1 + \mathfrak{k}], \dots , \lambda'[d_m + \mathfrak{k}]$, and let $W_1, \dots , W_m$ be associated Weil representations of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ of types $\lambda'[d_1 + \mathfrak{k}], \dots , \lambda'[d_m + \mathfrak{k}]$. Then $\widetilde{S} = S_1 \otimes \ldots \otimes S_m$ is a representation of $H(V/\mathfrak{k}V)^m$ and $\widetilde{W} = W_1 \otimes \ldots \otimes W_m$ is a representation of $\mathrm{Sp}\,(\overline{V}/\mathfrak{k}V)^m$. Further, it follows from the very definitions of $\widetilde{S}$ and $\widetilde{W}$ that

(a′) $\widetilde{S}$ is irreducible.

(b′) $\widetilde{S}$ lies over the linear character μ of Z.

(c′) $\widetilde{W}(g)\widetilde{S}(h)\widetilde{W}(g)^{-1} = \widetilde{S}({}^g h)$ for all $h \in H(V/\mathfrak{k}V)^m$ and $g \in \mathrm{Sp}\,(V/\mathfrak{k}V)^m$.

Observe that the number of times μ enters $\widetilde{S}$ is equal to $|R/\mathfrak{k}|^{nm}$. Note also that $[H(V/\mathfrak{k}V)^m : Z] = |R/\mathfrak{k}|^{2nm}$. We deduce from Lemma 2.1 that $\widetilde{S}$ is similar to $\overline{S}$. It follows from Schur's lemma that $\overline{W}$ is similar, up to multiplication by a linear character of $\mathrm{Sp}\,(V/\mathfrak{k}V)$, to the restriction of $\widetilde{W}$ to the diagonal embedding of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ into $\mathrm{Sp}\,(V/\mathfrak{k}V)^m$. But $\mathrm{Sp}\,(V/\mathfrak{k}V)$ is perfect. This proves the theorem, provided $\overline{S}$ exists.

We proceed to establish the existence of $\overline{S}$. For $s = 1, \dots , m$ we have the $R/\mathfrak{k}$-submodule $V_s = x_s V/\mathfrak{i}V$ of $\mathfrak{j}V/\mathfrak{i}V$. Let $[\ ,\ ]_s$ denote the form on V_s obtained by restricting to V_s the form $[\ ,\ ]$ defined on $\mathfrak{j}/\mathfrak{i}V$. Then

$[\ ,\ ]_s$ is alternating and $R/\mathfrak{k}$-bilinear, and also non-degenerate since $\operatorname{ann} x_s = \mathfrak{k}$.

For $s = 1, \dots, m$ the fact $x_s^2 = d_s t$ yields a group isomorphism $C_s : H(V/\mathfrak{k}V) \to H(V_s)$, defined by

$$C_s(r + \mathfrak{k}, v + \mathfrak{k}V) = (d_s r + \mathfrak{k}, x_s v + \mathfrak{i}V), \quad r \in R,\ v \in V.$$

Since $x_s x_{s'} = 0$ for $s \neq s'$, the submodules V_s of $\mathfrak{j}V/\mathfrak{i}V$ are orthogonal. Further, as $\{x_1 + \mathfrak{k}, \dots, x_m + \mathfrak{k}\}$ is an $R/\mathfrak{k}$-basis of $\mathfrak{j}V/\mathfrak{i}V$, we see that $V_1 \oplus \cdots \oplus V_s = \mathfrak{j}V/\mathfrak{i}V$. Thus the m maps C_s yield the group epimorphism $C : H(V/\mathfrak{k}V)^m \to H(\mathfrak{j}V/\mathfrak{i}V)$, defined by

$$C(h_1, \dots, h_m) = C_1(h_1) \cdots C_m(h_m), \quad h_i \in H(V/\mathfrak{k}V).$$

Given that $(0, \mathfrak{i}V)$ fixes $X(\mathfrak{i})$ and $\mathfrak{k}$ annihilates t, the map $D : H(\mathfrak{j}V/\mathfrak{i}V) \to \mathrm{GL}\,(X(\mathfrak{i}))$, defined by

$$D(r + \mathfrak{k}, v + \mathfrak{i}V) = S(rt, v)|_{X(\mathfrak{i})}, \quad r \in R,\ v \in \mathfrak{j}V$$

is a well-defined representation.

Define $\overline{S} : H(V/\mathfrak{k}V)^m \to \mathrm{GL}\,(X(\mathfrak{i}))$ to be the representation $\overline{S} = D \circ C$. We claim that $\overline{S}$ satisfies (a), (b) and (c).

Indeed, from Proposition 3.1 we know that the representation $(rt, v) \mapsto S(rt, v)|_{X(\mathfrak{i})}$ of $(\mathfrak{j}^2, \mathfrak{j}V)$ is irreducible. Therefore D is irreducible, and since C is surjective, it follows that $\overline{S}$ is also irreducible.

For $r_1, \dots, r_m \in R$ we have

$$\begin{aligned}\overline{S}((r_1 + \mathfrak{k}, 0), \dots, (r_m + \mathfrak{k}, 0)) &= D(d_1 r_1 + \cdots + d r_m + \mathfrak{k}, 0)\\ &= S((d_1 r_1 + \cdots + d r_m)t, 0)|_{X(\mathfrak{i})}\\ &= \lambda((d_1 r_1 + \cdots + d r_m)t) 1_{X(\mathfrak{i})}.\end{aligned}$$

Thus $\overline{S}$ lies indeed over μ.

We finally verify that $\overline{S}$ satisfies (c). For $s = 1, \dots, m$ let $e_s : H(V/\mathfrak{k}V) \to H(V/\mathfrak{k}V)^m$ be the embedding $e_s(h) = (1, \dots, h, \dots, 1)$, where $h \in H(V/\mathfrak{k}V)$ is in the s-position. Note that $\overline{S}(e_s(h)) =$

$D(C(e_s(h))) = D(C_s(h))$. Further, if $g \in \mathrm{Sp}\,(V/\mathfrak{k}V)$, then ${}^g e_s(h) = e_s({}^g h)$. Let $h = (0, v + \mathfrak{k}V)$ with $v \in V$ and let $g \in \mathrm{Sp}\,(V/\mathfrak{k}V)$. Then

$$\begin{aligned}
\overline{W}(B(g))\overline{S}(e_s(h))\overline{W}(B(g))^{-1} &= W(g)|_{X(\mathfrak{i})} D(0, x_s v + \mathfrak{i}V) W(g)|_{X(\mathfrak{i})}^{-1} \\
&= W(g)|_{X(\mathfrak{i})} S(0, x_s v)|_{X(\mathfrak{i})} W(g)|_{X(\mathfrak{i})}^{-1} \\
&= S(0, x_s g v)|_{X(\mathfrak{i})} \\
&= D(C(e_s(0, gv + \mathfrak{k}V))) \\
&= D(C(e_s({}^g(0, gv + \mathfrak{k}V)))) \\
&= D(C({}^g e_s(0, gv + \mathfrak{k}V))) \\
&= \overline{S}({}^g e_s(h)).
\end{aligned}$$

Since $H(V/\mathfrak{k}V)^m$ is generated by the images of the maps e_s, and the set $(0, V/\mathfrak{k}V)$ generates $H(V/\mathfrak{k}V)$, we infer that (c) holds. This completes the proof of the theorem. □

5. The bottom layer of X. As above, let $\mathfrak{i}$ be an ideal of R of square (0), and set $\mathfrak{j} = \mathrm{ann}\,\mathfrak{i}$, $\mathfrak{k} = (\mathfrak{i} : \mathfrak{j})$. Suppose that $\mathfrak{i} = \mathfrak{j}$. Then $\mathfrak{k} = R$, so $\mathrm{Sp}\,(V/\mathfrak{k}V)$ is the trivial group acting trivially on $X(\mathfrak{i})$. Since $\dim_{\mathbf{C}} X(\mathfrak{i}) = 1$ by Proposition 3.1, we see that the representation $\overline{W}$ of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ afforded by $X(\mathfrak{i})$ is trivial. For uniformity of terminology we agree to the following convention: the Weil representation of trivial group is the trivial representation, its type being primitive.

5.1 Theorem. *Let $\mathfrak{i}$ be an ideal of R of square* (0)*. Set $\mathfrak{j} = \mathrm{ann}\,\mathfrak{i}$ and $\mathfrak{k} = (\mathfrak{i} : \mathfrak{j})$. Then the representation $\overline{W}$ of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ afforded by $X(\mathfrak{i})$ is similar to a Weil representation of some type, primitive or not, if and only if $\mathfrak{j}/\mathfrak{i}$ is a principal $R/\mathfrak{k}$-module, in which case the type is primitive.*

Proof. If $\mathfrak{i} = \mathfrak{j}$ then $\mathfrak{j}/\mathfrak{i}$ is certainly principal, and we saw above that $\overline{W}$ is a Weil representation of primitive type. Assume for the remainder of the proof that $\mathfrak{i}$ is properly contained in $\mathfrak{j}$.

Sufficiency. Suppose that $\mathfrak{j}/\mathfrak{i}$ is generated by $r + \mathfrak{i}$ for some $r \in \mathfrak{j}$. Thus $\mathfrak{j}/\mathfrak{i}$ is a free $R/\mathfrak{k}$-module of rank 1. Further, $\mathfrak{j}^2 = Rt$ for $t = r^2$. It follows from Theorem 4.2 that $\overline{W}$ is similar to a Weil representation of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ of primitive type.

Necessity. Suppose that $X(\mathfrak{i})$ affords a Weil representation of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ of type μ. As indicated in Section 2, the congruence subgroup $\Gamma(\mathfrak{i}_\mu)$ of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ is in the kernel of this representation. But, by Theorem 3.3, the representation of $X(\mathfrak{i})$ afforded by $\mathrm{Sp}\,(V/\mathfrak{k}V)$ is faithful. It follows that $\Gamma(\mathfrak{i}_\mu)$ is the trivial group, whence μ is primitive. Thus, by Lemma 4.1, $R/\mathfrak{k}$ has a unique minimal ideal. This means there is only one ideal of R lying above $\mathfrak{k}$.

Now the very definition of $\mathfrak{k}$ yields $\mathfrak{k} = \cap_{x\in\mathfrak{j}}(\mathfrak{i} : (x))$, so the stated property of $\mathfrak{k}$ implies $\mathfrak{k} = (\mathfrak{i} : (t))$ for some $t \in \mathfrak{j}$. As a result, the homomorphism of $R/\mathfrak{k}$-modules

$$R/\mathfrak{k} \ni r + \mathfrak{k} \longrightarrow rt + \mathfrak{i} \in \mathfrak{j}/\mathfrak{i} \tag{10}$$

is injective. On the other hand, Proposition 3.1 yields

$$\deg X(\mathfrak{i}) = |\mathfrak{j}/\mathfrak{i}|^n,$$

while (4) and the assumption that $X(\mathfrak{i})$ affords a Weil representation of $\mathrm{Sp}\,(V/\mathfrak{k}V)$ of primitive type combine to give

$$\deg X(\mathfrak{i}) = |R/\mathfrak{k}|^n.$$

Hence $|\mathfrak{j}/\mathfrak{i}| = |R/\mathfrak{k}|$, so (10) must be a bijection. This means that $\mathfrak{j}/\mathfrak{i}$ is generated by t as an R/k-module, as required. □

Denote by $\mathfrak{s}$ the unique minimal ideal of R, as ensured by Lemma 4.1. Denote by l the nilpotency degree of $\mathfrak{m}$. Since $\mathfrak{m}^{l-1}$ is non-zero and annihilates $\mathfrak{m}$, we see that $\mathfrak{s} = \mathfrak{m}^{l-1}$. If $l = 1$, then R is a field and $\mathfrak{m} = (0)$, $\mathfrak{s} = R$. If $l \geq 2$ then $\mathfrak{s}$ is contained in $\mathfrak{m}$ and has square (0); further, $X(\mathfrak{i}) \subseteq X(\mathfrak{s})$ for any ideal $\mathfrak{i}$ of R of square (0), since $\mathfrak{s} \subseteq \mathfrak{i}$. If $l = 2$, then R has precisely three ideals, namely (0), $\mathfrak{s} = \mathfrak{m}$ and R. In particular, R is a principal ring. If $l > 2$, then $\mathfrak{s}$ is properly contained in $\mathfrak{m}$.

Suppose that R is not a field. It is shown in [**2**] that $X(\mathfrak{s})$, referred to as the bottom layer of X, is equal to the set of fixed points of $\Gamma((\mathfrak{s} : \mathfrak{m}))$ in X. Thus, as mentioned in the introduction the quotient $\mathrm{Sp}\,(V)$-module $X/X(\mathfrak{s})$ has two irreducible components, namely its ± 1-eigenspaces relative to the action of -1_V. Further, when R is a principal ring $X(\mathfrak{s})$ affords a Weil module of primitive type for the quotient symplectic group $\mathrm{Sp}\,(V/(\mathfrak{s}:\mathfrak{m})V)$, so by repeatedly applying this procedure one obtains all irreducible components of X. This was essentially the technique used in [**3**]. Our next result shows that when R is not principal this inductive procedure will never work.

5.2 Theorem. *Suppose that R is not a field. The representation of* $\mathrm{Sp}\,(V/(\mathfrak{s}:\mathfrak{m})V)$ *afforded by $X(\mathfrak{s})$ is similar to a Weil representation if and only if R is a principal ring, in which case its type is primitive.*

Proof. Sufficiency follows from Theorem 5.1 applied to $\mathfrak{i}=\mathfrak{s}$. As for necessity, if $l=2$, then R was noted above to be principal. Suppose next $l>2$. If $X(\mathfrak{s})$ affords a Weil representation, then Theorem 5.1 implies that $\mathfrak{m}/\mathfrak{s}$ is a principal R-module. Since $l>2$, we have $\mathfrak{s}=\mathfrak{m}^{l-1}\subseteq\mathfrak{m}^2$. Further,

$$\mathfrak{m}/\mathfrak{m}^2\cong(\mathfrak{m}/\mathfrak{s})/(\mathfrak{m}^2/\mathfrak{s}).$$

We infer that $\mathfrak{m}/\mathfrak{m}^2$ is a principal R-module. Thus R itself is a principal ring, as ensured by Proposition 8.8 of [**1**]. □

Denote by F_q the residue class field of R, that is $F_q=R/\mathfrak{m}$. Further, let $\mathrm{Sp}\,(2n,q)=\mathrm{Sp}\,(V/\mathfrak{m}V)$. The first occurrence of a non-principal ring takes place when $l=3$. In this case the next result shows that the decomposition problem for X is equivalent to the problem of decomposing the tensor product of $\dim_{F_q}\mathfrak{m}/\mathfrak{m}^2$ Weil modules for $\mathrm{Sp}\,(2n,q)$.

5.3 Theorem. *Suppose that $l=3$. Then the representation $\overline{W}$ of* $\mathrm{Sp}\,(2n,q)$ *afforded by $X(\mathfrak{s})$ is similar to tensor product of* $\dim_{F_q}\mathfrak{m}/\mathfrak{m}^2$ *Weil representations of primitive type.*

Proof. Apply Theorem 4.2 to the ideal $\mathfrak{i}=\mathfrak{s}$. In this case we have $\mathfrak{i}=\mathfrak{m}^2$, $\mathfrak{j}=\operatorname{ann}\mathfrak{i}=\mathfrak{m}$ and $\mathfrak{k}=(\mathfrak{i}:\mathfrak{j})=\mathfrak{m}$. Further, $\mathfrak{j}^2=\mathfrak{s}$ is a principal ideal, $R/\mathfrak{k}=F_q$ and $\mathfrak{j}/\mathfrak{i}=\mathfrak{m}/\mathfrak{m}^2$ is a free F_q-module of finite rank $m>0$. The result thus follows. □

For a unit d of R, let $\lambda[d]$ be the primitive linear character of R given by $r \mapsto \lambda(dr)$.

5.4 Proposition. *The complex conjugates of a Weil representation of type λ is a Weil representation of type $\lambda[-1]$.*

Proof. Let S^* and W^* be the complex conjugate of the Schrödinger and Weil representations S and W of type λ. Note that S^* is an irreducible representation of $H(V)$ satisfying

$$S^*(r,0) = \overline{\lambda(r)}1_X = \lambda(r)^{-1}1_X = \lambda(-r)1_X = \lambda[-1](r)1_X, \quad r \in R.$$

Since $\lambda[-1]$ is primitive, we infer that S^* is a Schrödinger representation of type $\lambda[-1]$. As W^* satisfies (2) relative to S^*, we conclude that W^* is a Weil representation of type $\lambda[-1]$. □

5.5 Theorem. *Let R_0 be any finite commutative quasi-Frobenius local ring of odd characteristic. Let ϕ be any complex irreducible character of* $\mathrm{Sp}\,(2n, R_0)$*. Then we can choose R so that R_0 is a quotient of R and the inflation of ϕ to* $\mathrm{Sp}\,(2n, R)$ *is equal to the character afforded by some* $\mathrm{Sp}\,(2n, R)$*-submodule of $X(\mathfrak{s})$.*

Proof. For each positive integer m consider the polynomial ring $P_m = R_0[X_1, Y_1, \dots, X_m, Y_m]$. Let I_m be the ideal of P_m generated by

$$X_i^2 - X_j^2, \quad X_i^2 + Y_i^2, \quad X_i^3, \quad X_iX_j, \quad Y_iY_j, \quad X_iY_k,$$

where $1 \leq i \neq j \leq m$ and $1 \leq k \leq m$. Set $R_m = P_m/I_m$ and consider the following elements of R_m

$$x_i = X_i + I_m, \quad y_i = Y_i + I_m, \quad t = X_1^2 + I_m, \quad 1 \leq i \leq m.$$

Then R_m is a free R_0-module of rank $2(m+1)$ with basis $\{1, x_1, y_1, \dots, x_m, y_m, t\}$. Further, the following relations hold in R_m

$$\begin{aligned} x_1^2 = -y_1^2 = \cdots = x_m^2 = -y_m^2 = t, \\ x_1^3 = y_1^3 = \cdots = x_m^3 = y_m^3 = 0, \\ x_ix_j = y_iy_j = x_iy_k = 0, \end{aligned} \tag{11}$$

where $1 \le i \neq j \le m$ and $1 \le k \le m$.

Denote by $\mathfrak{m}_0$ and $\mathfrak{s}_0$ the unique maximal and minimal ideals of R_0, respectively. Then R_m is a finite commutative quasi-Frobenius local ring of odd characteristic, with unique maximal ideal $\mathfrak{m}_0 \oplus R_0x_1 \oplus R_0y_1 \oplus \cdots \oplus R_0x_m \oplus R_0y_m \oplus R_0t$ and unique minimal ideal $\mathfrak{s}_0 t$. Consider the ideal $\mathfrak{i}_m = R_0t$ of R_m. Then

$$\mathfrak{i}_m^2 = (0),$$
$$\mathfrak{j}_m = \operatorname{ann}\mathfrak{i}_m = R_0x_1 \oplus R_0y_1 \oplus \cdots \oplus R_0x_m \oplus R_0y_m \oplus R_0t,$$
$$\mathfrak{k}_m = (\mathfrak{i}_m : \mathfrak{j}_m) = \mathfrak{j}_m.$$

Further, $R_m/\mathfrak{k}_m \cong R_0$, $\mathfrak{j}_m^2 = \mathfrak{i}_m$ is principal and $\mathfrak{j}_m/\mathfrak{i}_m \cong R_0x_1 \oplus R_0y_1 \oplus \cdots \oplus R_0x_m \oplus R_0y_m$ is a free R_0-module of rank $2m$.

Use the generator t of $\mathfrak{j}^2$ to define a non-degenerate symmetric $R_m/\mathfrak{k}_m$-bilinear form $(\ ,\ )_m$ on $\mathfrak{j}_m/\mathfrak{i}_m$, as indicated in Section 4. Then the relations (11) show that relative to the basis $\{x_1 + \mathfrak{i}_m, y_1 + \mathfrak{i}_m, \dots, x_m + \mathfrak{i}_m, y_m + \mathfrak{i}_m\}$ of $\mathfrak{j}_m/\mathfrak{i}_m$, the Gram matrix of $(\ ,\ )_m$ is equal to $\operatorname{diag}(1, -1, \dots, 1, -1)$.

Let $W_m : \operatorname{Sp}(2n, R_m) \to \operatorname{GL}(X_m)$ be a Weil representation of primitive type. From Theorem 3.3 we know that the congruence subgroup $\Gamma(\mathfrak{k}_m)$ of $\operatorname{Sp}(2n, R_m)$ acts trivially on $X_m(\mathfrak{i}_m)$. Further, by Theorem 4.2 and Proposition 5.4 the representation of $\operatorname{Sp}(2n, R_0)$ afforded by $X_m(\mathfrak{i}_m)$ via the canonical isomorphism $\operatorname{Sp}(2n, R_m)/\Gamma(\mathfrak{k}_m) \cong \operatorname{Sp}(2n, R_0)$ has character $(\psi\overline{\psi})^m$, where ψ is a Weil character of $\operatorname{Sp}(2n, R_0)$ of primitive type, and the bar indicates complex conjugation.

From Theorem 3.2 we see that $\varphi = \psi\overline{\psi}$ is the permutation character of $\operatorname{Sp}(2n, R_0)$ acting on a symplectic space V_0 of rank $2n$ over R_0. In particular, φ is a faithful character. Further, the number of times the trivial character $1_{\operatorname{Sp}(2n,R_0)}$ of $\operatorname{Sp}(2n, R_0)$ enters φ is equal to the number of $\operatorname{Sp}(2n, R_0)$-orbits of V_0, hence is at least two.

Let $(\phi_i)_{i\in I}$ be the family of all complex irreducible characters of $\operatorname{Sp}(2n, R_0)$. For each $i \in I$ the Burnside-Brauer theorem, cf. Section 4 of [**7**] ensures the existence of a non-negative integer m_i such that ϕ_i enters φ^{m_i}. Choose a positive integer a large enough so that ϕ is contained in $a\sum_{i\in I}\phi_i$. Next take a positive integer b so that $2^b > a$. Since ϕ_i enters φ^{m_i} and φ^b contains $a \cdot 1_{\operatorname{Sp}(2n,R_0)}$, we see that $a\phi_i$ is contained in φ^{m_i+b}. Let $m = \max\{m_i + b \,|\, i \in I\}$. For $i \in I$ the character φ^{m_i+b} is contained in φ^m since $1_{\mathrm{Sp}\,(2n,R_0)}$ enters $\varphi^{m-(m_i+b)}$. We deduce that $a\sum_{i\in I}\phi_i$, and hence ϕ, is contained in φ^m. On taking $R = R_m$ and $\mathfrak{i} = \mathfrak{i}_m$, we conclude that the $\mathrm{Sp}\,(2n, R_0)$-module $X(\mathfrak{i})$ has a submodule whose character is equal to ϕ. Since $X(\mathfrak{i}) \subseteq X(\mathfrak{s})$, the result follows. □

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