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LIÉNARD LIMIT CYCLES ENCLOSING PERIOD ANNULI, OR ENCLOSED BY PERIOD ANNULI

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ABSTRACT. We construct examples of polynomial Liénard systems with both centers and limit cycles. The first class of examples has limit cycles enclosed by period annuli. The second class has limit cycles surrounding central regions. In both cases we show that it is possible to construct polynomial systems having an arbitrary number of limit cycles with such properties. As a limit case, we construct an analytic Liénard system with infinitely many limit cycles surrounding a central region. We also show that for every n there exists a Liénard system of degree n with $n-2$ limit cycles.

1. Introduction.

Let

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y) \tag{1}$$

be an autonomous plane differential system. We assume $P(x,y)$, $Q(x,y)$ to be analytic real functions defined on all of the real plane. We say that a critical point O of (1) is a center if it has a punctured neighborhood covered with nontrivial cycles. If O is a center, the largest connected region covered with cycles surrounding O is called *central region* and will be denoted by N_O. Every connected region covered with nontrivial concentric cycles is usually called a *period annulus.* Period annuli are not necessarily contained in central regions. An example is given by the Hamiltonian system

$$\dot{x} = y, \qquad \dot{y} = -x(4x^2-1)(x^2-1), \tag{2}$$

which has centers at $(-1,0)$, $(0,0)$, $(1,0)$, and a period annulus enclosing such centers. Since when O is a center there exists a first integral

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defined on all of its central region [**7**], such systems are usually called *integrable*. The study of integrable systems is of physical interest for the existence of a quantity which is constant along the evolution of the system, as in conservative systems.

An isolated nontrivial cycle γ of (1) is said to be a *limit cycle*, since it is the ω-limit or the α-limit of all its neighboring orbits. Limit cycles of analytic systems are either asymptotically stable or negatively asymptotically stable or semistable. Hence, a limit cycle of an analytic system cannot be a component of the boundary of a period annulus. The existence of limit cycles in plane systems is usually proved by applying Poincaré-Bendixson theory, or by means of bifurcation techniques. In both cases the involved systems display some kind of dissipativity. Several results have been obtained by producing limit cycle bifurcation from the cycles of a central region. In this case the bifurcation process locally destroys the integrability of the system.

Even in polynomial systems, integrability and dissipativity can coexist in different regions. There are several examples of systems having both centers and attracting critical points. The more specific question whether centers and limit cycles can coexist in the same system has been considered in a few papers. It is known that in quadratic systems this cannot occur [**13**]. Examples of coexistence in higher degree systems were given in [**1, 4**], respectively for cubic and quartic systems. In [**12**] a cubic system with three centers and two limit cycles is given. In [**2**] appears a cubic system with a limit cycle surrounding three centers. In [**8**] a degree 7 system with a period annulus enclosing five centers and 44 limit cycles is constructed.

In this paper we are concerned with the coexistence of centers and limit cycles of analytic Liénard systems, a class of systems widely studied for their relevance in applications. The only paper dealing with such a problem seems to be [**3**], which contains an example of polynomial Liénard system of degree 9 with a center and a limit cycle not enclosing the central region. Here we shall deal with a special kind of configuration that occurs when a limit cycle is enclosed by a period annulus, or when a limit cycles encloses a period annulus.

The first case can be treated following the approach of [**8**], perturbing a reversible system by means of reversible perturbations. This allows to generate simultaneous bifurcations of couples of limit cycles enclosed by a period annulus. In this way, for every $k > 0$ we can show the existence of a Liénard system of degree $4k+2$ with $2k$ limit cycles and a central region, all enclosed by a period annulus. A slight modification of the proof allows to show that for every $n > 3$ there exist a Liénard system of degree n having $n-2$ limit cycles. The question of estimating the maximum number of limit cycles of polynomial Liénard systems is related to Hilbert's 16th problem.

The case of limit cycles enclosing a period annulus (in our case, a central region) is more involved. If the central region is bounded, then its boundary ∂N_O contains at least a critical point. The first return map associated to ∂N_O has a flat derivative so that bifurcation techniques based on its derivatives cannot be applied. This difficulty can be overcome by considering bifurcations at infinity. Starting with a Liénard system with a center, we apply a sequence of bifurcations from infinity that produce several limit cycles enclosing the central region. We show that for every $k > 0$ there exists a Liénard system of degree $6k+3$ with k limit cycles surrounding a central region. Such a result can be adapted to construct an analytic Liénard system with infinitely many limit cycles surrounding a central region.

2. Limit cycles enclosed by period annuli. We are concerned with the plane differential system,

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x). \tag{3}$$

equivalent to the generalized Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{4}$$

where f and g are analytic functions defined on $\mathbf{R}$. We call $F(x)$ and $G(x)$ the unique functions such that $F'(x) = f(x)$, $F(0) = 0$, $G'(x) = g(x)$, $G(0) = 0$. The critical points of (3) are the points $(x_0, 0)$, where $g(x_0) = 0$.

For the sake of brevity, we say that the Liénard system (3) is *reversible* if $f(-x) = -f(x)$, $g(-x) = -g(x)$. We are actually dealing with a special kind of reversibility, equivalent to the symmetry of the vector field $(y, -g(x) - yf(x))$ with respect to the y-axis. This kind of reversibility is equivalent to the property that a curve $t \mapsto (x(t), y(t))$

is a solution to (3) if and only if the curve $t \mapsto (-x(-t), y(-t))$ is a solution to (3). In particular, if a solution intersects the y-axis, then its orbit is symmetric with respect to the y-axis. The following lemma is a straightforward consequence of the continuous dependence on initial data.

Lemma 1. *Let $f(x)$ and $g(x)$ be odd. Then every cycle of* (3) *intersecting the y-axis has a neighborhood covered with cycles, all intersecting the y-axis.*

As a consequence, no limit cycles of a reversible Liénard system can meet the y-axis.

In the next proofs we shall repeatedly apply bifurcation procedures in order to construct systems with several limit cycles. A main point of such procedures is the possibility to perturb systems in such a way that the estimated number of limit cycles does not decrease after the perturbation. This is ensured, for instance, by Theorem III.1.2 in [**9**].

In this section, bifurcations will be obtained as a consequence of stability inversions of critical points. The stability of a critical point of a Liénard system is determined by the local sign of $f(x)$ and $G(x)$. We omit the proof of next lemma, which is an elementary consequence of La Salle invariance principle.

Lemma 2. *Let x_0 be an isolated zero of $g(x)$, with $G''(x_0) = g'(x_0) > 0$. If $f(x) > 0$ in a punctured neighborhood of x_0, then $(x_0, 0)$ is asymptotically stable. If $f(x) < 0$ in a punctured neighborhood of x_0, then $(x_0, 0)$ is negatively asymptotically stable.*

Let

$$\dot{x} = P(x,y) = \sum_{i+j=0}^{n} b_{i,j} x^i y^j, \qquad \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} c_{i,j} x^i y^j, \tag{5}$$

be a polynomial differential system of degree n. We consider the *coefficient vector* $C = (b_{00}, b_{10}, \ldots, b_{0n}, c_{00}, c_{10}, \ldots, c_{0n}) \in \mathbf{R}^N$, $N = n^2 + 3n + 2$, having as components the coefficients of (5). If $(x_0, y_0) \in \mathbf{R}^2$, then we write (x_0, y_0, C) for the $(N + 2)$-dimensional vector

$(x_0, y_0, b_{00}, b_{10}, \dots, b_{0n}, c_{00}, c_{10}, \dots, c_{0n})$. Let us denote by $\mathbf{P}_n$ the set of polynomial systems of degree $\leq n$, and by $\mathbf{D}_n$ the set of polynomial systems of degree $\leq n$, reversible with respect to the y-axis. We denote by S_C the polynomial system corresponding to the coefficient vector C. We set $y^+ = \{(x,y) \in \mathbf{R}^2 : x = 0,\ y > 0\}$.

Lemma 3. *Let $C \in \mathbf{R}^N$ be the coefficient vector of a reversible polynomial system, γ a cycle of* (5) *intersecting the y-axis at a point $z_0 = (0, y_0)$. Then there exists a compact neighborhood $U_0 \subset \mathbf{R}^{N+2}$ of $(0, y_0, C)$, such that if $(0, y_0, C') \in U_0$ and $S_{C'} \in \mathbf{D}_n$, then every orbit of $S_{C'}$ passing through U_0 is a cycle.*

Proof. Let $z(t, z_0, C)$ be the solution to S_C such that $z(0, z_0, C) = z_0$, T the period of $z(t, z_0, C)$. There exists $\nu > 0$ such that $z(T+\nu, z_0, C)$ is contained in the half-plane $x > 0$ together with a neighborhood V_0. By the theorem of continuous dependence on initial conditions and parameters, there exists a neighborhood W_0 of $(0, z_0, C)$ in the space R^{N+2}, such that for every point $(0, z_1, C') \in W_0$ the solution of the system $S_{C'}$ starting at z_1 satisfies $z(T+\nu, z_1, C') \in V_0$. W_0 can be taken so that all such solutions cross y^+.

If we restrict to the subclass $\mathbf{D}_n$, then by the reversibility, all the solutions starting at a point of W_0 are cycles of the corresponding system. □

Theorem 1. *For every integer $k > 0$ there exists a reversible polynomial Liénard system of degree $4k+2$ with a period annulus enclosing $2k$ limit cycles and a center.*

Proof. Let us set $n = 4k+2$. Let us consider system (2), which has five critical points at $(0,0)$, $(\pm 1/2, 0)$, $(\pm 1, 0)$. Let us denote by C_g the coefficient vector of (2), considered as an $(N+2) = (n^2+3n+4)$-dimensional vector. At each step of this proof, both $f(x)$ and $g(x)$ will be odd. In particular, $g(x)$ will not change, and since $g'(0) = 1$, the origin will be a center for all the systems considered.

Since $\lim_{x \to \pm\infty} G(x) = +\infty$, every orbit of (2) out of a compact set is a cycle, hence the system has an unbounded period annulus P. Let γ be one of the cycles contained in P, γ enclosing all the critical points of

(2), and U_0 the neighborhood of Lemma 3. Let $z(t, z_0)$ be the solution corresponding to γ.

We proceed by successive perturbations, considering the systems' coefficients as bifurcation parameters. All the parameters will be chosen small enough so that, by Lemma 3, the perturbed systems have a period annulus containing the point z_0.

Setting $f(x) = x(x-1)^{2k}(x+1)^{2k}$, let us consider the $(4k+2)$-degree system obtained by taking

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x). \tag{6}$$

By Lemma 2, $(1, 0)$ is asymptotically stable, while $(-1, 0)$ is negatively asymptotically stable.

Then we consider the system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y[f(x) - f_{\mu_1}(x)], \tag{7}$$

where $f_{\mu_1}(x) = \mu_1 x[(x-1)^{2k-2}(x+1)^{2k-2}]$. The order of $f_{\mu_1}(x)$ at ± 1 is lower than that of $f(x)$, hence the local sign of $f(x) - f_{\mu_1}(x)$ is that of $-f_{\mu_1}(x)$, for $\mu_1 \neq 0$. As μ_1 becomes positive, by Lemma 2 the points $(\pm 1, 0)$ change stability, and at least two limit cycles, one for each critical point, appear for small positive values of μ_1.

If $k = 1$, that is if $n = 6$, the procedure stops here. If $k > 1$, we proceed for $k - 1$ additional steps.

In the second step we consider the systems

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y[f(x) - f_{\mu_1}(x) + f_{\mu_2}(x)], \tag{8}$$

where $f_{\mu_2}(x) = \mu_2 x[(x-1)^{2k-4}(x+1)^{2k-4}]$. As μ_2 becomes positive, the points $(\pm 1, 0)$ change again stability, and two more limit cycles appear for small positive values of μ_2. Also, by Theorem III.1.2 in [**9**], for small positive values of μ_2 the new system has two limit cycles close to those ones appeared in the previous bifurcation.

We proceed adding more and more perturbations until we add a linear term

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y[f(x) - f_{\mu_1}(x) + \cdots \pm \mu_k x]. \tag{9}$$

After this last perturbation, we have $2k$ limit cycles. Each of the systems considered has a center at the origin, by the reversibility of all the systems constructed. Also, all such systems have a period annulus enclosing such limit cycles and the center. □

The lowest degree for which we obtain a period annulus enclosing a center and limit cycles is 6. In this case we get a center and two limit cycles.

In the next theorem we consider perturbations that change the stability properties of the origin. In this way we are led to consider nonreversible systems.

Theorem 2. *For every integer $n > 5$ there exists a polynomial Liénard system of degree n with $n-2$ limit cycles.*

Proof. Let us first assume n even. Let us set $n = 4k+2$.

By the previous theorem, there exists a reversible system of degree n,

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x) \tag{10}$$

having a center and $2k$ limit cycles. The polynomial $f(x)$ is odd, hence it has the form $f(x) = \sum_{j=0}^{r} a_{2j+1}x^{2j+1}$, $r = 2k = (n/2) - 1$. We consider successive perturbations obtained by adding even-degree monomials of the form $\lambda_{2j}x^{2j}$, $j = r, \dots, 0$. As in the previous theorem, we proceed by decreasing degrees. As a first perturbation, we take the system of the form

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y[f(x) + \lambda_{2r}x^{2r}], \tag{11}$$

with $\lambda_{2r} > 0$. The origin is asymptotically stable. In fact, the vector product of the unperturbed vector field (10) and the perturbed vector field (11) is $-\lambda_{2r}x^{2r}y^2 \le 0$. Hence the orbits of (11) cross the cycles of (10) towards the interior. By the LaSalle invariance principle, this gives both the stability and the attractivity of O.

The second perturbation is obtained by adding a term of degree $2r-2$:

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y[f(x) + \lambda_{2r}x^{2r} - \lambda_{2r-2}x^{2r-2}]. \tag{12}$$

Arguing as above, one can prove that the origin is negatively asymptotically stable, so that a limit cycle bifurcates out of the origin as λ_{2r-2} becomes positive.

Then we go on adding perturbations of lower and lower degree, until we add a constant term:

$$(13)\quad \dot{x} = y, \qquad \dot{y} = -g(x) - y[f(x) + \lambda_{2r}x^{2r} - \lambda_{2r-2}x^{2r-2} + \cdots \pm \lambda_0].$$

After this last perturbation, we have produced $r = 2k$ new limit cycles. Such perturbations can be produced without destroying the presence of previous limit cycles. Hence the last system has $4k = n - 2$ limit cycles.

Let us now consider the case of $n > 5$, n odd. Let us set $n = 4k + 3$.

By Theorem 1, there exists a reversible system of degree $n-1 = 4k+2$, having a center and $2k$ limit cycles. We can repeat the procedure of the first part of this proof, but starting with a perturbation $\lambda_{4k+2}x^{4k+2}$ of degree $n - 1 = 4k + 2$, instead of degree $4k$. This allows to produce the desired $2k + 1$ bifurcations, for the additional $2k + 1$ limit cycles. □

The above result is not sharp. In fact, it is known that there exist cubic Liénard systems with quadratic damping with two limit cycles [**5**]. Anyway, high degree systems still have to be studied in detail, and at present there are no sharper results for Liénard systems of arbitrary degree.

3. Limit cycles enclosing central regions. We say that an autonomous plane differential system is *ultimately bounded* (UB), if the system has a globally asymptotically stable compact set. Similarly, we say that a system is *negatively ultimately bounded* (NUB), if the system has a negatively globally asymptotically stable compact set.

For the reader's convenience, we report here a theorem by Graef [**6**] concerned with Liénard systems, in the form that will be used in this paper.

Theorem 3 (Graef). *Let $F(x)$ and $g(x)$ be Lipschitzian. Assume that there exists $\sigma \in R$ such that*

$xF(x) > 0$ *for* $|x| \geq \sigma$;

$xg(x) > 0$ for $|x| \geq \sigma$;

$F(x) \geq \text{const.} > 0$ for $x \geq \sigma$ ($F(x) \leq \text{const.} < 0$ for $x \leq -\sigma$) ;

$\int_0^{\pm\infty} [f(x) + |g(x)|] \, dx = \pm\infty$.

Then (3) *has a globally asymptotically stable compact set K.*

Graef's theorem can also be used to prove the existence of a negatively globally asymptotically stable compact set. In fact, $(x(t), y(t))$ is a solution to (3) if and only if $(x(-t), -y(-t))$ is a solution to

$$\dot{x} = y, \qquad \dot{y} = -g(x) + yf(x). \tag{14}$$

As a consequence, the solutions of (3) are bounded if and only if the solutions of (14) are negatively bounded, and vice versa.

It is easy to check that if $F(x)$ and $g(x)$ are odd-degree polynomials, with positive leading coefficients, then (3) is UB. Similarly, if $F(x)$ and $g(x)$ are odd-degree polynomials, $g(x)$ with positive leading coefficient, $F(x)$ with negative leading coefficient, then (3) is NUB.

In the following, we show that Liénard systems can have several limit cycles surrounding a center. The procedure we apply is based on repeated bifurcations at infinity. We report here the main result used, proved in [**10**], applied in [**11**] to study limit cycles bifurcating from infinity.

Theorem 4. *Let*

$$\dot{x} = P_\mu(x, y) \qquad \dot{y} = Q_\mu(x, y) \tag{15}$$

be a continuous family of plane differential systems for $\mu \in [0, \mu^*)$. Assume that there exists a compact set H containing all the critical points of (15), *for $\mu \in [0, \mu^*)$. If the system* (15) *is* NUB *for $\mu = 0$, and* UB *for $\mu \in (0, \mu^*)$, then a family of annuli M_μ, having limit cycles as boundary, bifurcates from infinity as μ becomes positive.*

The bifurcating annuli M_μ are asymptotically stable invariant annuli. They may reduce to limit cycles, when the inner and outer components of their boundary coincide.

Theorem 5. *For every $k > 0$ there exists a polynomial Liénard system of degree $6k+3$ with k limit cycles surrounding a central region.*

Proof. By [**3**, Theorem 6], (3) has a nondegenerate center at the origin if and only if $F(x)$ and $G(x)$ are polynomials of a polynomial $A(x)$, with $A'(0) = 0$, $A''(0) \neq 0$.

Let us choose $A(x) = x^2 + x^3$, $G(x) = A(x) + A(x)^2$. The critical points of all the systems that we consider in this proof are the points $(x_0, 0)$, where $g(x_0) = 0$. In this proof $G(x)$ is fixed, so that the bifurcating sets do not contain critical points, hence they are annuli, having a couple of limit cycles as boundary.

Let us set $F_0(x) = A(x)$, $f_0(x) = F_0'(x)$. Graef conditions hold for the system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf_0(x), \tag{16}$$

hence it is UB. Now let us set $F_1(x) = A(x) - \mu_1 A(x)^3$, $f_1(x) = F_1'(x)$, and consider the perturbed systems

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf_1(x). \tag{17}$$

For $\mu_1 > 0$, the system (17) is NUB, so that at least a limit cycle γ_1 bifurcates from infinity as μ_1 becomes positive. Since the integrability condition holds for every μ_1, the origin is a center, hence γ_1 surrounds its central region N_O. The system (17) has degree 9.

If $k = 1$, the procedure stops here. Otherwise we consider a new perturbation. Let us consider the system we obtain by setting $F_2(x) = A(x) - \mu_1 A(x)^3 + \mu_2 A(x)^5$, $f_2(x) = F_2'(x)$,

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf_2(x), \tag{18}$$

Such a system, by Graef condition, is UB. For small positive values of μ_2, a limit cycle γ_2 bifurcates from infinity. Also, for small positive values of μ_2, there exists a limit cycle γ_1^* of (18), close to γ_1. Again, the integrability condition holds, hence the origin is a center, with central region surrounded by γ_1^* and γ_2. The system (17) has degree 15.

Such a procedure can be repeated an arbitrary number of times, generating systems with a center at the origin, surrounded by k large amplitude limit cycles. Since at every step the system's degree increases by 6, the resulting system has degree $6k+3$. □

The above proof can be modified along the lines of the last theorem in [**11**], in order to show that there exists an analytic Liénard system with a center surrounded by infinitely many limit cycles. For a couple of vectors $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ let us set $z_1 \wedge z_2 = x_1 y_2 - x_2 y_1$. The next lemma was proved in [**11**]. We denote by $\delta'(t)$ the vector tangent to the curve $\delta(t)$.

Lemma 4. *Let M be an asymptotically stable annulus of the differential system*

$$\dot{z} = v(z),$$

$z = (x, y) \in R^2$, $v \in C^1(\mathbf{R}^2, \mathbf{R}^2)$, *having nontrivial cycles γ^i, γ^e as inner and outer components of its boundary. Then there exist C^1 curves δ^i enclosed by γ^i, δ^e enclosing γ^e, such that ${\delta^i}' \wedge v \neq 0$ on δ^i, ${\delta^e}' \wedge v \neq 0$ on δ^e.*

Theorem 6. *There exists an analytic Liénard system with a center enclosed by infinitely many limit cycles.*

We start as in the proof of Theorem 5, choosing $F_0(x) = A(x) = x^2 + x^3$, $G(x) = A(x) + A(x)^2$,

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y f_0(x).$$

As in Theorem 5, in this proof $G(x)$ does not change, so that the critical points of all the systems that we consider do not change as we consider new perturbations. This ensures that the bifurcating sets do not contain critical points. We apply the first perturbation, choosing $F_3(x) = A(x) - \lambda_3 A(x)^3$, $f_3(x) = F_3'(x)$ (we have changed indices' numbering, with respect to Theorem 5),

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y f_3(x), \qquad \lambda_3 > 0. \tag{19}$$

we produce a family of asymptotically stable invariant annuli M_3 bifurcating from infinity. Let us denote by v_3 the vector field associated to the system (19). By the above lemma, there exist C^1 curves δ_3^i, δ_3^e

such that $\delta_3^{e\prime} \wedge v_3 \neq 0$, $\delta_3^{i\prime} \wedge v_3 \neq 0$, respectively, on δ_3^e, δ_3^i. Due to the form of Liénard system, since M_3 is asymptotically stable, we have $\delta_3^{e\prime} \wedge v_3 < 0$ on δ_3^e, $\delta_3^{i\prime} \wedge v_3 > 0$, on δ_3^i. By the compactness of δ_3^e, δ_3^i, there exists $\varepsilon_3 > 0$ such that $\delta_3^{e\prime} \wedge v_3 < -\varepsilon_3 < 0$ on δ_3^e, $\delta_3^{i\prime} \wedge v_3 > \varepsilon_3 > 0$, on δ_3^i. Let us denote by N_3 the annulus having δ_3^i as inner boundary, and δ_3^e as outer boundary.

Now let us apply a second perturbation, choosing $F_5(x) = A(x) - \lambda_3 A(x)^3 + \lambda_5 A(x)^5$, $f_5(x) = F_5'(x)$,

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y f_5(x), \qquad \lambda_5 > 0. \tag{20}$$

Let us denote by v_5 the corresponding vector field. A family of negatively asymptotically stable invariant annuli M_5, bifurcates from infinity as λ_5 becomes positive. Let us choose λ_5 small enough to have $\delta_3^{e\prime} \wedge v_5 < -\varepsilon_3$, $\delta_3^{i\prime} \wedge v_5 > \varepsilon_3$, respectively, on δ_3^e, δ_3^i. By the previous lemma and the negative asymptotic stability of M_5, there exist also C^1 curves δ_5^e, δ_5^i and $\varepsilon_5 > 0$, such that $\delta_5^{e\prime} \wedge v_5 > \varepsilon_5 > 0$ on δ_5^e, $\delta_5^{i\prime} \wedge v_5 < -\varepsilon_5 < 0$ on δ_5^i.

By adding perturbations of higher and higher degree, we construct a sequence of (negatively) asymptotically stable invariant annuli M_{2k+1}, with C^1 curves δ_{2k+1}^e, δ_{2k+1}^i, defining annuli N_{2k+1} such that

(i) $N_{2k+1} \cap N_{2h+1} = \varnothing$, for $k \neq h$; N_{2k+1} positively (negatively) invariant with respect to v_{2k+1}, if M_{2k+1} is (negatively) asymptotically stable;

(ii) for $h \geq k$: $|\delta_{2k+1}^{e\prime} \wedge v_{2h+1}| > \varepsilon_{2k+1}$, $|\delta_{2k+1}^{i\prime} \wedge v_{2h+1}| > \varepsilon_{2k+1}$, respectively, on δ_{2k+1}^e, δ_{2k+1}^i.

Moreover, we can choose the parameters λ_{2k+3} small enough to satisfy

(iii) $|\lambda_{2k+3}|/|\lambda_{2k+1}| < 1/(2k+3)$.

The power series

$$\sum_{k=1}^{\infty} (-1)^k \lambda_{2k+1} x^{2k+1}$$

has radius of convergence ∞ because of condition (iii). Let us set $\Psi(x) = x + \sum_{k=1}^{\infty} (-1)^k \lambda_{2k+1} x^{2k+1}$. $\Psi(x)$ is an analytic function defined on all of $\mathbf{R}$. Let us set $F(x) = \Psi(A(x))$, $f(x) = F'(x)$. We claim that the system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y f(x) \tag{21}$$

has infinitely many limit cycles. Let us denote by v_∞ the corresponding vector field.

Assume M_{2k+1}, for some $k > 0$, to be asymptotically stable for v_{2k+1}, hence N_{2k+1} positively invariant with respect to v_{2k+1}. For $h \geq k$, we have $\delta^{e\prime}_{2k+1} \wedge v_{2h+1} < -\varepsilon_{2k+1}$ on δ^{e}_{2k+1}, so that

$$\begin{aligned}\delta^{e\prime}_{2k+1} \wedge v_\infty &= \delta^{e\prime}_{2k+1} \wedge \left(\lim_{h\to\infty} v_{2h+1}(z)\right)\\ &= \lim_{h\to\infty} \delta^{e\prime}_{2k+1} \wedge v_{2h+1}(z)\\ &\leq -\varepsilon_{2k+1} < 0.\end{aligned}$$

We can work similarly on δ^{i}_{2k+1}, proving that $\delta^{i\prime}_{2k+1} \wedge v_\infty \geq \varepsilon_{2k+1} > 0$. This proves that N_{2k+1} is positively invariant for v_∞. Since the critical points of (21) are in a fixed compact set, there exist infinitely many sets N_{2k+1} not containing critical points of (21). Hence the ω-limit set of a point on the boundary of such ∂N_{2k+1} is a limit cycle γ_{2k+1}.

In order to complete the proof we only have to show that the origin is a center of (21). We have $A(x) = -1 + \sqrt{(1+2x^2+2x^3)^2}/2 = -1 + \sqrt{1+4G(x)}/2$, hence $A(x)$ is an analytic function of $G(x)$ in a neighborhood of 0. As a consequence, $F(x) = \Psi(A(x))$ is an analytic function of $G(x)$. Since $g'(x) > 0$, by Theorem 1 in [**3**] the origin is a center of (21). □

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