ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 1, 2005

SPACES $L_2(\lambda)$ OF A POSITIVE VECTOR MEASURE λ AND GENERALIZED FOURIER COEFFICIENTS

S. OLTRA, E.A. SÁNCHEZ PÉREZ AND O. VALERO

ABSTRACT. Let L be a Banach lattice and consider a countably additive vector measure λ with values on L. Let $L_2(\lambda)$ be the Banach lattice of square integrable functions with respect to λ. In this paper we obtain several structure results for this space and the Fourier coefficients related to orthonormal sequences under the assumption that λ is positive. However, $L_2(\lambda)$ is not in general isomorphic to a Hilbert space. In fact, the norm of this space depends on the norm of L.

1. Introduction and basic results. Let L be a Banach lattice and let (Ω, Σ) be a measurable space. Let $\lambda : \Sigma \to L$ be a countably additive vector measure. If f is a measurable function, it is said that it is scalarly integrable if it is integrable with respect to each scalar measure like $\lambda_{x'}$ for every $x' \in L'$, where $\lambda_{x'}(A) := \langle \lambda(A), x' \rangle$, $A \in \Sigma$. If f is scalarly integrable, it is said that it is integrable with respect to λ (λ-integrable for short) if for every $A \in \Sigma$ there is a vector $\int_A f\, d\lambda \in L$ such that $\langle \int_A f\, d\lambda, x' \rangle = \int_A f\, d\lambda_{x'}$. The definition of integrability of scalar functions with respect to a vector measure was first given by Bartle, Dunford and Schwartz [**1**] and studied by Lewis [**11**] and [**12**].

The Banach lattice $L_1(\lambda)$, see for example [**4**], is defined by the equivalence classes of λ-integrable functions such that the set where they differ has zero semi-variation, with the natural order and the norm

$$\|f\|_\lambda = \sup \Big\{ \int_\Omega |f|\, d|\langle \lambda, x' \rangle| : x' \in B_{X'} \Big\}, \quad f \in L_1(\lambda),$$

where $|\langle \lambda, x' \rangle|$ denotes the variation of the scalar measure $\lambda_{x'}$. The following expression gives an equivalent norm,

$$|||f|||_\lambda = \sup_{A \in \Sigma} \| \int_A f\, d\lambda \|, \quad f \in L_1(\lambda),$$

2000 AMS *Mathematics Subject Classification.* Primary 46E30, 46G10, 42A65.
Key words and phrases. Vector measures, orthogonal functions, Hilbert spaces.
Received by the editors on December 13, 2001, and in revised form on July 31, 2002.

that satisfies the inequality $|||f|||_\lambda \leq \|f\|_\lambda \leq 2|||f|||_\lambda$ for every $f \in L_1(\lambda)$.

In this paper we obtain several properties of the space $L_2(\lambda)$ of square integrable functions with respect to λ, i.e., classes of measurable functions f such that f^2 are λ-integrable. It is a particular case of an $L_p(\lambda)$ space. These spaces can be defined for any countably additive vector measure λ and each $1 \leq p < \infty$ and have been studied in [**16**]. They can be considered as Banach function spaces over a Rybakov measure for λ.

The geometric structure of the space $L_2(\lambda)$ is up to a point similar to the Hilbert space, but the definition of the norm on $L_2(\lambda)$ depends on the norm of L. Under the assumption that the linear span of the range of λ contains an unconditional basis of L, we prove that it is possible to obtain a basis of functions for $L_2(\lambda)$ that are orthonormal with respect to the vector measure λ, in a sense that we will clarify through the paper. A Fourier calculus is also obtained for this function space in a natural way. In this context, we will define what we call the generalized Fourier coefficients of a function, that provide the projection of a function that minimizes a distance defined by the norm of $L_2(\lambda)$.

In all the paper, (Ω, Σ) will denote a measurable space. If L is a Banach lattice, L^+ will denote its positive cone. The reader can find information about measure spaces, scalar measures and integration with respect to scalar measures in [**8**] or in [**6**] and [**7**]. General vector measure theoretical questions with no explicit explanation can be answered with the help of [**5**]. Our main reference about Banach lattices and Banach (Köthe) function spaces are [**14**] and [**10**]. The results about the unconditional basis of Banach spaces that we use can be found in [**13**]. We refer to [**17**] for general questions about Banach space theory. More information about vector measure integration and related questions can be found in [**2–4, 9, 11, 12**] and [**15**]. If V is a linear space we will write $[v]$ to denote the linear space generated by the single vector $v \in V$.

The following definition gives a particular example of functions that are p-integrable with respect to a vector measure. In this paper we restrict our attention in order to obtain particular geometric results for the case of $L_2(\lambda)$ when λ is a positive measure.

Definition 1. Let X be a (real) Banach space and $\lambda : \Sigma \to X$ a countably additive vector measure. We say that a measurable function f is square integrable with respect to λ (square λ-integrable for short) if f^2 is a λ-integrable function.

In the rest of the paper, if λ is a countably additive vector measure, we will write μ for a finite measure that controls λ such that $\mu(A) \leq \|\lambda\|(A)$ for every $A \in \Sigma$. Such a measure always exists. For instance, we can define it as a Rybakov measure for λ, see [**5**].

We will use the following results throughout the paper. They are satisfied for countably additive vector measures and their proofs can be found or follow directly from the results of [**16**]. The proof of Lemma 2 can be found in Remark 3 of [**16**].

Lemma 2. *Every square λ-integrable function is λ-integrable.*

The proof of Proposition 3 follows directly from the inequality $2|fg| \leq f^2 + g^2$ that holds for measurable functions f and g and the fact that $L_1(\lambda)$ is an ideal of measurable functions.

Proposition 3. *Let X be a* (*real*) *Banach space and $\lambda : \Sigma \to X$ a countably additive vector measure. If f, g are square λ-integrable, then the pointwise product function fg is λ-integrable. Consequently, the set of all the square λ-integrable functions defines a linear space.*

We will denote the linear space of square λ-integrable functions by $L_2(\lambda)$. In fact, $L_2(\lambda)$ endowed with the norm given by the expression

$$\sup_{A \in \Sigma} \left\| \int_A f^2 \, d\lambda \right\|^{1/2}, \quad f \in L_2(\lambda),$$

is a Banach lattice, see [**16**].

Definition 4. Let L be a Banach lattice. We will say that a countably additive vector measure $\lambda : \Sigma \to L$ that satisfies that $\lambda(A) \in L^+$ for every $A \in \Sigma$ is a positive vector measure.

From now on we restrict our attention to the case of positive vector measures, since in this case we can obtain a simple representation of the natural norm of the space $L_2(\lambda)$. Note that a vector measure is positive if and only if every positive function $f \in L_1(\lambda)$ has a positive integral $\int_\Omega f\, d\lambda \in L^+$.

Definition 5. If λ is a positive vector measure, we denote by $\|.\|_{\lambda,2}$ to the nonnegative function $\|.\|_{\lambda,2} : L_2(\lambda) \to R$ defined by the formula

$$\|f\|_{\lambda,2} := \Big\| \int_\Omega f^2 \, d\lambda \Big\|^{1/2}, \quad f \in L_2(\lambda).$$

It is clear that $\|f\|_{\lambda,2} = \sup_{A\in\Sigma} \| \int_A f^2 \, d\lambda \|^{1/2}$ for every $f \in L_2(\lambda)$ if λ is a positive measure. Thus, Lemma 6 and Proposition 7 are direct consequences of the results of [**16**].

Lemma 6. *Let λ be a positive vector measure. The function $\|.\|_{\lambda,2}$ is a norm on $L_2(\lambda)$.*

Proposition 7. *Let λ be a positive vector measure. The space $(L_2(\lambda), \|.\|_{\lambda,2})$ is a Banach function space over μ, where μ is a positive finite control measure for λ that satisfies $\mu(A) \le \|\lambda\|(A)$ for every $A \in \Sigma$. Moreover, the set of simple functions is dense in $L_2(\lambda)$.*

In the rest of the paper we will consider the set $R^+_{L'} = \{x' \in (L')^+ : x'$ defines a Rybakov measure$\}$. It is easy to check that $R^+_{L'}$ is dense in $(L')^+$ using the fact that the set of Rybakov measures is dense in X', see [**5**]. Since λ is positive we directly obtain the equality

$$\sup \Big\{ \Big(\int_\Omega h^2 \, d\langle \lambda, x'\rangle \Big)^{1/2} : x' \in R^+_{L'} \cap B_{L'} \Big\} = \|h\|_{\lambda,2}, \quad f \in L_2(\lambda).$$

Note that each $x' \in R^+_{L'} \cap B_{L'}$ defines a norm of a Hilbert space of classes of μ-a.e. equal functions $L_2(\langle \lambda, x'\rangle)$, since the μ-null sets are the same as the $\langle \lambda, x'\rangle$-null sets.

We finish this section with an example of the structure with which we are dealing. We will dedicate Section 3 to the study of particular cases as the following one.

Example 8. Let $1 < p < \infty$ and consider the sequence space $(l_p, \|.\|_p)$ with its natural coordinatewise order. Let us denote by $(e_i)_{i=1}^{\infty}$ to the canonical basis of l_p. Consider a measurable space $([0,\infty), \Sigma_0)$, where Σ_0 is a σ-algebra, and a set of probability measures μ_i, $i \in N$, that are zero on measurable subsets which do not intersect the interval $[i-1, i]$, for every $i \in N$. Then, a direct calculation shows that the vector measure

$$\lambda_0(A) := \sum_{i=1}^{\infty} \frac{\mu_i(A \cap [i-1, i])}{2^{i/p}} e_i, \quad A \in \Sigma,$$

is countably additive. In fact, it is positive. Thus we can define the space $(L_2(\lambda_0), \|.\|_{\lambda_0,2})$. In this case the norm of the space is given by

$$\|f\|_{\lambda_0,2} = \Big\| \int_{[0,\infty)} f^2 \, d\lambda_0 \Big\|^{1/2} = \bigg(\sum_{i=1}^{\infty} \frac{|\int_{[i-1,i]} f^2 \, d\mu_i|^p}{2^i} \bigg)^{1/2p}, \quad f \in L_2(\lambda_0).$$

Thus, it is clear that $(L_2(\lambda_0), \|.\|_{\lambda_0,2})$ is not isomorphic to a Hilbert space. However, note that for each $x' \in l_{p'}^{+}$ that defines a Rybakov measure the Hilbert space $L_2(\langle \lambda_0, x' \rangle)$ is well-defined.

2. Projections and generalized Fourier coefficients for functions in $L_2(\lambda)$. Although the spaces of square λ-integrable functions are not in general isomorphic to Hilbert spaces, they have several properties that are similar to the ones that follow for Hilbert spaces. In particular, it is possible to establish an analogue of the classical projection procedure *via* the Fourier coefficients related to an orthonormal basis. In fact, we can obtain an approximation technique for the spaces $L_2(\lambda)$ that follows the lines of the classical functional analysis related to Hilbert spaces.

From now on we will deal with positive vector measures. This assumption is necessary if we want the following definition to make sense. Moreover, the use of the simpler expression $\|.\|_{\lambda,2}$ for the norm of $L_2(\lambda)$ is only possible if λ is positive, see Lemma 6. Let $\lambda : \Sigma \to L$ be a positive vector measure defined on the Banach lattice L.

Definition 9. Let $(f_i)_{i=1}^{\infty}$ be a sequence of square λ-integrable functions. We say that it is a λ-orthonormal sequence if

1) $\|f_i\|_{\lambda,2} = 1$ for every $i \in N$.

2) $\int_\Omega f_i f_j \, d\lambda = 0$ if $i \neq j$.

We use this kind of sequence in order to obtain the approximation results for certain functions in $L_2(\lambda)$. Let $g \in L_2(\lambda)$ and consider a λ-orthonormal sequence $(f_i)_{i=1}^\infty$. An element $x' \in (L')^+$ that defines a Rybakov measure also defines a Hilbert space $L_2(\langle \lambda, x' \rangle)$. Moreover, since for each $x' \in (L')^+ \cap B_{L'}$

$$\Big| \int_\Omega f_i f_j \, d\langle \lambda, x' \rangle \Big| \leq \Big\| \int_\Omega f_i f_j \, d\lambda \Big\| \quad i, j \in N,$$

and $\int_\Omega f_i^2 \, d\langle \lambda, x' \rangle \neq 0$ for every $i \in N$, we obtain that the sequence $(f_i)_{i=1}^\infty$ is orthogonal in $L_2(\langle \lambda, x' \rangle)$. This motivates the following definition.

Definition 10. Let $g \in L_2(\lambda)$ and let $(f_i)_{i=1}^\infty$ be a λ-orthonormal sequence. Consider an element $x' \in (L')^+$ that defines a Rybakov measure, i.e., $x' \in R_{L'}^+$, following the notation of Section 1. We denote by $\alpha_i(x')$ the Fourier coefficient of g in $L_2(\langle \lambda, x' \rangle)$,

$$\alpha_i(x') := \frac{\int_\Omega g f_i \, d\langle \lambda, x' \rangle}{\int_\Omega f_i^2 \, d\langle \lambda, x' \rangle}.$$

Thus, we can consider $\alpha_i(.)$ as functions from $R_{L'}^+$ into R. We will say that the function $\alpha_i(.)$, $i \in N$, is a generalized Fourier coefficient of $g \in L_2(\lambda)$ with respect to the sequence $(f_i)_{i=1}^\infty$.

Definition 11. We will say that a function g is projectable with respect to the λ-orthonormal sequence $(f_i)_{i=1}^\infty$ if

$$\int_\Omega g f_i \, d\lambda \in \left[\int_\Omega f_i^2 \, d\lambda \right] \quad \text{for all} \quad i \in N.$$

Note that a projectable function g satisfies that for every finite subsequence $(f_{i_k})_{k=1}^m$ of the λ-orthonormal sequence $(f_i)_{i=1}^\infty$ there are scalars β_k such that

$$\int_\Omega f_{i_k}\Big(g - \sum_{k=1}^{m} \beta_k f_{i_k}\Big)\, d\lambda = 0, \quad k = 1, \dots, m.$$

In the case that λ is a positive *scalar* measure, this is exactly the orthogonality condition that is satisfied for the Fourier coefficients and defines the best approximation to the function g in the finite dimensional subspace span$\{f_{i_k} : k = 1, \dots, m\}$ with respect to the norm of the corresponding Hilbert space. In this section we show that this also holds for positive vector measures.

Proposition 12. *Let $g \in L_2(\lambda)$ and consider a λ-orthonormal sequence $(f_i)_{i=1}^\infty$. Then g is projectable with respect to $(f_i)_{i=1}^\infty$ if and only if $\alpha_i(.)$ is a constant function in $R^+_{L'}$, for every $i \in N$.*

Proof. Suppose that g is projectable and take an element $x' \in R^+_{L'}$. Then, if $i \in N$, there is a real number β_i such that $\int_\Omega g f_i \, d\lambda = \beta_i \int_\Omega f_i^2 \, d\lambda$. Thus,

$$\alpha_i(x') = \frac{\int_\Omega g f_i \, d\langle \lambda, x'\rangle}{\int_\Omega f_i^2 \, d\langle \lambda, x'\rangle} = \frac{\langle \int_\Omega g f_i \, d\lambda, x'\rangle}{\langle \int_\Omega f_i^2 \, d\lambda, x'\rangle} = \beta_i.$$

Since β_i does not depend on the element $x' \in R^+_{L'}$, we obtain that $\alpha_i(.)$ is a constant function.

Conversely, suppose that $\alpha_i(.)$ is a constant function. Then, we have that there is a constant β_i such that

$$\Big\langle \int_\Omega g f_i \, d\lambda - \beta_i \int_\Omega f_i^2 \, d\lambda, x' \Big\rangle = 0 \quad \text{for all} \quad x' \in R^+_{L'}.$$

Since span$\{x' : x' \in R^+_{L'}\}$ is dense in L', we obtain the result

$$\Big\| \int_\Omega g f_i \, d\lambda - \beta_i \int_\Omega f_i^2 \, d\lambda \Big\| = 0. \qquad \square$$

Theorem 13. *Let L be a weakly sequentially complete Banach lattice. Let $\lambda : \Sigma \to L$ be a positive vector measure. If $g \in L_2(\lambda)$*

is projectable with respect to the λ*-orthonormal sequence* $(f_i)_{i=1}^{\infty}$*, and* α_i*,* $i \in N$*, are its (constant) Fourier coefficients, then*

1) *The sequence of the partial sums* $g_n := \sum_{i=1}^{n} \alpha_i f_i$*, converges in* $L_2(\lambda)$ *to a function that we denote by* $\sum_{i=1}^{\infty} \alpha_i f_i$*.*

2) *The function* $\sum_{i=1}^{\infty} \alpha_i f_i$ *is the unique function of the set*

$$S = \Big\{ h \in L_2(\lambda) : \lim_n \Big\| h - \sum_{i=1}^{n} \beta_i f_i \Big\|_{\lambda,2} = 0,\ (\beta_i)_{i=1}^{\infty} \in R^N \Big\}$$

that satisfies that

$$\inf_{h \in S} \|g - h\|_{\lambda,2} = \Big\| g - \sum_{i=1}^{\infty} \alpha_i f_i \Big\|_{\lambda,2}.$$

Proof. 1) Let $g \in L_2(\lambda)$ be a projectable function. We just need to show that the sequence $(g_n)_{n=1}^{\infty}$ is Cauchy. Since L is weakly sequentially complete the Banach lattice $L_1(\lambda)$ is so by Theorem 3 of [**4**]. Consider a natural number n and a real number $\varepsilon > 0$. Then there is an element $x'_\varepsilon \in R^+_{L'} \cap B_{X'}$ such that

$$\Big\| \int_\Omega g_n^2 \, d\lambda \Big\| \le \int_\Omega g_n^2 \, d\langle \lambda, x'_\varepsilon \rangle + \varepsilon,$$

since λ is positive. Moreover, since $(f_i)_{i=1}^{\infty}$ is λ-orthonormal and $L_2(\langle \lambda, x'_\varepsilon \rangle)$ is a Hilbert space, we obtain

$$\begin{aligned}
\Big\| \int_\Omega g_n^2 \, d\lambda \Big\| &= \int_\Omega \sum_{i=1}^{n} \alpha_i^2 f_i^2 \, d\langle \lambda, x'_\varepsilon \rangle + \varepsilon \\
&= \sum_{i=1}^{n} \frac{(\int_\Omega g f_i \, d\langle \lambda, x'_\varepsilon \rangle)^2}{\int_\Omega f_i^2 \, d\langle \lambda, x'_\varepsilon \rangle} + \varepsilon \\
&\le \int_\Omega g^2 \, d\langle \lambda, x'_\varepsilon \rangle + \varepsilon \\
&\le \Big\| \int_\Omega g^2 \, d\lambda \Big\| + \varepsilon < \infty.
\end{aligned}$$

Since these inequalities hold for every natural number n, we obtain that the increasing sequence $(\sum_{i=1}^{n} \alpha_i^2 f_i^2)_{n=1}^{\infty}$ of λ-integrable functions is norm-bounded in $L_1(\lambda)$. But this Banach lattice is weakly sequentially complete, and then the sequence has a strong limit by Theorem 1.c.4 of [**14**].

Now note that, for each pair of natural numbers $m > n$, $g_m - g_n = \sum_{i=n+1}^{m} \alpha_i f_i$, and

$$\int_\Omega (g_m - g_n)^2 \, d\lambda = \int_\Omega \sum_{i=n+1}^{m} \alpha_i^2 f_i^2 \, d\lambda.$$

Since $(\sum_{i=1}^{n} \alpha_i^2 f_i^2)_{n=1}^{\infty}$ converges in $L_1(\lambda)$, it is clear that for each $\varepsilon > 0$ there is a natural number n_0 such that for every $n, m \geq n_0$, $m > n$,

$$\Big\| \int_\Omega \sum_{i=n+1}^{m} \alpha_i^2 f_i^2 \, d\lambda \Big\| < \varepsilon.$$

This means that $(g_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L_2(\lambda)$.

To prove 2), suppose that there is another sequence $(\beta_i)_{i=1}^{\infty} \in R^N$ such that $\sum_{i=1}^{\infty} \beta_i f_i \in L_2(\lambda)$ and

$$\Big\| g - \sum_{i=1}^{\infty} \beta_i f_i \Big\|_{\lambda,2} < \Big\| g - \sum_{i=1}^{\infty} \alpha_i f_i \Big\|_{\lambda,2}.$$

Then there is a natural number n such that

$$\Big\| g - \sum_{i=1}^{n} \beta_i f_i \Big\|_{\lambda,2} < \Big\| g - \sum_{i=1}^{n} \alpha_i f_i \Big\|_{\lambda,2}.$$

But $\sum_{i=1}^{n} \alpha_i f_i$ defines the best approximation to the subspace generated by $(f_i)_{i=1}^{n}$ in each Hilbert space $L_2(\langle \lambda, x' \rangle)$, $x' \in R_{L'}^{+} \cap B_{X'}$, since $\alpha_i(x') = \alpha_i$ for every $i = 1, \ldots, n$. This contradicts the above inequality. A similar argument shows that $\sum_{i=1}^{\infty} \alpha_i f_i$ is the unique function that satisfies this property. □

Note that we only need weak sequential completeness to assure the convergence of the series defined by the λ-orthonormal sequence $(f_i)_{i=1}^{\infty}$ and the associated Fourier coefficients. Thus, we do not need this property for finite sequences $(f_i)_{i=1}^{n}$, $n \in N$.

3. Applications. Positive vector measures with values in Banach spaces with an unconditional basis. Let L be a reflexive Banach space with a (normalized) unconditional basis $(e_i)_{i=1}^{\infty}$ that has an unconditional constant equal to one. Let $(e'_i)_{i=1}^{\infty}$ be the corresponding biorthogonal functionals. It is well-known that L is a Banach lattice when we consider the order defined by the positive cone

$$L^+ := \Big\{ \sum_{i=1}^{\infty} \eta_i e_i : \eta_i \geq 0 \text{ for every } i \in N \Big\},$$

see Chapter 1.a in [**14**]. In particular, the basis constant of the sequence $(e_i)_{i=1}^{\infty}$ is also one, and then the basis is monotone. Since the space L is reflexive, we obtain that the basis $(e_i)_{i=1}^{\infty}$ is shrinking, and then the biorthogonal functionals $(e'_i)_{i=1}^{\infty}$ form a basis of L' (Theorems 1.c.9 and 1.c.13 of [**13**]). Moreover, the basis is boundedly complete, and then L is weakly sequentially complete, and we can apply the approximation results of Section 2, see Theorems 1.c.10 and 1.c.13 of [**13**]. Examples of such spaces are of course the spaces of sequences l_p, $1 < p < \infty$. For the aim of simplicity we will suppose that the biorthogonal functionals also have norm one.

In this section we deal with positive vector measures with values in such spaces L. We will show that in this case we can construct a λ-orthonormal sequence that is complete, in the sense that we can write each element of the space $L_2(\lambda)$ as a series defined by elements of the sequence. As a consequence, we directly obtain that each function $g \in L_2(\lambda)$ is projectable with respect to this λ-orthonormal sequence.

Definition 14. Consider a positive vector measure $\lambda : \Sigma \to L$. Let us denote by $Rg(L_2(\lambda))$ the set

$$Rg(L_2(\lambda)) := \Big\{ \int_{\Omega} f^2 \, d\lambda : f \in L_2(\lambda) \Big\}.$$

We will say that the vector measure λ is range complete if $(e_i)_{i=1}^{\infty} \subset Rg(L_2(\lambda))$. If the corresponding spaces $L_2(\langle \lambda, e'_i \rangle)$, $i \in N$, are separable we will say that λ is coordinatewise separable.

Lemma 15. *Let λ be a positive range complete vector measure. Consider the class of sets $B_i := \{g : \int_{\Omega} g^2 \, d\lambda \in [e_i]\}$, $i \in N$. Then*

1) *the set* B_i *is a closed subspace of* $L_2(\lambda)$.

2) *If* $f \in B_i$ *and* $g \in B_j$ *for* $i \neq j$, *then*

$$\int_\Omega fg\, d\langle \lambda, e'_k \rangle = 0$$

for every $k \in N$. *Thus,* $\int_\Omega fg\, d\lambda = 0$.

3) *If* λ *is a coordinatewise separable vector measure, then* B_i *is a complemented subspace of* $L_2(\lambda)$.

Proof. 1) Let us show first that B_i is a linear space. Let $f, g \in B_i$. Consider $j \in N$ such that $i \neq j$. Then, by the Hölder inequality related to the positive scalar measure $\langle \lambda, e'_j \rangle$, we obtain

$$\begin{aligned} \Big| \Big\langle \int_\Omega gf\, d\lambda, e'_j \Big\rangle \Big| &= \Big| \int_\Omega gf\, d\langle \lambda, e'_j \rangle | \\ &\leq \Big(\int_\Omega g^2\, d\langle \lambda, e'_j \rangle \Big)^{1/2} \Big(\int_\Omega f^2\, d\langle \lambda, e'_j \rangle \Big)^{1/2} = 0. \end{aligned}$$

Thus, $\int_\Omega gf\, d\lambda \in [e_i]$ and consequently, $\int_\Omega (f+g)^2\, d\lambda \in [e_i]$, and then B_i is a linear space.

Now let us show that it is closed. Consider a convergent sequence $(g_n)_{n=1}^\infty \subset B_i$ with limit g. Take $j \neq i$. Then we can consider $(g_n)_{n=1}^\infty$ as a sequence in the Hilbert space $L_2(\langle \lambda, e'_j \rangle)$. Then it is clear that

$$0 = \lim_n \int_\Omega g_n^2\, d\langle \lambda, e'_j \rangle = \int_\Omega g^2\, d\langle \lambda, e'_j \rangle.$$

Since this holds for every $j \neq i$, we obtain that $\int_\Omega g^2\, d\lambda \in [e_i]$.

The statement 2) has been implicitly proved above. For the proof of 3), consider $i \in N$. As a consequence of 2) it is easy to see that for every $g \in B_i$, $\|g\|_{\lambda,2} = (\int_\Omega g^2\, d\langle \lambda, e'_i \rangle)^{1/2}$. Now, let us define the inclusion $I_i : B_i \to L_2(\langle \lambda, e'_i \rangle)$, $I(g) = g$. The operator I_i is an isometry, and then we can identify the subspace B_i with a closed subspace of the separable Hilbert space $L_2(\langle \lambda, e'_i \rangle)$. Let $(f_{i,j})_{j=1}^\infty$ be an orthonormal basis of the (complemented) separable Hilbert subspace $I_i(B_i)$. We can consider these functions as elements of $L_2(\lambda)$, and then the projection $P_i : L_2(\lambda) \to L_2(\lambda)$, $P_i(g) := \sum_{j=1}^{\infty} \alpha_{i,j} f_{i,j}$, where

$$\alpha_{i,j} := \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle,$$

is well-defined, since $\langle \int_\Omega f_{i,j}^2 \, d\lambda, e_i' \rangle = 1$ for every $j \in N$. Let us show that P_i is continuous, and then the subspace B_i is complemented. Note that

$$\|g\|_{L_2(\langle \lambda, e_i' \rangle)} = \Big(\sum_{j=1}^{\infty} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle \Big)^{1/2}$$

for every $g \in L_2(\lambda)$. We can apply Parseval's equality for the space $L_2(\langle \lambda, e_i' \rangle)$ to prove the inequalities

$$\begin{aligned} \|P_i(g)\|_{\lambda,2} &= \Big(\int_\Omega \Big(\sum_{j=1}^{\infty} \alpha_{i,j} f_{i,j} \Big)^2 d\langle \lambda, e_i' \rangle \Big)^{1/2} \\ &= \Big(\sum_{j=1}^{\infty} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle^2 \Big)^{1/2} \le \|g\|_{\lambda,2}. \end{aligned}$$

Therefore, the space B_i is complemented. □

Theorem 16. *Let λ be a positive range complete and coordinatewise separable vector measure. The sequence $(f_{i,j})_{i,j=1}^{\infty}$ defines a λ-orthonormal basis for the space $L_2(\lambda)$. In particular, each function $g \in L_2(\lambda)$ can be written as a series*

$$\sum_{i=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{i,j} f_{i,j}.$$

where $\alpha_{i,j}$ are the generalized Fourier coefficients of the function g defined in Section 2.

Proof. It is clear by the construction of the proof of Lemma 15 that the functions $f_{i,j}$, $i, j \in N$, define a λ-orthonormal sequence in $L_2(\lambda)$. Then we just need to show that the limit

$$\lim_n \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} f_{i,j}$$

with respect to the norm $\|.\|_{\lambda,2}$ exists and is the function g. An explicit calculation of the norm of the difference between a partial sum of the series above and g gives

$$\begin{aligned}
&\Big\| \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j} f_{i,j} - g \Big\|_{\lambda,2}^2 \\
&\qquad = \Big\| \int_\Omega \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j} f_{i,j} - g \Big)^2 \, d\lambda \Big\| \\
&\qquad = \Big\| \sum_{i=1}^{n}\sum_{j=1}^{n} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle^2 \int_\Omega f_{i,j}^2 \, d\lambda + \int_\Omega g^2 \, d\lambda \\
&\qquad\quad - 2 \sum_{i=1}^{n}\sum_{j=1}^{n} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle \int_\Omega g f_{i,j}\, d\lambda \Big\|.
\end{aligned}$$

Note that each element of the space L' can be written as a series $\sum_{i=1}^{\infty} \eta_i e_i'$ since the basis $(e_i)_{i=1}^{\infty}$ is shrinking, and then we can write the above expression as

$$\begin{aligned}
&\sup_{\|\sum_{i=1}^{\infty} \eta_i e_i'\| \le 1} \Big\{ \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle^2 \Big) \eta_i \\
&\qquad + \sum_{i=1}^{\infty} \Big\langle \int_\Omega g^2 \, d\lambda, e_i' \Big\rangle \eta_i - 2 \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle^2 \Big) \eta_i \Big\} \\
&= \sup_{\|\sum_{i=1}^{\infty} \eta_i e_i'\| \le 1} \Big\{ \sum_{i=1}^{n} \Big(\Big\langle \int_\Omega g^2 \, d\lambda, e_i' \Big\rangle - \sum_{j=1}^{n} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle^2 \Big) \eta_i \\
&\qquad + \sum_{i=n+1}^{\infty} \Big\langle \int_\Omega g^2 \, d\lambda, e_i' \Big\rangle \eta_i \Big\}.
\end{aligned}$$

Let us denote

$$\varepsilon_i(n) = \Big\langle \int_\Omega g^2 \, d\lambda, e_i' \Big\rangle - \sum_{j=1}^{n} \Big\langle \int_\Omega g f_{i,j}\, d\lambda, e_i' \Big\rangle^2 .$$

If $i \in N$, it is clear by the structure of the Hilbert space $L_2(\langle \lambda, e_i' \rangle)$ that $\lim_n \varepsilon_i(n) = 0$. Let $\delta > 0$, and take an m_0 such that

$$\sup_{\|\sum_{i=1}^{\infty} \eta_i e_i'\| \le 1} \Big\{ \Big\langle \int_\Omega g^2 \, d\lambda, \sum_{i=m_0+1}^{\infty} \eta_i e_i' \Big\rangle \Big\} \le \frac{\delta}{2}.$$

Since $\varepsilon_i(n)$ converges to 0, we can also find a natural number m_1 such that

$$\sup_{\|\sum_{i=1}^{\infty} \eta_i e'_i\| \leq 1} \Big\{ \sum_{i=0}^{m_0} \eta_i \varepsilon_i(m_1) \Big\} \leq \frac{\delta}{2}.$$

This implies the result. □

Finally we can apply the results of Section 2 to each $g \in L_2(\lambda)$, since every function in this space is obviously projectable with respect to the λ-orthonormal sequence constructed in Theorem 16. Therefore, we obtain the following approximation result as an application of Theorem 16 and Theorem 13 for any subsequence of $(f_{i,j})_{i,j=1}^{\infty}$, which always defines a λ-orthonormal sequence. Note that L is weakly sequentially complete, since it is reflexive.

Corollary 17. *Under the conditions of Theorem* 16*, if* $(f_{i_k,j_r})_{k=1,r=1}^{\infty}$ *is a subsequence of* $(f_{i,j})_{i,j=1}^{\infty}$*, the minimum of the error*

$$\Big\| g - \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \beta_{i_k,j_r} f_{i_k,j_r} \Big\|_{\lambda,2}$$

is only obtained for $\beta_{i_k,j_r} = \alpha_{i_k,j_r}$*,* $k = 1, \dots, \infty$*,* $r = 1, \dots, \infty$*. The same result holds for finite subsequences.*

Acknowledgments. The authors gratefully acknowledge the many helpful suggestions of the referee.

REFERENCES

1. R.G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.

2. G.P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), 317–330.

3. ———, *When L^1 of a vector measure is an AL-space*, Pacific J. Math. **162** (1994), 287–303.

4. ———, *Banach space properties of L^1 of a vector measure*, Proc. Amer. Math. Soc. **123** (1995), 3797–3806.

5. J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys Monographs, vol. 15, Amer. Math. Soc., Providence, RI, 1977.

6. D.H. Fremlin, *Measure theory*, Vol. 1, Torres Fremlin, Colchester, 2000.

7. ———, *Measure theory*, Vol. 2, Torres Fremlin, Colchester, 2001.

8. P.R. Halmos, *Measure theory*, Springer, New York, 1974.

9. I. Kluvánek and G. Knowles, *Vector measures and control systems*, North-Holland, Amsterdam, 1975.

10. H.E. Lacey, *The isometric theory of classical Banach spaces*, Springer, Berlin, 1974.

11. D.R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. **33** (1970), 157–165.

12. ———, *On integrability and summability in vector spaces*, Illinois J. Math. **16** (1972), 294–307.

13. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I, Springer, Berlin, 1977.

14. ———, *Classical Banach spaces* II, Springer, Berlin, 1979.

15. S. Okada and W.J. Ricker, *The range of the integration map of a vector measure*, Arch. Math. **64** (1995), 512–522.

16. E.A. Sánchez Pérez, *Compactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue-Bochner spaces*, Illinois J. Math. **45** (2001), 907–923.

17. P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge Univer. Press, Cambridge, 1991.

Escuela Politécnica Superior de Alcoy, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Pl. Ferrándiz Carbonell, 03801 Alcoy, Spain
E-mail address: soltra@mat.upv.es

E.T.S. Ingenieros de Caminos, Canales y Puertos, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Camino de Vera, 46071 Valencia, Spain
E-mail address: easancpe@mat.upv.es

Departamento de Matemáticas e Informática, Edificio Anselm Turmeda, Carretera de Valldemossa, km. 7.5, Universidad de las Islas Baleares, 07122 Palma de Mallorca, Baleares, Spain
E-mail address: o.valero@uib.es