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ON EXTENSIONS OF SIMPLE REAL GENUS ACTION

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ABSTRACT. May has proved recently [**7**] that if a finite simple group G is generated by two elements of order 2 and s, and acts faithfully on a bordered Klein surface X of least possible genus, then $[\mathrm{Aut}\,(X) : G]$ divides 4 and he asked if $[\mathrm{Aut}\,(X) : G] = 4$ can actually occur. The aim of this note is to give a positive answer to this question. First we give necessary and sufficient conditions for the action of G to be so extendible and then we show that $\mathrm{PSL}\,(2, p)$ satisfy these conditions for arbitrary prime p with $p \equiv \pm 1 \bmod 8$.

1. The *real genus* $\rho(G)$ of a finite group G is the minimum algebraic genus of any compact bordered Klein surface on which G acts faithfully as a group of automorphisms. A *real genus action* of G is an action of G on a bordered Klein surface of algebraic genus $g = \rho(G)$. These notions were introduced by May in [**6**]. In [**7**] May proved that if G is a simple finite group with the real genus action on X and G is generated by two elements of order 2 and s, then G is normal in the group $\mathrm{Aut}\,(X)$ of all automorphisms of X, $[\mathrm{Aut}\,(X) : G]$ divides 4 and finally $\mathrm{Aut}\,(X)$ embeds faithfully in $\mathrm{Aut}\,(G)$. In [**7**] May also posed several open problems. The one he considered the most interesting was whether the case $[\mathrm{Aut}\,(X) : G] = 4$ can actually occur. Here we shall give necessary and sufficient conditions for the action of G to be so extended and then we show that $\mathrm{PSL}\,(2, p)$ for $p \equiv \pm 1 \bmod 8$ satisfies these conditions.

2. We shall use the same approach, notations and terminology as in [**6**] and [**7**]. May remarked that in such exceptional cases $|G| = 3(\rho(G) - 1)$ and $\mathrm{Aut}\,(X)$ must be an M^*-group. So $G = \Delta/\Gamma$, where Γ is a bordered surface NEC group and Δ is an NEC-group with signature $(0; +; [3, 3]; \{(-)\})$, since, by [**2**], these are the only NEC groups with

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area $2\pi/3$ which admit bordered surface groups as normal subgroups. A group Δ has the presentation $\langle x_1, x_2, e, c \mid x_1^3, x_2^3, x_1x_2e, c^2, ece^{-1}c\rangle$ and for the canonical epimorphism $\theta : \Delta \to G$,

$$\theta(x_1) = a,\ \theta(x_2) = b,\ \theta(c) = 1,\ \theta(e) = (ab)^{-1}, \tag{1}$$

where a and b are two elements of order 3. Now $[\mathrm{Aut}\,(X) : G] = 4$ if and only if there is a group Λ with signature $(0; +; [-]; \{(2, 2, 2, 3)\})$ containing Δ and Γ as normal subgroups. Recall that the group Λ has the presentation $\langle c_0, c_1, c_2, c_3 \mid c_0^2, c_1^2, c_2^2, c_3^2, (c_0c_1)^2, (c_1c_2)^2, (c_2c_3)^2, (c_0c_3)^3\rangle$. Using Theorem 2.3.3 and Remark 2.3.6 of [**1**] one can show that c_1 or c_2 belongs to Δ. Furthermore, in the first case $x_1 = c_0c_3$, $x_2 = c_2c_3c_0c_2$, $e = (c_2c_0)^2$ and $c = c_1$ also belong to Δ; they obey all canonical relations of Δ and

$$\begin{aligned} &c_3x_1c_3 = x_1^{-1}, \quad c_3x_2c_3 = x_2^{-1}, \quad c_3cc_3 = x_1^{-1}cx_1,\\ &c_2x_1c_2 = x_2^{-1}, \quad c_2x_2c_2 = x_1^{-1}, \quad c_2cc_2 = c. \end{aligned} \tag{2}$$

So these elements generate a normal subgroup in Λ of index 4 and therefore they form a canonical set of generators for Δ. The second case leads us to the same actions. Here $x_1 = c_3c_0$, $x_2 = c_1c_0c_3c_1$, $e = (c_1c_3)^2$, $c = c_2$ and

$$\begin{aligned} &c_0x_1c_0 = x_1^{-1}, \quad c_0x_2c_0 = x_2^{-1},\\ &c_1x_1c_1 = x_2^{-1}, \quad c_1x_2c_1 = x_1^{-1}. \end{aligned} \tag{3}$$

Thus if Γ is normal in Λ then the maps $x_1 \mapsto x_1^{-1}$, $x_2 \mapsto x_2^{-1}$ and $x_1 \mapsto x_2^{-1}$, $x_2 \mapsto x_1^{-1}$ induce automorphisms of G. This gives the only if part of the following

Theorem. *Let G be a simple group generated by two elements of order* 2 *and s. Then G is a subgroup of index* 4 *in* $\mathrm{Aut}\,(X)$ *for some Klein surface X of genus $g = \rho(G)$ if and only if G admits two generators a and b of order* 3 *for which the maps $\varphi(a) = a^{-1}$, $\varphi(b) = b^{-1}$ and $\psi(a) = b^{-1}$, $\psi(b) = a^{-1}$ induce automorphisms of G.*

The above conditions are also sufficient. Indeed let Λ and Δ be a pair of NEC groups as above, where $c_1 \in \Delta$. Let a and b be a pair of generators for G which satisfy the assumption. We define Γ as the kernel of epimorphism given by (1). The automorphisms φ, ψ are induced by automorphisms $\widetilde{\varphi}, \widetilde{\psi}$ of Δ defined by

$$\widetilde{\varphi}(x_1) = x_1^{-1}, \quad \widetilde{\varphi}(x_2) = x_2^{-1},$$
$$\widetilde{\psi}(x_1) = x_2^{-1}, \quad \widetilde{\psi}(x_2) = x_1^{-1},$$

which preserve Γ. The images of c_2 and c_3 generate Λ/Δ. Furthermore if $w \in \Gamma$ then $c_3 w c_3 = \widetilde{\varphi}(w) \in \Gamma$ and $c_2 w c_2 = \widetilde{\psi}(w) \in \Gamma$. So Γ is normal in Λ.

3. Now by Theorem 2.16 of [**3**], see also [**8**], the group $\mathrm{PSL}\,(2,p)$, where p is arbitrary prime with $p \equiv \pm 1 \bmod 8$, can be generated by two elements x, y of order 3 with the same trace. On the other hand Macbeath showed [**4**, Theorem 3] that two generating pairs (A, B) and (A_1, B_1) of $\mathrm{PSL}\,(2,p)$ for which $\mathrm{tr}\,A = \mathrm{tr}\,A_1$, $\mathrm{tr}\,B = \mathrm{tr}\,B_1$ and $\mathrm{tr}\,AB = \mathrm{tr}\,A_1B_1$ are conjugate within the larger group $\mathrm{PSL}\,(2, \overline{\mathrm{F}}_p)$, i.e. $A_1 = XAX^{-1}$ and $B_1 = XBX^{-1}$ for some $X \in \mathrm{PSL}\,(2, \overline{\mathrm{F}}_p)$, where $\overline{\mathrm{F}}_p$ is the algebraic closure of F_p. So the above maps φ, ψ do indeed induce automorphisms of $\mathrm{PSL}(2,p)$. The second part of this paragraph was inspired by the proof of Theorem 3 in [**9**].

4. It is known [**5**] that every finite simple group except $U_3(3)$ can be generated by two elements, one of which is an involution. So, in particular, all results of May from [**7**] and the above theorem hold true for all simple groups but $U_3(3)$ without the above generation assumption. This solves another problem of May posed in [**7**].

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