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DEGENERATE ELLIPTIC SYSTEMS

S.K. AFYAN

ABSTRACT. We solve the Riemann-Hilbert boundary value problem for a linearly elliptic system of two second order differential equations in a simply connected domain in the plane, which is degenerate on the whole boundary of the domain and reduced to a simple (canonical) form, whose characteristic equation has simple roots (to within low order terms).

1. Introduction. Degenerate elliptic equations and systems have extensive applications in mechanics. Such systems play a significant role in the theory of small bending surfaces and the membrane theory of shells with variable curvature [**16**].

The degenerate equations also occur in the study of magnetohydrodynamic streams, when the velocity exceeds the velocity of sound, as well as in the study of the motion of water in an open channel, when the stream velocity is greater than the velocity of the spreading surface waves [**8, 14**].

Such elliptic systems are reasonably well understood. It is known that the Dirichlet problem is Fredholm for one elliptic equation (this means that the homogeneous problem and the corresponding conjugate problem have the same finite number of linearly independent solutions), but this is not true for elliptic *systems*. For example, the Dirichlet problem for Bitsadze's system

$$u_{xx} - u_{yy} - 2v_{yy} = 0, \quad 2u_{xy} + v_{xx} - v_{yy} = 0$$

is neither Fredholm nor Noetherian (the given problem is called Noetherian if the reciprocal conjugate homogeneous problem has a finite number of linearly independent solutions) [**7**].

The main question in this context is: For what kind of boundary conditions will the problem be Noetherian for the degenerate elliptic system?

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To decide which are the right boundary conditions, we shall reduce the system to a canonical form, to be defined later.

In this paper we consider the equation

$$L(u) = A(z)u_{xx}+B(z)u_{xy}+C(z)u_{yy}+a(z)u_x+b(z)u_y+c(z)u = 0, \tag{0.1}$$

where $A(z)$, $B(z)$, $C(z)$, $a(z)$, $b(z)$ and $c(z)$ are given 2×2-dimension matrices on a simply connected domain $\mathcal{D}$ in the $z = x+iy$ plane from class $C^1(\mathcal{D})\cap C_\alpha(\overline{\mathcal{D}})$ and $\overline{\mathcal{D}}$ is closing of $\mathcal{D}$, $0 < \alpha \leq 1$, and $u = (u_1, u_2)$ is an unknown column.

We suppose that $\det A(z) \neq 0$ in $\mathcal{D}$ and that the characteristic equation

$$\det \mathcal{N}(z,\lambda) \equiv \det[A(z)\lambda^2 + B(z)\lambda + C(z)] = 0 \tag{0.2}$$

has no real roots in $\mathcal{D}$ and has real roots at Γ, which is the boundary of $\mathcal{D}$. This means that equation (0.1) is elliptic in $\mathcal{D}$ and degenerates at Γ.

We assume that the equation (0.2) has only simple roots.

That the setting of any boundary conditions is essentially connected with the type and character of degeneration was pointed out by Keldish and Bitsadze.

It is, in particular, important whether the determinant of the matrix $\mathcal{N}(z,\lambda(z))$, where $\lambda(z)$ is the characteristic root, is zero or not. To simplify the computation of this determinant, we reduce the equation (0.1) to some canonical form

$$\frac{\partial}{\partial\zeta_1}\left(\overline{\frac{\partial v}{\partial\zeta_2}} - r(z)\frac{\partial v}{\partial\zeta_2}\right) + P(v) + Q(\bar{v}) = 0, \tag{0.3}$$

or

$$\frac{\partial}{\partial\zeta_1}\left(\frac{\partial v}{\partial\zeta_2} - r(z)\overline{\frac{\partial v}{\partial\zeta_2}}\right) + P(v) + Q(\bar{v}) = 0, \tag{0.4}$$

where $|r(z)| < 1$ in $\mathcal{D}$, $|r(z)| = 1$ on Γ, $P(v)$, $Q(\bar{v})$ are first order linear differential operators,

$$\frac{\partial}{\partial\zeta_1} = \frac{\partial}{\partial x} - \lambda_1(z)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial\zeta_2} = \frac{\partial}{\partial x} - \lambda_2(z)\frac{\partial}{\partial y},$$

and $\lambda_1(z) \neq \lambda_2(z)$ are continuous differentiable roots of equation (0.2).

For the sake of simplicity we consider the case in which $P(v)$, $Q(\bar{v}) = 0$.

The case $\lambda_1(z) = \lambda_2(z) = i$ and $P(v), Q(\bar{v}) \neq 0$ was considered in [**2, 3**]. In the general case we can obtain analogously a similar result.

Thus for the equation

$$\frac{\partial}{\partial \zeta_1}\left(\frac{\overline{\partial v}}{\partial \zeta_2} - r(z)\frac{\partial v}{\partial \zeta_2}\right) = 0 \tag{0.5}$$

we consider the following Riemann-Hilbert problem. To find a solution of equation (0.5) of class $C^1_\alpha(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ satisfying the boundary condition

$$\operatorname{Re}[\overline{\mu(t)}v] = f(t), \quad t \in \Gamma, \tag{0.6}$$

where $\mu(t)$ and $f(t)$ are given functions of class $C_\alpha(\Gamma)$.

1. Reducing of system 0.1 to a canonical form. Assume that the characteristic equation (0.3) has only simple roots.

Lemma 1. *If $\lambda_1 \in C^1(\mathcal{D})$, $\operatorname{Im}\lambda_1(z) > 0$, is a simple root of* (0.3) *and*

$$|\operatorname{Im} M_{11}(z)\overline{M_{21}}(z)| + |\operatorname{Im} M_{12}(z)\overline{M_{22}}(z)| \neq 0, \tag{1.1}$$

in $\mathcal{D}$ where $M_{jk}(z)$ are elements of the matrix $M(z) = \mathcal{N}(z, \lambda_1(z))$. Then the system (0.1) *is equivalent to the equation*

$$\begin{aligned}\left[\frac{\partial}{\partial x} - \lambda_1(z)\frac{\partial}{\partial}\right]\Big[d_{11}(z)\frac{\partial u_1}{\partial x} + d_{21}(z)\frac{\partial u_1}{\partial y} + d_{12}(z)\frac{\partial u_2}{\partial x}\\ + d_{22}(z)\frac{\partial u_2}{\partial y}\Big] + L_1(u_1) + L_2(u_2) = 0,\end{aligned} \tag{1.2}$$

where the functions $d_{jk}(z)$ are from the class $C^1(\mathcal{D})$ and L_1, L_2 are first order linear differential operators.

Proof. Since $\lambda_1(z)$ is a simple root of (0.3), we have

$$\operatorname{rank} M(z) = \operatorname{rank}\mathcal{N}(z, \lambda_1(z)) \leq 1.$$

Using (1.1) we obtain rank $M(z) = 1$. Hence the system of equations

$$\begin{aligned} \alpha_1(z)M_{11}(z) + \alpha_2(z)M_{21}(z) = 0, \\ \alpha_1(z)M_{12}(z) + \alpha_2(z)M_{22}(z) = 0 \end{aligned} \tag{1.3}$$

has a nontrivial, continuous differentiable solution $(\alpha_1(z), \alpha_2(z))$. For definiteness, we assume that $\alpha_1(z) \neq 0$. With

$$k_1(z) := \frac{M_{21}(z)}{\alpha_1(z)}, \qquad k_2(z) := \frac{M_{22}(z)}{\alpha_1(z)},$$

we have from (1.3)

$$\begin{aligned} M_{11}(z) &= -k_1(z)\alpha_2(z), \qquad & M_{21}(z) &= k_1(z)\alpha_1(z), \\ M_{12}(z) &= -k_2(z)\alpha_2(z), \qquad & M_{22}(z) &= k_2(z)\alpha_1(z). \end{aligned}$$

Using these expressions, inequality (1.1) can be written as

$$(|k_1(z)|^2 + |k_2(z)|^2) \operatorname{Im} \overline{\alpha_1}(z)\alpha_2(z) \neq 0.$$

Since $|k_1(z)|^2 + |k_2(z)|^2 \neq 0$, we have

$$\operatorname{Im} \overline{\alpha_1(z)}\alpha_2(z) \neq 0. \tag{1.4}$$

Using (1.4) it is easy to see that the equation (0.1) is equivalent to

$$\alpha_1(z)[L_{11}(u_1) + L_{12}(u_2)] + \alpha_2(z)[L_{21}(u_1) + L_{22}(u_2)] = 0, \tag{1.5}$$

where

$$L_{jk}(u_k) = A_{jk}u_{kxx} + B_{jk}u_{kxy} + C_{jk}u_{kyy} + a_{jk}u_{kx} + b_{jk}u_{ky} + C_{jk}u_k.$$

Substituting L_{jk} into (1.5), we obtain

(1.6)
$$\begin{aligned} A_1(z)\frac{\partial^2 u_1}{\partial x^2} + B_1(z)\frac{\partial^2 u_1}{\partial x \partial y} + C_1(z)\frac{\partial^2 u_1}{\partial y^2} + A_2(z)\frac{\partial^2 u_2}{\partial x^2} + B_2(z)\frac{\partial^2 u_2}{\partial x \partial y} \\ + C_2(z)\frac{\partial^2 u_2}{\partial y^2} + L_1(u_1) + L_2(u_2) = 0, \end{aligned}$$

where L_1, L_2 are differential operators of first order and

$$A_j(z) = \alpha_1 A_{1j} + \alpha_2 A_{2j}, \qquad B_j(z) = \alpha_1 B_{1j} + \alpha_2 B_{2j},$$
$$C_j(z) = \alpha_1 C_{1j} + \alpha_2 C_{2j}, \quad j = 1, 2.$$

From (1.3) it follows that $\lambda_1(z)$ is a solution of the equations

$$\alpha_1(z)\mathcal{N}_{1j}(z,\lambda) + \alpha_2(z)\mathcal{N}_{2j}(z,\lambda) = 0, \quad j = 1, 2,$$

and so

$$\begin{aligned} \alpha_1(z)\mathcal{N}_{1j}(z,\lambda) &+ \alpha_2(z)\mathcal{N}_{2j}(z,\lambda) \\ &= [\lambda - \lambda_1(z)][d_{1j}(z)\lambda + d_{2j}(z)] = 0, \quad j = 1, 2, \end{aligned} \tag{1.7}$$

where

$$d_{1j}(z) = A_j(z), \quad d_{2j}(z) = \lambda_1(z)A_j(z) + B_j(z), \quad j = 1, 2.$$

Therefore, equation (1.6) has the representation (1.2). This establishes the following.

Lemma 2. *If $\lambda_1(z) \neq \lambda_2(z)$, $\operatorname{Im}\lambda_j(z) > 0$, $j = 1, 2$, are continuous differentiable solutions of characteristic equation* (0.3) *and the condition* (1.1) *is valid, then the system* (0.1) *reduces, by means of a linearly nondegenerate transformation*

$$v = \overline{\beta_2(z)}u_1 - \overline{\beta_1(z)}u_2 \tag{1.8}$$

to a (*complex*) *canonical form*

$$\frac{\partial}{\partial \zeta_1}\left(\overline{\frac{\partial v}{\partial \zeta_2}} - r(z)\frac{\partial v}{\partial \zeta_2}\right) + P(v) + Q(\bar{v}) = 0, \tag{1.9}$$

or

$$\frac{\partial}{\partial \zeta_1}\left(\frac{\partial v}{\partial \zeta_2} - r(z)\overline{\frac{\partial v}{\partial \zeta_2}}\right) + P(v) + Q(\bar{v}) = 0, \tag{1.10}$$

where $|r(z)|<1$ *in* $\mathcal{D}$, $|r(z)|=1$ *on* Γ *and* P, Q *are first order linear differential operators and*

$$\frac{\partial}{\partial\zeta_1}=\frac{\partial}{\partial x}-\lambda_1(z)\frac{\partial}{\partial y},\qquad \frac{\partial}{\partial\zeta_2}=\frac{\partial}{\partial x}-\lambda_2(z)\frac{\partial}{\partial y} \tag{1.11}$$

are generalized differential operators.

Proof. Arguing as in the proof of Lemma 1, we get $\operatorname{rank}\mathcal{N}(z,\lambda_j(z))=1$, $j=1,2$. Under the condition (1.4) we have that the matrix

$$\omega(z)=\begin{Vmatrix}\alpha_1(z) & \alpha_1(z)\\ \alpha_2(z) & \alpha_2(z)\end{Vmatrix}$$

is nondegenerate everywhere in $\mathcal{D}$. Therefore,

$$\operatorname{rank}\omega(z)\mathcal{N}(z,\lambda_2(z))=1. \tag{1.12}$$

Since the coefficients of the square trinomial $\mathcal{N}_{jk}(z,\lambda)$ are real, using (1.7), (1.12) and $\lambda_1\neq\lambda_2$, $\lambda_1\neq\bar{\lambda}_2$, we obtain

$$\operatorname{rank}\begin{Vmatrix}d_{11}(z)\lambda_2(z)+d_{21}(z) & d_{12}(z)\lambda_2(z)+d_{22}(z)\\ \overline{d_{11}(z)}\lambda_2(z)+\overline{d_{21}(z)} & \overline{d_{12}(z)}\lambda_2(z)+\overline{d_{22}(z)}\end{Vmatrix}=1. \tag{1.13}$$

Hence, there exists a nonzero vector $(\beta_1(z),\beta_2(z))$ in $\overline{\mathcal{D}}$ such that

$$\begin{aligned}
&d_{11}(z)\lambda_2(z)+d_{21}(z)=d_1(z)\beta_2(z),\\
&d_{12}(z)\lambda_2(z)+d_{22}(z)=-d_1(z)\beta_1(z),\\
&\overline{d_{11}(z)}\lambda_2(z)+\overline{d_{21}(z)}=d_2(z)\beta_2(z),\\
&\overline{d_{12}(z)}\lambda_2(z)+\overline{d_{22}(z)}=-d_2(z)\beta_1(z),
\end{aligned} \tag{1.14}$$

where $|d_1(z)|^2+|d_2(z)|^2\neq 0$ in $\overline{\mathcal{D}}$. From (1.11) it follows that

$$\begin{aligned}
\frac{\partial}{\partial x}&=\frac{1}{\overline{\lambda_2(z)}-\lambda_2(z)}\left[\overline{\lambda_2(z)}\frac{\partial}{\partial\zeta_2}-\lambda_2(z)\frac{\bar{\partial}}{\partial\zeta_2}\right],\\
\frac{\partial}{\partial y}&=\frac{1}{\overline{\lambda_2(z)}-\lambda_2(z)}\left[\frac{\partial}{\partial\zeta_2}-\frac{\bar{\partial}}{\partial\zeta_2}\right].
\end{aligned} \tag{1.15}$$

In (1.2), replacing $\partial u/\partial x$, $\partial u/\partial y$ by $\partial u/\partial \zeta_2$, $\overline{\partial u}/\partial \zeta_2$ and using (1.14), we obtain

$$\frac{\partial}{\partial \zeta_1}\left[\frac{\bar{d}_2}{\bar{\lambda}_2 - \lambda_2}\frac{\partial v}{\partial \zeta_2} - \frac{d_1}{\bar{\lambda}_2 - \lambda_2}\frac{\overline{\partial v}}{\partial \zeta_2}\right] + L_3(v) + L_4(\bar{v}) = 0, \tag{1.16}$$

where v is as in (1.8).

We now show that nondegeneracy of (1.8), that is, for any $z \in \overline{\mathcal{D}}$, we have

$$\begin{vmatrix} \beta_1(z) & \beta_2(z) \\ \overline{\beta_1(z)} & \overline{\beta_2(z)} \end{vmatrix} \neq 0. \tag{1.17}$$

Suppose this is not true, i.e., that this determinant vanishes at some z_0. Since $(\beta_1(z_0), \beta_2(z_0)) \neq (0,0)$ (let $\beta_2(z_0) \neq 0$ for definiteness),

$$\beta_1(z_0) = k\beta_2(z_0), \quad \overline{\beta_1(z_0)} = k\overline{\beta_2(z_0)}, \tag{1.18}$$

where k is a real number. For the sake of simplicity we consider the particular case in which $a_{jk}(z_0) = b_{jk}(z_0) = c_{jk}(z_0) = 0$. Then $L_3(v)$ and $L_4(\bar{v})$ vanish. Using

$$v = \bar{\beta}_2(z_0)(u_1 - ku_2),$$

the equation (1.16) can be rewritten in the form

$$\frac{\partial}{\partial \zeta_1}\left[\frac{\overline{d_2(z_0)\beta_2(z_0)}}{\overline{\lambda_2(z_0)} - \lambda_2(z_0)}\frac{\partial v_1}{\partial \zeta_2} - \frac{d_2(z_0)\overline{\beta_2(z_0)}}{\overline{\lambda_2(z_0)} - \lambda_2(z_0)}\frac{\partial v_1}{\partial \bar{\zeta}_2}\right] = 0, \tag{1.19}$$

where

$$v_1 = u_1 - ku_2, \quad v_2 = u_2. \tag{1.20}$$

The equation (1.19) is equivalent to the system of two real equations of the form

$$\begin{aligned} \nu_1\frac{\partial^2 v_1}{\partial x^2} + \gamma_1\frac{\partial^2 v_1}{\partial x \partial y} + \mu_1\frac{\partial^2 v_1}{\partial y^2} &= 0, \\ \nu_2\frac{\partial^2 v_2}{\partial x^2} + \gamma_2\frac{\partial^2 v_2}{\partial x \partial y} + \mu_2\frac{\partial^2 v_2}{\partial y^2} &= 0, \end{aligned} \tag{1.21}$$

where ν_j, γ_j, μ_j are real numbers. The system (1.21) is nonelliptic with respect to v_1 and v_2. However, this is impossible since (1.21) was obtained from an elliptic system by multiplying both sides by the nondegenerate matrix $\omega(z_0)$ and applying the nondegenerate transformation (1.20). This contradiction proves that (1.8) is nondegenerate.

Since $d_1(z)$ and $d_2(z)$ are not both zero, the equation (1.16) can be represented either as (1.9) or as (1.10). This completes the proof. □

2. Riemann-Hilbert type boundary value problem. We consider the homogeneous canonical equation for the case in which the first order derivatives and the function itself are missing:

$$\frac{\partial}{\partial \zeta_1}\left(\overline{\frac{\partial v}{\partial \zeta_2}} - r(z)\frac{\partial v}{\partial \zeta_2}\right) = 0, \tag{2.1}$$

where the $r(z)$ function is from $C^1_\alpha(\overline{\mathcal{D}})$.

Riemann-Hilbert type problem. We look for a solution of (2.1) in the class $C^1_\alpha(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ satisfying the boundary condition

$$\operatorname{Re}[\overline{\mu(t)}v] = f(t), \quad t \in \Gamma, \tag{2.2}$$

where $\mu(t)$ and $f(t)$ are functions defined in $C_\alpha(\Gamma)$.

The equation (2.1) can be reduced to the Beltrami equation. Indeed, using (1.11) and denoting

$$w = \frac{\partial v}{\partial \zeta_2} - \overline{r(z)}\,\overline{\frac{\partial v}{\partial \zeta_2}} \tag{2.3}$$

we rewrite (2.1) in the form

$$\frac{\partial w}{\partial \bar z} - q_1(z)\frac{\partial w}{\partial z} = 0, \tag{2.4}$$

where

$$q_1(z) = \frac{i\overline{\lambda_1(z)} - 1}{i\overline{\lambda_1(z)} + 1}.$$

It is easy to verify that $|q_1(z)| < 1$ in $\mathcal{D}$ and $|q_1(z)| = 1$ on Γ.

A method for finding the general solution of the equation (2.4) when $|q_1(z)| \leq q_0 < 1$ on the $\mathcal{D}$ was put forward in Vekua [**16**, p. 116] and Boyarskiĭ [**10**].

The following theorem generalizes their result.

Theorem 1. *Let $q(z) \in C^1(\mathcal{D}) \cap C_\alpha(\overline{\mathcal{D}})$, $0 < \alpha \leq 1$, $|q(z)| < 1$ in domain $\mathcal{D}$ and $|q(z)| = 1$ on the boundary $\mathcal{D}$. Then there is a homeomorphic solution $W(z)$ of Beltrami equation*

$$\frac{\partial w}{\partial \bar{z}} - q(z)\frac{\partial w}{\partial z} = 0 \tag{2.5}$$

and the general solution has the form

$$w = \Phi(W(z)), \tag{2.6}$$

where Φ is an arbitrary analytic function in $\mathcal{D}_1 = W(\mathcal{D})$.

Proof. We try to seek some homeomorphic solution $W(z)$ of (2.5) of the form

$$W(z) = z - \frac{1}{\pi}\iint_{\mathcal{D}} \frac{q(\zeta)f(\zeta)}{\zeta - z}\, d\xi\, d\eta \equiv z + T(qf), \tag{2.7}$$

where f is an unknown function from class $C^\alpha(\overline{\mathcal{D}})$ [**16**]. Substituting $W(z)$ into (2.5), we obtain ($q(z) \neq 0$)

$$f - \Pi(qf) = 1, \tag{2.8}$$

where

$$\Pi(qf) = -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{q(\zeta)f(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta. \tag{2.9}$$

This integral is understood in the sense of Cauchy. We now prove that the equation (2.8) has a solution in the space $L_p(\mathcal{D})$, $p = 2/(1-\alpha)$.

Indeed, since

$$L_p(\Pi g) \leq \Lambda_p L_p(g), \quad \Lambda_p = L_p(\Pi),$$

we have

$$L_p(\Pi(qf)) \leq \Lambda_p L_p(qf).$$

It is easy to see that there exists a point $z^* \in \mathcal{D}$ such that

$$L_p(qf) = |q(z^*)| \cdot L_p(f).$$

Since $|q(z)| < 1$ in $\mathcal{D}$ and $|q(z)| = 1$ only on Γ, we have $|q(z^*)| < 1$. Hence

$$L_p(\Pi(qf)) \leq \Lambda_p |q(z^*)| L_p(f). \tag{2.10}$$

It is known that the norm Λ_p is a continuous function of p and that $\Lambda_2 = 1$ [**16**]. Therefore there exists an $\varepsilon > 0$ such that $\Lambda_p \cdot |q(z^*)| < 1$ for $p \leq 2 + \varepsilon$. It follows from (2.10) that $f \to \Pi(qf)$ is a contractive mapping. According to Banach's contractive mapping principle, there exists a solution of equation (2.8) which belongs to class $L_p(\mathcal{D})$ for each $p \leq 2 + \varepsilon$ and the solution can be found by means of the successive approximation method:

$$f_0 = 1, \quad f_{n+1} = \Pi(qf_n) + 1, \quad n = 0, 1, \dots ,$$

Substituting the obtained solution f into (2.7) we get a homeomorphic solution of (2.5) and the general solution is well known to be of the form (2.6) (see also [**16**]). This completes the proof.

Now substituting $w(z)$ from (2.6) into (2.3), when $q = q_1$, $W = H_1$, we get

$$\frac{\partial v}{\partial \zeta_2} - \overline{r(z)}\,\overline{\frac{\partial v}{\partial \zeta_2}} = \Phi(H_1(z)).$$

It follows that

$$(1 - |r(z)|^2)\frac{\partial v}{\partial \zeta_2} = \Phi(H_1(z)) + \overline{r(z)\Phi(H_1(z))}. \tag{2.11}$$

As z tends to $t \in \Gamma$, we obtain

$$\Phi(\tau) + \overline{r_1(\tau)\Phi(\tau)} = 0, \quad \tau \in \Gamma_1 = H_1(\Gamma), \tag{2.12}$$

where

$$\tau = H_1(t), \quad r_1(\tau) = r(H_1^{-1}(\tau)).$$

Since any simply connected domain can be conformally mapped onto the unit disk, we will consider the case where $\mathcal{D}_1$ is the unit disk. Determination of $\Phi(\xi)$ by the boundary condition (2.7) is reduced to the famous conjugation problem with respect to an unknown piecewise analytic function

$$\Omega(\xi) = \begin{cases} \Phi(\zeta) & \text{for } |\xi| < 1, \\ \Phi(1/\bar{\zeta}) & \text{for } |\zeta| > 1. \end{cases}$$

Indeed, the boundary condition (2.7) has the form

$$\Omega^+(\tau) + \overline{r_1(\tau)}\Omega^-(\tau) = 0, \tag{2.13}$$

where $\Omega^+(\tau)$ and $\Omega^-(\tau)$ are the limits of $\Omega(\xi)$ from inside and outside of Γ_1 in point τ of Γ_1, respectively.

Let m be the integer defined by

$$m = \frac{1}{2\pi}\,\Delta_\Gamma \arg \overline{r(t)}, \tag{2.14}$$

where $\Delta_\Gamma \arg \overline{r(t)}$ is increase of $\arg \overline{r(t)}$, when t is rotated over Γ once in the positive direction. Assume that $H_1(z)$ conserves the orientation of Γ, then we have also

$$m = \frac{1}{2\pi}\,\Delta_{\Gamma_1} \arg \overline{r_1(t)}. \tag{2.14$'$}$$

Using the solution of problem (2.13) [**9**], we find all analytic functions satisfying the boundary condition (2.12) [**5**]

$$\begin{gathered} \Phi(\zeta) = 0 \quad \text{for } m \le -1, \\ \Phi(\xi) = \exp\left[\frac{1}{2\pi i}\int_{\Gamma_1} \frac{\ln(\tau^{-m}\overline{r_1(\tau)})}{\tau - \zeta}\,d\tau\right]\sum_{k=0}^{m} C_k \zeta^k \quad \text{for } m \ge 0, \end{gathered} \tag{2.15}$$

where C_k are complex constants satisfying the conditions $C_{m-k} = \overline{C_k}$, $k = 0, 1, \ldots, m$.

Let's return to equation (2.11) which can be easily reduced to the Beltrami equation

$$\frac{\partial v}{\partial \bar{z}} + q_2(z)\frac{\partial v}{\partial z} = h(z), \tag{2.16}$$

where

$$q_2(z) = \frac{1 - i\lambda_2(z)}{1 + i\lambda_2(z)}, \quad h(z) = \frac{\Phi(H_1(z)) + \overline{r(z)\Phi(H_1(z))}}{(1 - |r(z)|^2)(1 + i\lambda_2(z))}$$

for $m \geq 0$ and $h(z) = 0$ for $m \leq -1$. It is easy to verify that $|q_2(z)| < 1$ in $\mathcal{D}$ and $|q_2(z)| = 1$ on Γ. By Lemma 2 [**5**], $h(z) \in C_\alpha(\overline{\mathcal{D}})$.

In the case $m \leq -1$, $h(z) = 0$, all solutions of equation (2.11) have the form

$$v(z) = \psi(H_2(z)) \tag{2.17}$$

where $H_2(z)$ is a homeomorphic solution of (2.16), when $h(z) = 0$, ψ is an analytic function in $\mathcal{D}_2 = H_2(\mathcal{D})$ and $v \in C^1_\alpha(\mathcal{D})$ if $q_2 \in C_\alpha(\mathcal{D})$, $0 < \alpha \leq 1$ (see (2.4) and (2.5)).

Substituting $v(z)$ from (2.17) into the boundary condition (2.2), we obtain

$$\operatorname{Re}(\overline{\nu(\tau)}\psi(\tau)) = g(\tau), \quad \tau \in \Gamma_2 = H_2(\Gamma), \tag{2.18}$$

where $\nu(\tau) = \mu(H_2^{-1}(\tau))$, $g(\tau) = f(H_2^{-1}(\tau))$.

Without loss of generality, we will consider the case in which $\mathcal{D}_2$ is the unit disk and the map $H_2(z)$ preserves the orientation of Γ. Define σ by

$$\sigma = \frac{1}{\pi}\,\Delta_{\Gamma_2} \arg \mu(\tau). \tag{2.19}$$

It is known [**12**] that the Riemann-Hilbert problem (2.18) for $\sigma \geq 0$ has a solution, and the general solution contains $\sigma + 1$ arbitrary real constants as a linear; and for $\sigma < 0$ that problem has a unique solution if it satisfies $-\sigma - 1$ some conditions of orthogonality.

In the case $m \geq 0$, $h(z) \neq 0$, the function $v(z)$ satisfies the nonhomogeneous equation (2.16). The right part of (2.16) can be represented in the form

$$h(z) = \sum_{k=0}^{m} d_k h_k(z),$$

where d_k are real constants and $h_k(z)$ are certain linearly independent functions, with respect to the field of real numbers, from the class $C_\alpha(\overline{\mathcal{D}})$. The general solution of the linear equation (2.16) will have the form

$$v(z) = \psi(H_2(z)) + \sum_{k=0}^{m} d_k f_k(z), \tag{2.20}$$

where $f_k(z)$ is a particular solution of (2.16) with right part $h_k(z)$. The solution $f_k(z)$ may be obtained in the form

$$z - \frac{1}{\pi} \iint_{\mathcal{D}} \frac{q_2(\zeta)\rho(\zeta)}{\zeta - z}\, d\xi\, d\eta,$$

where the function ρ is a solution of equation

$$\rho - \frac{1}{\pi} \iint_{\mathcal{D}} \frac{q_2(\zeta)\rho(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta = h_k - q_2.$$

Substituting $v(z)$ from (2.20) into (2.2), we obtain

$$\operatorname{Re}\,(\overline{\nu(\tau)}\psi(\tau)) = g(\tau) - \sum_{k=0}^{m} d_k g_k(\tau), \tag{2.21}$$

where $g_k(\tau) = \operatorname{Re}\,[\overline{\nu(\tau)} f_k(H_2^{-1}(\tau))]$.

For $\sigma \geq 0$, function $\psi(z)$ is determined up to $\sigma + m + 2$ real constants.

For $\sigma < 0$, the problem (2.16) is solvable for $\psi(z)$ if and only if the right part of (2.21) satisfies some conditions of orthogonality. Therefore, the numbers $d_0, d_1, \ldots, d_m$ must satisfy some algebraic system of $-\sigma - 1$ linear equations. Hence the problem (2.1)–(2.2) will have $\sigma + m + 2$ many solutions, $\sigma > -m - 2$.

We have thus proved the following result.

Theorem 2. *Let* $r(z) \in C^1_\alpha(\overline{\mathcal{D}})$ *and* $\mu(t)$, $f(t) \in C_\alpha(\Gamma)$. *Then the problem* (2.1)–(2.2) *has*

1) $m + \sigma + 2$ *linearly independent solutions, over the field of real numbers, in the cases*

a) $m \geq 0$, $\sigma \geq 0$,

b) $m \geq 0$, $\sigma < 0$, $m + 1 \geq -\sigma - 1$;

2) $\sigma + 1$ *linearly independent solutions in the case* $m < 0$, $\sigma \geq 0$;

3) *a unique solution in the case* $m < 0$, $\sigma < 0$, *if and only if the right part of* (2.2) *satisfies some conditions of orthogonality.*

3. Conclusion. It is well known that the stating of boundary conditions for degenerate elliptic systems essentially depends upon the set in which the degeneration takes place. For instance, there is a system, degenerate at an internal point of the domain, for which the Dirichlet problem is Fredholm. For another system the same inhomogeneous problem is solvable without any condition, in spite of the fact that the homogeneous problem has linearly independent solutions. For a third one the homogeneous problem has linearly independent solutions of infinite number, and the inhomogeneous one is solvable if and only if an infinite number of linearly independent conditions are satisfied [**6, 13**]. When the degeneration takes place only at a portion of the boundary of the domain, then the considerations of various authors have the following similarity: the part of the boundary, where degeneration takes place, is either completely or partially freed of the boundary condition [**11**].

We considered the case when degeneration takes place at the whole boundary of the domain and when the characteristic equation corresponding to (0.1) has i as double roots inside the domain and an arbitrary number of roots at the boundary.

In these cases equation (0.1) is reduced to the following complex canonical form

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial v}{\partial \bar{z}} - q(z)\overline{\frac{\partial v}{\partial \bar{z}}}\right) + P(v) + Q(\bar{v}) = 0, \tag{3.1}$$

where the following nondegenerate transformation was used

$$v = \alpha(z)u_1 + \beta(z)u_2,$$

and $P(u)$ and $Q(\bar{u})$ are linear differential operators of the first order. In the papers [**1–5**] we investigated three cases of degeneration: (1) degeneration at the whole boundary; (2) at any closed curve inside the domain; (3) both degeneration cases simultaneously. Various problems, which depended on the behavior of the coefficient $q(z)$, were discussed. It was shown that those problems are Noetherian. Besides, those problems are reduced to the form $v = Lv + f$ where L is a completely continuous operator in the corresponding Banach space. Also the indexes (i.e., the difference between the numbers of linearly independent solutions of the homogeneous problem and that of solvability conditions for the inhomogeneous one) for those problems were computed or evaluated.

In the present paper we again consider the degeneration at the whole boundary, when the characteristic equation of (0.1) has different and variable roots inside the domain, and an arbitrary number of solutions at the boundary. Equations (0.3) and (0.4) are the canonical forms of (0.1) in the complex expression. Equation (0.3), for instance, can be written in the form

$$\left(\frac{\partial}{\partial \bar{z}} - q_1(z)\frac{\partial}{\partial z}\right)\left(\frac{\partial v}{\partial \bar{z}} - q_2(z)\frac{\partial v}{\partial z}\right) + P(v) + Q(\bar{v}) = 0. \tag{3.2}$$

This equation is essentially different from (3.1), since the Beltrami operator is applied twice in it. For this reason I have not succeeded in reducing the Riemann-Hilbert problem for (3.2) to the form $v = Lv+f$ in Banach spaces where L is a complete continuous operator. But in the particular case, when $P(v) = 0$, $Q(\bar{v}) = 0$, the problem (2.1)–(2.2) was solved explicitly (if it had a solution).

Given its complexity, the solution formula of that problem was not included in the formulation of Theorem 2 but it shows up in the proof of Theorem 2.

In stating the problem (2.1)–(2.2) at the whole boundary of the degenerate set, we give only the combination of the components of the sought vector function and not the vector function's values. This means that the set in which the system is degenerated is partially free from boundary conditions.

It is to be noted that the cases $P(v) \neq 0$, $Q(\bar{v}) \neq 0$ and also the simple root cases were not investigated.

It seems that, in the latter case, stating the problem also depends on whether the characteristic matrix vanishes or not.

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Department of Physics, Yerevan State University, 375025 Yerevan, Armenia
E-mail address: afyan@sun.ysu.am