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CONVOLUTION AND FOURIER TRANSFORM OVER THE SPACES $\mathcal{K}'_{p,k}$, $p > 1$

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ABSTRACT. We introduce the space $\mathcal{K}_{p,k}, p > 1$ that is the vector space of all C^∞-functions f such that $e^{k|x|^p}\partial^\alpha f$ vanishes at infinity for all $\alpha \in N^n$ and its dual $\mathcal{K}'_{p,k}$. For $f, g \in \mathcal{K}'_{p,2^p k}$, $k \in Z$, $k < 0$, we study the linear functional $f \otimes g$ on $\mathcal{K}_{p,k}$ defined by

$$\langle f \otimes g, \phi \rangle = \langle f(x), \langle g(y), \phi(x+y) \rangle \rangle, \quad \phi \in \mathcal{K}_{p,k}.$$

Also, we show a representation theorem and an inversion formula for the usual distributional Fourier transform over the spaces $\mathcal{K}'_{p,k}$, $k \in Z$, $k < 0$.

1. Introduction.

For spaces of functions and distributions we use the notations and terminology of Horvath [**3**]. In particular, $\mathcal{S}_k$ is the space of all infinitely differentiable functions f on R^n such that $(1+|x|^2)^k \partial^\alpha f(x)$ vanishes at infinity for all $\alpha \in N^n$.

We denote $\mathcal{K}_p$, $p \geq 1$, the space of all functions $\phi \in C^\infty(R^n)$ such that

$$\nu_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leq k}} e^{k|x|^p} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, \dots ,$$

where $D^\alpha = (i^{-1}\partial/\partial x_1)^{\alpha_1} \cdots (i^{-1}\partial/\partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The space $\mathcal{K}_p$ with semi-norm ν_k, $k = 1, 2, \dots$ is a Frechet space and the space of C^∞-functions with compact support $\mathcal{D}$ is a dense subset of $\mathcal{K}_p$. By $\mathcal{K}'_p$ we mean the space of continuous linear functionals on $\mathcal{K}_p$. For further details, we refer to [**4**].

We introduce the spaces $\mathcal{K}_{p,k}(R^n)$, $p > 1$, that are defined as the vector spaces of all functions f defined on R^n which possess continuous

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partial derivatives of all orders and satisfy the condition that if $\alpha \in N^n$ and $\varepsilon > 0$, then there exists $C = C(f, \alpha, \varepsilon) > 0$ such that

$$e^{k|x|^p} |\partial^\alpha f(x)| \leq \varepsilon,$$

for $|x| > C(f, \alpha, \varepsilon)$.

In what follows, we shall write $\mathcal{K}_{p,k}$ instead of $\mathcal{K}_{p,k}(R^n)$ and always assume $p > 1$. For every $\alpha \in N^n$, we define on $\mathcal{K}_{p,k}$ the semi-norms

$$q_{k,\alpha}(f) = \max_{x \in R^n} e^{k|x|^p} |\partial^\alpha f(x)|.$$

The space $\mathcal{K}_{p,k}$ equipped with the countable family of semi-norms is a locally convex space. The space of C^∞-functions with compact support $\mathcal{D}$ is a dense subspace of $\mathcal{K}_{p,k}$. By $\mathcal{K}'_{p,k}$, we mean the space of continuous linear functionals on $\mathcal{K}_{p,k}$.

In this paper, we will study convolution and Fourier transform over $\mathcal{K}'_{p,k}$ as in the case over $\mathcal{S}'_k$ in [**1**] and [**2**]. We will prove that for $f, g \in \mathcal{K}'_{p,2^p k}$, $\phi \in \mathcal{K}_{p,k}$, $k \in Z$, $k < 0$, the linear functional $f \otimes g$ defined by

$$\langle f \otimes g, \phi \rangle = \langle f(x), \langle g(y), \phi(x+y) \rangle \rangle, \quad \phi \in \mathcal{K}_{p,k},$$

has sense as the application of the functional $f \in \mathcal{K}'_{p,2^p k}$ to $\langle g(y), \phi(x+y) \rangle \in \mathcal{K}_{p,2^p k}$. We will also show a representation theorem for the usual distributional Fourier transform over the space $\mathcal{K}'_{p,k}$, $k \in Z$, $k < 0$. Its inversion formula is also obtained, which enables us to prove that $\mathcal{K}'_{p,2^p k}$ is a commutative convolution algebra with a unit element.

2. Convolution over $\mathcal{K}'_{p,2^p k}$. First we will prove that for $f, g \in \mathcal{K}'_{p,2^p k}$, $\phi \in \mathcal{K}_{p,k}$, $k \in Z$, $k < 0$, the linear functional $f \otimes g$ defined by

$$\langle f \otimes g, \phi \rangle = \langle f(x), \langle g(y), \phi(x+y) \rangle \rangle \tag{1}$$

has sense as the application of the functional $f \in \mathcal{K}'_{p,2^p k}$ to $\langle g(y), \phi(x+y) \rangle \in \mathcal{K}_{p,2^p k}$. It is also obtained that $f \otimes g \in \mathcal{K}'_{p,k}$.

We define the convolution $f \otimes g$ over $\mathcal{K}'_{p,2^p k}$ on $\mathcal{K}_{p,k}$ by (1).

For the proof of the above results, we need the following several lemmas.

Lemma 2.1. *Let $x \in R^n$ be a fixed vector, $\phi \in \mathcal{K}_{p,k}$, $k \in Z$, $k < 0$. Then $\phi(x+y) \in \mathcal{K}_{p,2^p k}$.*

Proof. Since $\phi \in \mathcal{K}_{p,k}$, for all $\varepsilon > 0$ and $\alpha \in N^n$, there exists $A(\phi, \alpha, \varepsilon) > 0$ such that

$$e^{k|z|^p} |\partial^\alpha \phi(z)| \leq \varepsilon,$$

for $|z| > A(\phi, \alpha, \varepsilon)$. Then, since $k < 0$, if we take $B(\phi, \alpha, \varepsilon, x) = A(\phi, \alpha, \varepsilon) + |x|$, then for $|y| > B(\phi, \alpha, \varepsilon, x)$,

$$\begin{aligned} e^{2^p k|y|^p} \, |\partial^\alpha \phi(x+y)| &= e^{2^p k|y|^p} \, e^{-k|x+y|^p} \, e^{k|x+y|^p} \, |\partial^\alpha \phi(x+y)| \\ &= e^{2^p k(|y|^p - (2^p)^{-1}|x+y|^p)} \, e^{k|x+y|^p} \, |\partial^\alpha \phi(x+y)| \\ &\leq e^{2^p k(|y|^p - (|x|^p + |y|^p))} \, e^{k|x+y|^p} \, |\partial^\alpha \phi(x+y)| \\ &= e^{-2^p k|x|^p} \, e^{k|x+y|^p} \, |\partial^\alpha \phi(x+y)| \\ &\leq e^{-2^p k|x|^p} \varepsilon. \end{aligned} \tag{2}$$

Therefore, for each fixed vector $x \in R^n$, $\phi(x+y) \in \mathcal{K}_{p,2^p k}$. □

Lemma 2.2. *If $g \in \mathcal{K}'_{p,2^p k}$ and $\phi \in \mathcal{K}_{p,k}$ with $k \in Z$, $k < 0$, then, for all $m \in N^n$,*

$$\partial^m \langle g(y), \phi(x+y) \rangle = \langle g(y), \partial^m \phi(x+y) \rangle. \tag{3}$$

Proof. We will prove (3) by induction on $|m|$. Assume $|m| = 1$. For each fixed $x \in R^n$ and each fixed $i = 1, 2, \dots n$, set $h_i = (h_{i,1}, h_{i,2}, \dots, h_{i,n}) \in R^n$ given by $h_{i,i} = \triangle x_i \neq 0$ and $h_{i,j} = 0$ for $j \neq i$. Now consider

$$\begin{aligned} &\frac{1}{\triangle x_i} \{ \langle g(y), \phi(x+y+h_i) \rangle - \langle g(y), \phi(x+y) \rangle \} \\ &\qquad - \langle g(y), \frac{\partial}{\partial x_i} \phi(x+y) \rangle = \langle g(y), \theta_{h_i,x}(y) \rangle, \end{aligned}$$

where

$$\theta_{h_i,x}(y) = \frac{1}{\triangle x_i} \{ \phi(x+y+h_i) - \phi(x+y) \} - \frac{\partial}{\partial x_i} \phi(x+y).$$

We will prove that $\theta_{h_i,x} \to 0$, in $\mathcal{K}_{p,2^p k}$ for $|h_i| \to 0$, which assures that

$$\frac{\partial}{\partial x_i} \langle g(y), \phi(x+y) \rangle = \Big\langle g(y), \frac{\partial}{\partial x_i} \phi(x+y) \Big\rangle.$$

First, we will check that $\theta_{h_i,x}(y) \in \mathcal{K}_{p,2^p k}$. For all $\alpha \in N^n$ and $y \in R^n$,

$$\begin{aligned}\partial^\alpha \phi(x+y+h_i) = \partial^\alpha \phi(x+y) &+ \triangle x_i \frac{\partial}{\partial x_i} \partial^\alpha \phi(x+y) \\ &+ \int_0^{\triangle x_i} (\triangle x_i - \xi) \frac{\partial^2}{\partial x_i^2} \partial^\alpha \phi(x+y+t_{i,\xi})\, d\xi,\end{aligned}$$

where $t_{i,\xi} = (t_{i,1,\xi}, t_{i,2,\xi}, \dots, t_{i,n,\xi})$ with $t_{i,j,\xi} = \xi$ for $j = i$ and $t_{i,j,\xi} = 0$ for $j \neq i$. Therefore,

$$\partial^\alpha \theta_{h_i,x}(y) = \int_0^{\triangle x_i} (\triangle x_i - \xi) \frac{\partial^2}{\partial x_i^2} \partial^\alpha \phi(x+y+t_{i,\xi})\, d\xi.$$

Since $\phi \in \mathcal{K}_{p,k}$, given $\varepsilon > 0$ and $\alpha \in N^n$, there exist $A(\phi, \alpha, \varepsilon) > 0$ such that if $|z| > A(\phi, \alpha, \varepsilon)$, then

$$e^{k|z|^p} \left| \frac{\partial^2}{\partial z_i^2} \partial^\alpha \phi(z) \right| < \varepsilon.$$

Now, for $|t| \leq |h_i| < 1$,

$$\begin{aligned}(4) \quad & e^{2^p k|y|^p} \left| \frac{\partial^2}{\partial y_i^2} \partial^\alpha \phi(x+y+t) \right| \\ &= e^{2^p k|y|^p} e^{-k|x+y+t|^p} e^{k|x+y+t|^p} \left| \frac{\partial^2}{\partial y_i^2} \partial^\alpha \phi(x+y+t) \right|.\end{aligned}$$

Since $\phi \in \mathcal{K}_{p,k}$, we have that for $|t| \leq 1$ and $|x+y+t| > A(\phi, \alpha, \varepsilon)$,

$$e^{k|x+y+t|^p} \left| \frac{\partial^2}{\partial y_i^2} \partial^\alpha \phi(x+y+t) \right| < \varepsilon.$$

If we let $B(\phi,\alpha,\varepsilon,x) = A(\phi,\alpha,\varepsilon) + |x| + 1$, since $k < 0$, we have that for $|y| > B(\phi,\alpha,\varepsilon,x)$, (4) is less than or equal to

$$\begin{aligned} e^{2^p k|y|^p} e^{-k|x+y+t|^p}\varepsilon &\le e^{2^p k(|y|^p-(2^p)^{-1}|x+y+t|^p)}\varepsilon \\ &\le e^{2^p k(|y|^p-|y|^p-|x+t|^p)}\varepsilon \\ &= e^{-2^p k|x+t|^p}\varepsilon \\ &\le e^{-2^p k(2^p|x|^p+2^p|t|^p)}\varepsilon \\ &\le e^{-2^{2p}k} e^{-2^{2p}k|x|^p}\varepsilon. \end{aligned}$$

So, for $|y| > B(\phi,\alpha,\varepsilon,x)$,

$$\begin{aligned} e^{2^p k|y|^p}|\partial^\alpha \theta_{h_i,x}(y)| &\le \frac{e^{-2^{2p}k}e^{-2^{2p}k|x|^p}\varepsilon}{|\triangle x_i|}\int_0^{\triangle x_i}(\triangle x_i - \xi)\,d\xi \\ &= \frac{|\triangle x_i|}{2}\,e^{-2^{2p}k}e^{-2^{2p}k|x|^p}\varepsilon, \end{aligned} \tag{5}$$

and thus $\theta_{h_i,x}(y) \in \mathcal{K}_{p,2^p k}$. On the other hand, for $|y| \le B(\phi,\alpha,\varepsilon,x)$ and $|y| \le 1$,

$$e^{2^p k|y|^p}\left|\frac{\partial^2}{\partial y_i^2}\,\partial^\alpha \phi(x+y+t)\right| \le M_1,$$

for some constant M_1. Setting $M_2 = \max\{M_1, e^{-2^{2p}k}e^{-2^{2p}k|x|^p}\varepsilon\}$ and taking into account (5), for all $y \in R^n$,

$$\begin{aligned} e^{2^p k|y|^p}|\partial^\alpha \theta_{h_i,x}(y)| &\le \frac{M_2}{\triangle x_i}\int_0^{\triangle x_i}(\triangle x_i - \xi)\,d\xi \\ &= \frac{|\triangle x_i|}{2}\,M_2, \end{aligned}$$

which tends to 0 as $|h_i| \to 0$. This proves the conclusion for $|m| = 1$. Now, the result of this lemma follows by induction on $|m|$. □

Lemma 2.3. *If $g \in \mathcal{K}'_{p,2^p k}, \phi \in \mathcal{K}_{p,k}$, $k \in Z$, $k < 0$, then $\langle g(y), \phi(x+y)\rangle \in \mathcal{K}_{p,2^p k}$.*

Proof. From Lemma 2.2, one has that $\langle g(y), \phi(x+y)\rangle$ is smooth. It remains to prove that, for any $m \in N^n$ and any $\varepsilon > 0$, there exists

$B > 0$ such that if $|x| > B$, then $e^{2^p k|x|^p}|\partial^m\langle g(y),\phi(x+y)\rangle| \le \varepsilon$. In fact, from Lemma 2.2 and [**3**, Remark of Proposition 2, p. 97] there exist a positive constant C and a nonnegative integer r such that

$$|\langle g,\phi\rangle| \le C \max_{0\le s\le r} q_{k,\alpha_s}(\phi), \tag{6}$$

for $\phi \in \mathcal{K}_{p,k}$.

Here C and r depend on g but not on ϕ. First, we will show that this lemma holds for $\phi \in \mathcal{D}(R^n)$. Since $\mathcal{D} \subset \mathcal{K}_{p,k}$, by (6), for any $m \in N^n$ and $\phi \in \mathcal{D}$,

$$\begin{aligned} e^{2^p k|x|^p}&|\partial_x^m\langle g(y),\phi(x+y)\rangle| \\ &= e^{2^p k|x|^p}|\langle g(y),\partial_x^m\phi(x+y)\rangle| \\ &\le C\max_{0\le s\le r}\max_{y\in R^n} e^{2^p k|x|^p}e^{2^p k|y|^p}|\partial_x^m\partial_y^{\alpha_s}\phi(x+y)| \\ &\le C\max_{0\le s\le r} e^{2^p k|x|^p}M_{m,\alpha_s}, \end{aligned}$$

where $M_{m,\alpha_s} = \max_{z\in R^n}|\partial^{m+\alpha_s}\phi(z)|$. Since $k<0$, this lemma holds for $\phi\in\mathcal{D}$. Next, since $\mathcal{D}$ is a dense subset of $\mathcal{K}_{p,k}$, for $\phi\in\mathcal{K}_{p,k}$, there exists a sequence $\{\phi_j\}\subset\mathcal{D}$ with $\phi_j\to\phi$ in $\mathcal{K}_{p,k}$ as $j\to\infty$. Hence for any $\varepsilon>0$ and any $\alpha\in N^n$, there exist $j_0 = j_0(\varepsilon,\alpha)\in N$ such that

$$\max_{z\in R^n} e^{k|z|^p}|\partial^\alpha\{\phi_j(z)-\phi(z)\}| \le \frac{\varepsilon}{2C},$$

for $j \ge j_0$. So, for any $\varepsilon>0$ and any $\alpha\in N^n$, if $j\ge j_0 = \max\{j_0(\varepsilon,m+\alpha_s)\}$, $s=0,1,\dots r$,

$$\begin{aligned} e^{2^p k|x|^p}\partial_x^m&|\{\langle g(y),\phi_j(x+y)\rangle - \langle g(y),\phi(x+y)\rangle\}| \\ &\le C\max_{0\le s\le r}\max_{y\in R^n} e^{2^p k|x|^p}e^{2^p k|y|^p} \\ &\quad\times|\partial_y^{\alpha_s}\partial_x^m\{\phi_j(x+y)-\phi(x+y)\}| \\ &= C\max_{0\le s\le r}\max_{y\in R^n} e^{2^p k|x|^p}e^{2^p k|y|^p}e^{-k|x+y|^p}e^{k|x+y|^p} \\ &\quad\times|\partial^{m+\alpha_s}\{\phi_j(x+y)-\phi(x+y)\}| \\ &= C\max_{0\le s\le r}\max_{y\in R^n} e^{k(2^p|x|^p+2^p|y|^p-|x+y|^p)}e^{k|x+y|^p} \\ &\quad\times|\partial^{m+\alpha_s}\{\phi_j(x+y)-\phi(x+y)\}| \\ &\le C\max_{0\le s\le r}\max_{z\in R^n} e^{k|z|^p}|\partial^{m+\alpha_s}\{\phi_j(z)-\phi(z)\}| \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{7}$$

Also, since $\langle g(y), \phi_{j_0}(x+y)\rangle \in \mathcal{K}_{p,2^p k}$, for any $\varepsilon > 0$ and $m \in N^n$, there exist $A(\varepsilon, m, \phi_{j_0})$ such that

$$e^{k|x|^p}|\partial_x^m \langle g(y), \phi_{j_0}(x+y)\rangle| < \frac{\varepsilon}{2},$$

for $|x| > A(\varepsilon, m, \phi_{j_0})$. Hence taking $B = A(\varepsilon, m, \phi_{j_0})$, for $|x| > B$, then, by (7) and the above fact,

$$\begin{aligned} e^{2^p k|x|^p}&|\partial_x^m \langle g(y), \phi(x+y)\rangle| \\ &\leq e^{2^p k|x|^p}|\partial_x^m \langle g(y), \phi_{j_0}(x+y)\rangle| \\ &\quad + e^{2^p k|x|^p}|\{\partial_x^m \langle g(y), \phi(x+y)\rangle - \partial_x^m \langle g(y), \phi_{j_0}(x+y)\rangle\}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus the result follows. □

Lemma 2.4. *Assume that $k \in Z$, $k < 0$, $g \in \mathcal{K}'_{p,2^p k}$ and $\phi_j \to 0$ in $\mathcal{K}_{p,k}$ for $j \to \infty$. Then $\langle g(y), \phi_j(x+y)\rangle \to 0$ in $\mathcal{K}_{p,2^p k}$ as $j \to \infty$.*

Proof. By (6) in the proof of Lemma 2.2 above,

$$e^{2^p k|x|^p}|\partial^m \langle g(y), \phi_j(x+y)\rangle| \leq C \max_{0\leq s\leq r} q_{2^p k, m+\alpha_s}(\phi_j).$$

From the above fact the result of this lemma follows immediately. □

Now, we conclude that

Theorem 2.5. *If $f, g \in \mathcal{K}'_{p,2^p k}$, $k \in Z$, $k < 0$, then $f \otimes g \in \mathcal{K}'_{p,k}$.*

Proof. Let $\{\phi_j\} \subset \mathcal{K}_{p,k}$ such that $\phi_j \to 0$ in $\mathcal{K}_{p,k}$ as $j \to \infty$. By Lemmas 2.1 and 2.3

$$\langle f \otimes g, \phi_j\rangle = \langle f(x), \langle g(y), \phi_j(x+y)\rangle\rangle$$

has sense, and by Lemma 2.4 and $f \in \mathcal{K}'_{p,2^p k}$, $\langle f \otimes g, \phi_j\rangle$ tends to zero as $j \to \infty$. □

3. Fourier transform over $\mathcal{K}'_{p,k}$. In this section, we will state a representation theorem for the usual distributional Fourier transform over the space $\mathcal{K}_{p,k}$, $k \in Z$, $k < 0$. Its inversion formula is also obtained, which enables us to prove that $\mathcal{K}'_{p,2^p k}$ is a commutative convolution algebra with unit element.

If we only replace $(1+|x|^2)^k$ and $\mathcal{S}_k$ by $e^{k|x|^p}$ and $\mathcal{K}_{p,k}$, respectively, we can show exactly like Theorem 2.1 in [**2**] the following representation theorem for the usual distributional Fourier transform over the space $\mathcal{K}_{p,k}$, $k \in Z$, $k < 0$, i.e., let $f \in \mathcal{K}'_{p,k}$, $k \in Z$, $k < 0$. Then for all $\phi \in \mathcal{K}_p$, the Parseval equality

$$\langle f, \mathcal{F}\phi \rangle = \langle \mathbb{T}_{\langle f(x), e^{ixy} \rangle}, \phi(y) \rangle,$$

follows, where $\mathbb{T}_{\langle f(x), e^{ixy} \rangle}$ is the member of $\mathcal{K}'_p$ given by

$$\langle \mathbb{T}_{\langle f(x), e^{ixy} \rangle}, \phi(y) \rangle = \int_{R^n} \langle f(x), e^{ixy} \rangle \phi(y)\, dy,$$

and $\mathcal{F}\phi$ denotes the classical Fourier transform of ϕ, namely,

$$(\mathcal{F}\phi)(t) = \int_{R^n} \phi(y) e^{ixy}\, dy, \quad t \in R^n.$$

Hence the usual distributional Fourier transform is represented over $\mathcal{K}'_{p,k}$, $k \in Z$, $k < 0$, for each $y \in R^n$, as the application of the functional $f \in \mathcal{K}'_{p,k}$ to the function $x \mapsto e^{ixy} \in \mathcal{K}_{p,k}$, $x \in R^n$, i.e.,

$$(\mathcal{F}f)(y) = \langle f(x), e^{ixy} \rangle, \quad y \in R^n.$$

Theorem 3.1. *Let f be a function defined on R^n such that $e^{k|x|^p} f(x)$ is integrable on R^n for some $k \in Z$, $k < 0$. Then the linear functional over $\mathcal{K}_{p,k}$ given by*

$$\langle T_f, \phi \rangle = \int_{R^n} f(x)\phi(x)\, dx, \quad \phi \in \mathcal{K}_{p,k}, \tag{8}$$

is an element of $\mathcal{K}'_{p,k}$.

Moreover, the distributional Fourier transform of T_f *given by equation* (8) *agrees with the classical Fourier transform of the function* f.

Proof. For $\phi \in \mathcal{K}_{p,k}$,

$$|\langle T_f, \phi\rangle| \le \int_{R^n} |e^{-k|x|^p} f(x)||e^{k|x|^p} g(x)|\, dx \\ \le q_{k,0}(\phi) \int_{R^n} |e^{-k|x|^p} f(x)|\, dx$$

From the hypothesis, the continuity of T_f follows immediately. The equality

$$(\mathcal{F}T_f)(y) = \langle T_f(x), e^{ixy}\rangle = \int_{R^n} f(x) e^{ixy}\, dx$$

concludes the proof. □

Now, in order to obtain an inversion formula for the Fourier transform over the space $\mathcal{K}_{p,k}$, we need the following lemmas. We denote by $C(a;R)$ the n-tube

$$[a_1-R, a_1+R]\times\cdots\times[a_n-R, a_n+R], \quad a = (a_1, \dots, a_n) \in R^n, \quad R > 0.$$

By applying the methods used by Zemanian [**5**] in the proof of Lemma 3.5-1 and only replacing $(1+|x|^2)^k$ by $e^{k|x|^p}$ in Lemma 2.2 in [**2**], we can obtain the following Lemma 3.2.

Lemma 3.2. *Let* $\phi \in \mathcal{K}_p$ *and* $f \in \mathcal{K}'_{p,k}$, *where* $k \in Z$, $k < 0$, $x \in R^n$. *Then, for any* $Y > 0$,

$$\int_{C(0;Y)} \langle f(x), e^{ixy}\rangle \phi(y)\, dy = \left\langle f(x), \int_{C(0;Y)} \phi(y) e^{ixy}\, dy \right\rangle.$$

Lemma 3.3. *Let* $\phi_1, \dots, \phi_n \in \mathcal{D}(R)$, $x = (x_1, \dots, x_n) \in R$, $t = (t_1, \dots, t_n) \in R$. *Then, for any* $k \in Z$, $k < 0$, *one has*

$$\frac{1}{\pi^n} \int_{R^n} \prod_{j=1}^{n} \phi_j(x_j + t_j) \frac{\sin Y t_j}{t_j}\, dt \quad \longrightarrow \quad \phi_1(x_1) \cdots \phi_n(x_n)$$

in $\mathcal{K}_{p,k}$ *as* $Y \to +\infty$.

Proof. If we only replace $(1+|x|^2)^k$ by $e^{k|x|^p}$, we can first prove exactly like Lemma 3.1 in [**2**] that for $\phi \in \mathcal{D}(R^n)$, and $p \in N$, $\alpha \in R$, $\alpha < 0$, $Y > 0$, then

$$\Psi_Y(x) = \frac{1}{\pi}\int_{\infty}^{\infty} \phi(t+x)\,\frac{\sin Yt}{t}\,dt \in \mathcal{K}_{p,k},$$

and

$$\max_{x\in R} e^{\alpha|x|^p}|D^p\{\Psi_Y(x)-\phi(x)\}| \;\longrightarrow\; 0, \tag{9}$$

for $Y \to +\infty$.

Now note that, since $k < 0$, for any $(x_1, \dots, x_n) \in R$,

$$\begin{aligned}
e^{k|x|^p} &= e^{k(({x_1}^2+{x_2}^2+\cdots+{x_n}^2)^{1/2})^p}\\
&\le e^{k((1/\sqrt{n}(|x_1|+|x_2|+\cdots+|x_n|))^p}\\
&\le \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}\,(|x_1|^p+|x_2|^p+\cdots+|x_n|^p)\right)\\
&= \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}\,|x_1|^p\right)\\
&\quad\times \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}\,|x_2|^p\right)\cdots\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}\,|x_n|^p\right).
\end{aligned}$$

Consider

$$\left|\frac{e^{k|x|^p}}{\pi^n}\,\partial^p\left(\int_{R^n}\prod_{j=1}^n \phi_j(x_j+t_j)\,\frac{\sin Yt_j}{t_j}\,dt - \phi_1(x_1)\cdots\phi_n(x_n)\right)\right| \tag{10}$$

for $x = (x_1, x_2, \dots, x_n) \in R^n$ and $p = (p_1, p_2, \dots, p_n) \in N^n$. Writing, for $j = 1, 2, \dots, n$,

$$\Psi_{j,Y}(x_j) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi_j(x_j+t_j)\frac{\sin Yt_j}{t_j}\,dt,$$

it follows that (10) can be written as

$$\exp(e^{k|x|})\big|\big[\partial_1^{p_1}\Psi_{1,Y}(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n) - \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_n^{p_n}\phi_n(x^n)\big]\big|$$

$$= e^{k|x|^p}\big|\big[\partial_1^{p_1}\Psi_{1,Y}(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)$$
$$- \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)$$
$$+ \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)$$
$$- \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)$$
$$+ \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)$$
$$\cdots$$
$$+ \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_{n-1}^{p_{n-1}}\phi_{n-1}(x_{n-1})\partial_n^{p_n}\Psi_{n,Y}(x_n)$$
$$- \partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_{n-1}^{p_{n-1}}\phi_{n-1}(x_{n-1})\partial_n^{p_n}\phi_n(x_n)\big]\big|$$

$$\leq \left|\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_1|^p\right)\partial_1^{p_1}(\Psi_{1,Y}(x_1)-\phi_1(x_1))\right.$$
$$\left.\times\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_2|^p\right)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_n|^p\right)\right|$$
$$+\left|\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_1|^p\right)\partial_1^{p_1}\phi_1(x_1)\right.$$
$$\times\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_2|^p\right)\partial_2^{p_2}(\Psi_{2,Y}(x_2)-\phi_2(x_2))$$
$$\left.\cdots\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_n|^p\right)\partial_n^{p_n}\Psi_{n,Y}(x_n)\right|$$
$$+\cdots+\left|\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_1|^p\right)\partial_1^{p_1}\phi_1(x_1)\right.$$
$$\left.\times\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_2|^p\right)\partial_2^{p_2}\phi_2(x_2)\right|$$
$$\cdots\left|\exp(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_{n-1}|^p\right)\partial_{n-1}^{p_{n-1}}\phi_{n-1}(x_{n-1})$$
$$\left.\times\exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}}|x_n|^p\right)\partial_n^{p_n}(\Psi_{n,Y}(x_n)-\phi_n(x_n))\right|.$$

By (9) and taking $\alpha = k(2^p)^{n-1}/\sqrt[p]{n}$, it follows that

$$(11) \qquad \max_{x_j \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_j|^p\right) |\partial_j^{p_j}(\Psi_{j,Y}(x_j) - \phi_j(x_j))| \longrightarrow 0,$$

as $Y \to +\infty$, for $1 \le j \le n$. Also, for $1 \le j \le n$,

$$\begin{aligned} &\max_{x_j \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_j|^p\right) |\partial_j^{p_j}(\Psi_{j,Y}(x_j))| \\ &\quad \le \max_{x_j \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_j|^p\right) \partial_j^{p_j} |(\Psi_{j,Y}(x_j) - \phi_j(x_j))| \\ &\quad\quad + \max_{x_j \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_j|^p\right) |\partial_j^{p_j}(\phi_j(x_j))|. \end{aligned}$$

Since $\phi_j \in \mathcal{D}(R^n)$, there exists a $Q_j > 0$ such that

$$\max_{x_j \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_j|^p\right) |\partial_j^{p_j}(\phi_j(x_j))| \le Q_j.$$

Taking into account (11), there exists a $P_j > 0$, $1 \le j \le n$, such that

$$\max_{x_j \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_j|^p\right) |\partial_j^{p_j}(\Psi_{j,Y}(x_j))| \le P_j,$$

for any $Y \ge 0$, and so,

$$\begin{aligned} &q_{k,p}(\Psi_{1,Y}(x_1)\Psi_{2,Y}(x_2)\cdots\Psi_{n,Y}(x_n) - \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)) \\ &\quad \le \max_{x_1 \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_1|^p\right) |\partial_1^{p_1}(\Psi_{1,Y}(x_1) - \phi_1(x_1))| \cdot P_1 \cdots P_n \\ &\quad\quad + Q_1 \cdot \max_{x_2 \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_2|^p\right) \\ &\quad\quad \times |\partial_2^{p_2}(\Psi_{2,Y}(x_2) - \phi_2(x_2))| \cdot P_3 \cdots P_n \\ &\quad\quad + \cdots + Q_1 \cdots Q_{n-1} \cdot \max_{x_n \in R} \exp\left(\frac{k(2^p)^{n-1}}{\sqrt[p]{n}} |x_n|^p\right) \\ &\quad\quad \times |\partial_n^{p_n}(\Psi_{n,Y}(x_n) - \phi_n(x_n))| \end{aligned}$$

By using (11), we obtain the result. □

Theorem 3.4. *Let $f \in \mathcal{K}'_{p,k}$, $k \in Z$, $k < 0$, and set by $F(y) = (\mathcal{F}f)(y)$, $y \in R^n$. Then for any $\phi_1, \phi_2, \dots, \phi_n \in \mathcal{D}(R)$, $t = (t_1, t_2, \dots, t_n) \in R^n$, and $\phi(t) = \phi_1(t_1)\phi_2(t_2)\cdots\phi_n(t_n)$, one has*

$$\langle f(t), \phi(t)\rangle = \lim_{Y\to+\infty} \left\langle \frac{1}{(2\pi)^n} \int_{C(0;Y)} F(y)e^{-ity}\, dy, \phi(t)\right\rangle.$$

Proof. Applying Fuming's theorem, Lemma 3.2 and Lemma 3.3,

$$\begin{aligned}
&\left\langle \frac{1}{(2\pi)^n} \int_{C(0;Y)} F(y)e^{-ity}\, dy,\ \phi(t)\right\rangle \\
&= \left\langle \frac{1}{(2\pi)^n} \int_{C(0;Y)} \langle f(x),\, e^{ixy}\rangle e^{-ity}\, dy,\ \phi(t)\right\rangle \\
&= \frac{1}{(2\pi)^n} \int_{R^n} \phi(t)\, dt \int_{C(0;Y)} \langle f(x),\, e^{ixy}\rangle e^{-ity}\, dy \\
&= \frac{1}{(2\pi)^n} \int_{C(0;Y)} \langle f(x),\, e^{ixy}\rangle\, dy \int_{R^n} \phi(t)e^{-ity}\, dt \\
&= \left\langle f(x), \frac{1}{(2\pi)^n} \int_{C(0;Y)} e^{ixy}\, dy \int_{R^n} \phi(t)e^{-ity}\, dt\right\rangle \\
&= \left\langle f(x), \frac{1}{(2\pi)^n} \int_{C(0;Y)} e^{i(x-t)y}\, dy \int_{R^n} \phi_1(t_1)\cdots\phi_n(t_n)\, dt\right\rangle \\
&= \left\langle f(x), \frac{1}{\pi^n} \int_{R^n} \prod_{j=1}^{n} \phi_j(x_j+t_j)\frac{\sin Yt_j}{t_j}\, dt\right\rangle \\
&\longrightarrow\ \langle f(x),\ \phi(x)\rangle
\end{aligned}$$

as $Y \to +\infty$. □

Let $f, g \in \mathcal{K}'_{p,2^pk}$, $k \in Z$, $k < 0$ and $F(y) = G(y)$, for any $y \in R^n$, where $F(y) = (\mathcal{F}f)(y)$, and $G(y) = (\mathcal{F}g)(y)$. Then, using Lemma 3.3, we have

$$\langle f(x), \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle = \langle g(x), \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle,$$

for all $\phi_1, \phi_2, \cdots \phi_n \in \mathcal{D}(R)$. Let $\phi \in \mathcal{D}(R^n)$. By [**3**, Proposition 1, p. 369], there exists a sequence whose terms are products of the form

$\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}$, being $\phi_{i_j} \in \mathcal{D}(R)$, for $j = 1, 2, \ldots, n$ and $i_j \in N$, which converges to $\phi \in \mathcal{D}(R^n)$. Since convergence in $\mathcal{D}$ implies convergence in $\mathcal{K}_{p,2^pk}$, it follows that $\langle f, \phi \rangle = \langle g, \phi \rangle$ for any $\phi \in \mathcal{D}(R^n)$. Since $\mathcal{D}$ is dense in $\mathcal{K}_{p,2^pk}$, it follows that $f = g$ in $\mathcal{K}'_{p,2^pk}$. Also, for all $y \in R^n$,

$$\begin{aligned}(\mathcal{F}(f \otimes g))(y) &= \langle (f \otimes g)(x), e^{ixy} \rangle \\ &= \langle f(t), \langle g(x), e^{iy(x+t)} \rangle \rangle \\ &= \langle f(t), e^{ity} \rangle \langle g(x), e^{ixy} \rangle \\ &= F(y) \cdot G(y).\end{aligned}$$

Hence it follows that for $f, g, h \in \mathcal{K}'_{p,2^pk}$, $k \in Z$, $k < 0$,

$$f \otimes g = g \otimes f$$

and

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h$$

in $\mathcal{K}'_{p,2^pk}$. Furthermore the Dirac delta belongs to $\mathcal{K}'_{p,2^pk}$ and

$$f \otimes \delta = \delta \otimes f = f.$$

This shows that $\mathcal{K}'_{p,2^pk}$, $k \in Z$, $k < 0$ is a commutative convolution algebra with unit element.

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