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THE TRANSCENDENCE OF CERTAIN INFINITE SERIES

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ABSTRACT. The paper deals with two criteria concerning the transcendence of the sums of infinite series. The terms of these series consist of positive rational numbers which converge rapidly to zero. Some examples are provided which make use of sequences defined recursively.

1. Introduction. There are several methods one may use to prove the transcendence of infinite series. One of them is Mahler's method. A nice survey of such results can be found in Nishioka's book [**5**]. Another technique makes use of the Roth's theorem [**8**]. One can also use Nyblom's theorem which can be found in [**6**]. Later he proved in [**7**] that if $\lambda > 2$ is a fixed real number and $\{a_k\}_{k=1}^{\infty}$ is a sequence of integers greater than unity and such that

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k^{\lambda+1}} > 1 \tag{1}$$

then the series $\sum_{k=1}^{\infty} 1/a_k$ converges to a transcendental number. If we want to describe a general criterion for the transcendence of series which converge quickly then it is useful to introduce the concept of transcendental sequences.

Definition 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the number $\sum_{n=1}^{\infty} 1/(a_n c_n)$ is transcendental, then the sequence $\{a_n\}_{n=1}^{\infty}$ is called transcendental.

This definition is due to Hančl [**3**]. Some criteria for transcendental sequences can be found in the same paper or in [**4**]. One interesting

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result concerning Liouville series (which are special cases of transcendental series) is due to Erdős [**2**]. The transcendence of other special series are proved in [**1**] also.

2. Main results. The main results of this paper are two theorems which show that sums of certain infinite series are transcendental. The following theorem is similar to Sándor's Proposition 1 published in [**9**] except using Roth's theorem at the end.

Theorem 2.1. *Let δ be a positive real number. Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be two sequences of positive integers such that*

$$\limsup_{k\to\infty} \frac{a_{k+1}}{(a_1 a_2 \dots a_k)^{2+\delta}} \cdot \frac{1}{b_{k+1}} = \infty \tag{2}$$

and

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \cdot \frac{b_k}{b_{k+1}} > 1. \tag{3}$$

Then the number

$$\xi = \sum_{k=1}^{\infty} \frac{b_k}{a_k} < \infty$$

is transcendental.

Example 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = 3$ and for every $n = 1, 2, \dots$

$$a_{n+1} = \begin{cases} (a_1 a_2 \dots a_n)^3 & \text{if } 25 \mid n \\ 2a_n & \text{otherwise.} \end{cases}$$

Let us take $\delta = 1/2$ in Theorem 2.1. Then we obtain the fact that the number

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is transcendental.

Remark. Nyblom's condition (1) together with Lemma 2.1 in [**7**] imply the conditions of Theorem 2.1 in the case $b_n = 1$.

Theorem 2.2. *Let δ and ε be positive real numbers. Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be two sequences of positive integers such that*

$$\limsup_{k\to\infty} \frac{a_{k+1}}{(a_1a_2\dots a_k)^{2+2/\varepsilon+\delta}} \cdot \frac{1}{b_{k+1}} = \infty, \tag{4}$$

and for every sufficiently large k

$$\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}} \geq \sqrt[1+\varepsilon]{\frac{a_k}{b_k}} + 1. \tag{5}$$

Then the number

$$\xi = \sum_{k=1}^{\infty} \frac{b_k}{a_k} < \infty$$

is transcendental.

Example 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = 11$ and for every $n = 1, 2, \dots$

$$a_{n+1} = \begin{cases} (a_1a_2\dots a_n)^6 & \text{if } 100 \mid n, \\ [a_n + 2\sqrt{a_n} + 2] & \text{otherwise.} \end{cases}$$

If we substitute $\delta = \varepsilon = 1$ in Theorem 2.2 we obtain that the number

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is transcendental.

Open problem. Is there any sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers and a positive real number ε such that $a_n > 2^{(2+\varepsilon)^n}$ and the number $\sum_{n=1}^{\infty} 1/a_n$ is algebraic?

3. Proofs.

Proof (*of Theorem* 2.1). Let M be a sufficiently large positive real number. Equation (2) implies that there exist infinitely many positive integers k such that

$$\frac{1}{M(a_1a_2\dots a_k)^{2+\delta}} > \frac{b_{k+1}}{a_{k+1}}\,. \tag{6}$$

From (3) we obtain that for each real number $A > 1$ we can find a positive integer k_0 such that for every k with $k > k_0$

$$\frac{1}{A} \cdot \frac{b_k}{a_k} > \frac{b_{k+1}}{a_{k+1}} .$$

From this and using mathematical induction we get for every k with $k > k_0$

$$\frac{1}{A^p} \cdot \frac{b_k}{a_k} > \frac{b_{k+p}}{a_{k+p}} .$$

This implies that for infinitely many positive integers k

$$\begin{aligned} \left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| &= \left|\sum_{i=1}^{\infty} \frac{b_i}{a_i} - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| = \left|\sum_{i=k+1}^{\infty} \frac{b_i}{a_i}\right| \\ &= \left|\frac{b_{k+1}}{a_{k+1}} + \frac{b_{k+2}}{a_{k+2}} + \cdots\right| \\ &< \left|\frac{b_{k+1}}{a_{k+1}} \cdot \left(1 + \frac{1}{A} + \frac{1}{A^2} + \cdots\right)\right| \\ &= \left|\frac{b_{k+1}}{a_{k+1}} \cdot \frac{A}{A-1}\right| . \end{aligned}$$

From this, (6) and the fact that M is sufficiently large, $(M > A/(A–1))$, we obtain

$$\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| < \left|\frac{b_{k+1}}{a_{k+1}} \cdot \frac{A}{A-1}\right| \le \left|\frac{M}{M(a_1 a_2 \dots a_k)^{2+\delta}}\right| = \frac{1}{(a_1 a_2 \dots a_k)^{2+\delta}} .$$

Consider a sequence of rational approximations p_k/q_k to ξ generated from the k-th partial sums, expressed in reduced form. Since $(p_k, q_k) = 1$ the lowest common multiple of the k integers in the set $\{a_1, a_2 \dots, a_k\}$, denoted $lcm\{a_1, a_2 \dots, a_k\}$, must be greater than or equal to q_k but as $a_1 a_2 \cdots a_n \ge lcm\{a_1, a_2 \dots, a_k\}$, we have that $q_k \le a_1 a_2 \cdots a_n$. Thus from the previous inequality we deduce that

$$\left|\xi - \frac{p_k}{q_k}\right| = \left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| < \frac{1}{(a_1 a_2 \cdots a_k)^{2+\delta}} < \frac{1}{q_k^{2+\delta}}$$

and this is valid for infinitely many positive integers k. From this and Roth's theorem we obtain the transcendence of the number ξ. $\square$

Proof (*of Theorem* 2.2). Let M be a sufficiently large positive real number. From (4) we obtain that for infinitely many k

$$\frac{1}{M(a_1 a_2 \dots a_k)^{2+2/\varepsilon+\delta}} > \frac{b_{k+1}}{a_{k+1}}.$$

Hence

$$\frac{1}{M^{\varepsilon/(1+\varepsilon)}(a_1 a_2 \dots a_k)^{2+\delta\varepsilon/(1+\varepsilon)}} > \left(\frac{b_{k+1}}{a_{k+1}}\right)^{\varepsilon/(1+\varepsilon)}. \tag{7}$$

Now by induction and using (5) we have for every sufficiently large k and every positive integer s

$$\sqrt[1+\varepsilon]{\frac{a_{k+s}}{b_{k+s}}} \geq \sqrt[1+\varepsilon]{\frac{a_k}{b_k}} + s.$$

Thus

$$\frac{a_{k+s}}{b_{k+s}} \geq \left(\sqrt[1+\varepsilon]{\frac{a_k}{b_k}} + s\right)^{1+\varepsilon}. \tag{8}$$

We also have for all real $z > 1$

$$\sum_{r=0}^{\infty} \frac{1}{(z+r)^{1+\varepsilon}} < \int_{z-1}^{\infty} \frac{dx}{x^{1+\varepsilon}} = \frac{1}{\varepsilon(z-1)^{\varepsilon}}. \tag{9}$$

Now we use a method analogous to that used in the proof of Theorem 2.1. From (8) and (9) we obtain that for infinitely many k

$$\begin{aligned}\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| = \left|\sum_{i=k+1}^{\infty} \frac{b_i}{a_i}\right| &\leq \left(\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}}\right)^{-(1+\varepsilon)} \\ &\quad + \left(\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}} + 1\right)^{-(1+\varepsilon)} + \cdots \\ &< \sum_{r=0}^{\infty}\left(\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}} + r\right)^{-(1+\varepsilon)} \\ &< \frac{1}{\varepsilon}\left(\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}} - 1\right)^{-\varepsilon}.\end{aligned}$$

Inequality (5) implies that $\lim_{k\to\infty}(a_k/b_k) = \infty$. Hence there exists a positive real constant c which does not depend on k such that

$$\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| < \frac{1}{\varepsilon\left(\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}} - 1\right)^{\varepsilon}} < \frac{c}{\varepsilon\left(\sqrt[1+\varepsilon]{\frac{a_{k+1}}{b_{k+1}}}\right)^{\varepsilon}} = \frac{c}{\varepsilon} \cdot \left(\frac{b_{k+1}}{a_{k+1}}\right)^{\varepsilon/(1+\varepsilon)}. \tag{10}$$

Let p_k/q_k be a sequence of rational approximations to ξ where $(p_k, q_k) = 1$. From this, (10) and (7) we obtain for infinitely many positive integers k that

$$\begin{aligned}\left|\xi - \frac{p_k}{q_k}\right| = \left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| &< \frac{c}{\varepsilon} \cdot \left(\frac{b_{k+1}}{a_{k+1}}\right)^{\varepsilon/(1+\varepsilon)} \\ &< \frac{c}{\varepsilon M^{\varepsilon/(1+\varepsilon)}} \cdot \frac{1}{(a_1 a_2 \dots a_k)^{2+\delta\varepsilon/(1+\varepsilon)}} \\ &< \frac{1}{q_k^{2+\delta\varepsilon/(1+\varepsilon)}}\end{aligned}$$

since M may be chosen a sufficiently large real number and again as in Theorem 2.1, $q_k < a_1 a_2 \cdots a_k$. This and Roth's theorem imply that the number ξ is transcendental. □

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