ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

APPROXIMATION OF SOBOLEV-TYPE CLASSES WITH QUASI-SEMINORMS

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ABSTRACT. Since the Sobolev set W_p^r, $0 < p < 1$, in general is not contained in L_q, $0 < q \le \infty$, we limit ourselves to the set $W_p^r \cap L_\infty$, $0 < p < 1$. We prove that the Kolmogorov n-width of the latter set in L_q, $0 < q < 1$ is asymptotically 1, that is, the set cannot be approximated by n-dimensional linear manifolds in the L_q-norm. We then describe a related set, the width of which is asymptotically n^{-r}.

1. Introduction and function classes. Very little is known about the exact order of any width of nontrivial classes of functions in the L_q-metric for $0 < q < 1$. Recall that, for $1 \le p, q \le \infty$, the orders of most widths of the classical Sobolev classes W_p^r in L_q are well known. In contrast, for $0 < p < 1$, the behavior of any of the widths of these classes in L_q, $0 < q \le \infty$, are not known. In general, the class W_p^r, $0 < p < 1$, is not contained in L_q, but even if we overcome this difficulty by taking, say, the smaller set $W_p^r \cap L_\infty$, $0 < p < 1$, we will show that it cannot be approximated well in L_q for any $0 < q \le \infty$. We remind the reader that, for the approximation of $f \in L_p$, $0 < p < 1$, by polynomials and by splines with either equidistant knots or knots on the Chebyshev partition, there are known Jackson-type estimates involving the moduli of smoothness of f in the L_p-quasi-norm, see, e.g., [**1**]. However, there are no simple relations between the moduli of smoothness and the derivatives of f, if they exist. Moreover, the moduli of smoothness are not equivalent to K-functionals which are identically zero, see, e.g., [**3**, Theorem 2.1]. Thus, we introduce new classes V_p^r, $0 < p < 1$, which we feel are the proper replacement of the Sobolev classes for $0 < p < 1$, and we obtain the exact orders of their Kolmogorov, linear, and pseudo-dimensional widths in L_q, $0 < q < 1$. We also obtain for these classes exact orders of best approximation in L_q, $0 < q < 1$, by rational functions and free-knot splines.

1991 AMS *Mathematics Subject Classification.* Primary 41A46.
Key words and phrases. n-widths in L_q, $0 < q < 1$, Sobolev type classes.
Received by the editors on March 3, 2003, and in revised form on September 17, 2003.

Let $I = (a,b)$ be a finite open finite interval, $r \in \mathbf{N}$, and $0 < p \le \infty$. By $\mathcal{W}_p^r := \mathcal{W}_p^r(I)$ we denote the usual Sobolev space of all functions $x : I \to \mathbf{R}$ such that $x^{(r-1)} \in AC_{loc}(I)$ equipped with the (quasi-)seminorm

$$\|x\|_{\mathcal{W}_p^r} := \|x^{(r)}\|_{L_p}.$$

In Section 2 we state our result on estimates of various widths of the subset

$$W_{p,\infty}^r := \left\{ x \in \mathcal{W}_p^r \mid \sum_{s=0}^{r} \|x^{(s)}\|_{L_p} \le 1, \quad \|x\|_{L_\infty} \le 1 \right\}, \quad 0 < p < 1,$$

in L_q, $0 < q < 1$. We show that they stay away from 0, as $n \to \infty$.

For $r \in \mathbf{N}$, $0 < p \le \infty$, we denote by $\mathcal{V}_p^r := \mathcal{V}_p^r(I)$, the space of all functions $x : I \to \mathbf{R}$ such that $x^{(r-1)} \in AC_{loc}(I)$ for which the (quasi-)seminorm

$$\|x\|_{\mathcal{V}_p^r} := \begin{cases} \left(\int_I \left| \int_{t_0}^t |x^{(r)}(\tau)| \, d\tau \right|^p dt \right)^{1/p}, & 0 < p < \infty, \\ \sup_{t \in I} \left| \int_{t_0}^t |x^{(r)}(\tau)| \, d\tau \right|, & p = \infty, \end{cases}$$

where t_0 is the midpoint of I, is finite. In Section 2 we give estimates of various widths of the unit ball V_p^r of $\mathcal{V}_p^r$, in L_q, $0 < q < 1$. We show that they tend to 0 when $n \to \infty$.

After a section of auxiliary lemmas, we prove the two main results in Sections 4 and 5. Finally in Section 6 we discuss the inclusion and noninclusion relations between $\mathcal{V}_p^r$ and $\mathcal{W}_p^r$.

2. Various widths and the main results. Let X be a real linear space of vectors x with norm $\|x\|_X$ and W any nonempty subset in X. Recall that the Kolmogorov n-width of W is defined by

$$d_n(W)_X^{kol} := \inf_{M^n} \sup_{x \in W} \inf_{y \in M^n} \|x - y\|_X,$$

where the lefthand infimum is taken over all affine subsets M^n of (algebraic) dimension $\le n$. The linear n-width of W is defined by

$$d_n(W)_X^{lin} := \inf_{M^n} \inf_{A} \sup_{x \in W} \|x - Ax\|_X,$$

where the lefthand infimum is taken over all affine subsets M^n of dimension $\le n$, and the middle infimum is taken over all linear continuous maps A from affine subsets $M = M(W)$ containing W into M^n.

Finally, we will also have estimates for yet another width, the pseudo-dimensional width which was introduced by Maiorov and Ratsaby [**7**–**9**], using the concept of pseudo-dimension due to Pollard [**12**]. Namely, let $M = M(T)$ be a set of real-valued functions $x(t)$ defined on the set T, and denote

$$\mathrm{Sgn}\ a := \begin{cases} 1 & a > 0 \\ 0 & a \le 0. \end{cases}$$

The pseudo-dimension $\dim_{ps} M$ of the set M is the largest integer n such that there exist points $t_1, \dots, t_n \in T$ and a vector $(y_1, \dots, y_n) \in \mathbf{R}^n$, for which

$$\mathrm{card}\,\{(\mathrm{Sgn}\,(x(t_1) + y_1), \dots, \mathrm{Sgn}\,(x(t_n) + y_n)) \mid x \in M\} = 2^n.$$

If n can be arbitrarily large, then $\dim_{ps} M := \infty$.

The pseudo-dimensional n-width of W is defined by

$$d_n(W)_X^{psd} := \inf_{M^n} \sup_{x \in W} \inf_{y \in M^n} \|x - y\|_X,$$

where the lefthand infimum is taken over all subsets M^n in a normed space X of real-valued functions such that $\dim_{ps} M^n \le n$.

The following properties of the pseudo-dimension are known, see [**4**].

If M is an arbitrary affine subset in a space of real-valued functions and $\dim M < \infty$, then

$$\dim_{ps} M = \dim M. \tag{2.1}$$

Let $P_n := P_n(I)$ be the space of algebraic polynomials p_n of degree $\le n$. Denote by $R_n := R_n(I)$ the manifold of rational functions $r_n = p_n/q_n$ where $p_n, q_n \in P_n$. Also denote by $\Sigma_{r,n} = \Sigma_{r,n}(I)$, the manifold of all piecewise polynomials $\sigma_{r,n}$, of order r and with $n-1$ knots in I, i.e., $\sigma_{r,n} \in \Sigma_{r,n}$, if for some points $a = t_0 < t_1 < \cdots < t_n = b$ it is a polynomial of degree $\le r-1$ on each interval (t_{i-1}, t_i), $i = 1, \dots, n$.

The rational functions r_n are defined arbitrarily at the poles, and the piecewise polynomials $\sigma_{r,n}$ are assigned arbitrary values at the knots.

It is known that

$$\dim_{ps} R_n \asymp \dim_{ps} \Sigma_{r,n} \asymp n. \tag{2.2}$$

It follows by (2.1) that if W is a nonempty subset of X, a normed space of real-valued functions, then

$$d_n(W)_X^{psd} \le d_n(W)_X^{kol} \le d_n(W)_X^{lin}. \tag{2.3}$$

Given $W \subset X$, let

$$E(W, R_n)_X := \sup_{x \in W} \inf_{r_n \in R_n} \|x - r_n\|_X,$$
$$E(W, \Sigma_{r,n})_X := \sup_{x \in W} \inf_{\sigma_{r,n} \in \Sigma_{r,n}} \|x - \sigma_{r,n}\|_X.$$

It follows from (2.2) that there exist an absolute integer $\alpha > 0$ and an integer $\beta = \beta(r) > 0$, such that

$$d_{\alpha n}(W)_X^{psd} \le E(W, R_n)_X, \tag{2.4}$$

$$d_{\beta n}(W)_X^{psd} \le E(W, \Sigma_{r,n})_X. \tag{2.5}$$

We are ready to state our first result.

Theorem 1. *Let $r \in \mathbf{N}$ and $0 < p < 1$. For any $0 < q \le \infty$,*

$$d_n(W^r_{p,\infty})^{psd}_{L_q} \asymp d_n(W^r_{p,\infty})^{kol}_{L_q} \asymp d_n(W^r_{p,\infty})^{lin}_{L_q} \asymp 1, \tag{2.6}$$

and

$$E\big(W^r_{p,\infty}, \Sigma_{r,n}\big)_{L_q} \asymp E(W^r_{p,\infty}, R_n)_{L_q} \asymp 1. \tag{2.7}$$

On the other hand we show

Theorem 2. *Let $r \in \mathbf{N}$ and $0 < p, q < 1$, be such that $r - 1 - 1/p + 1/q > 0$. Then*

$$d_n(V^r_p)^{psd}_{L_q} \asymp d_n(V^r_p)^{kol}_{L_q} \asymp d_n(V^r_p)^{lin}_{L_q} \asymp n^{-r}, \tag{2.8}$$

and

$$E(V_p^r, \Sigma_{r,n})_{L_q} \asymp E(V_p^r, R_n)_{L_q} \asymp n^{-r}. \tag{2.9}$$

3. Auxiliary lemmas. The following lemma follows immediately from [**6**, Lemma 2.2, p. 489], also see [**9**, Claim 1].

Lemma A. *Let $m \in \mathbf{N}$ and $V_m := \{v \mid v := (v_1, \dots, v_m), v_i = \pm 1, i = 1, \dots, m\}$. Then there exists a subset $F_m \subset V_m$ of cardinality $\geq 2^{m/16}$ such that for any $\hat{v}, \check{v} \in F_m$, where $\hat{v} \neq \check{v}$, the distance $\|\hat{v} - \check{v}\|_{l_1^m} \geq m/2$.*

Given $\varepsilon > 0$, points x_i, $i = 1, \dots, n$, in a linear normed space X are called ε-distinguishable if $\|x_i - x_j\|_X \geq \varepsilon$ for all $i \neq j$. Let H be any nonempty subset of X, the maximal integer $n \in \mathbf{N}$, such that there exist n ε-distinguishable points $h_i \in H$, is called the ε-packing number $M_\varepsilon(H)_X$ of H in X. If n can be arbitrarily large, then $M_\varepsilon(H)_X := \infty$.

The next lemma follows directly from [**5**, Corollary 3], also see [**9**, Lemma 1].

Lemma B. *Let $H_{n,a} := \{h\}$ be a set of Lebesgue-measurable functions h on $(0,1)$ such that $\|h\|_{L_\infty} \leq a < \infty$ and $\dim_{ps} H_{n,a} \leq n < \infty$. Then for any $\varepsilon > 0$,*

$$M_\varepsilon(H_{n,a})_{L_1} \leq e(n+1)(4ea/\varepsilon)^n.$$

We prove the following

Lemma 1. *Let $I := (0,1)$, and let $a > 0$, $\varepsilon > 0$, and $m \in \mathbf{N}$, such that $m \geq 16(8 + \log_2(a/\varepsilon))$, be given. Suppose that a set $\Phi_m = \{\varphi\} \subset L_\infty$ exists, of cardinality $\geq 2^{m/16}$ such that*

$$\|\varphi\|_{L_\infty} \leq a, \quad \varphi \in \Phi_m,$$

and for some $0 < q < 1$,

$$\|\hat{\varphi} - \check{\varphi}\|_{L_q} \geq \varepsilon, \quad \hat{\varphi} \neq \check{\varphi}, \quad \hat{\varphi}, \check{\varphi} \in \Phi_m.$$

Then for any $n \in \mathbf{N}$ *such that* $n \le \big(16(8+\log_2(a/\varepsilon))\big)^{-1} m$ *we have*

$$d_n(\Phi_m)^{psd}_{L_q} \ge 2^{-2-1/q}(2^q-1)^{1/q}\varepsilon.$$

Proof. Let $H_n \subset L_q$ be such that $\dim_{ps} H_n \le n$. Denote

$$\delta := E(\Phi_m, H_n)_{L_q}. \tag{3.1}$$

With any $\varphi \in \Phi_m$ we associate an element $h_\delta(\varphi;\cdot) \in H_n$, such that

$$\|\varphi(\cdot) - h_\delta(\varphi;\cdot)\|_{L_q} \le 2\delta, \tag{3.2}$$

and denote by

$$H_{\delta,n} := H_{\delta,n}(I) := \{h_\delta(\varphi;\cdot), \varphi \in \Phi_m\},$$

the collection of these functions. Now we let

$$h_{\delta,a}(\varphi;t) := \begin{cases} -a & \text{for } t : h_\delta(\varphi;t) < -a, \\ h_\delta(\varphi;t) & \text{for } t : |h_\delta(\varphi;t)| \le a, \\ a & \text{for } t : h_\delta(\varphi;t) > a, \end{cases}$$

and denote by

$$H_{\delta,n,a} := H_{\delta,n,a}(I) := \{h_{\delta,a}(\varphi;\cdot), \varphi \in \Phi_m\},$$

the collection of the truncated functions. Clearly

$$\|h_{\delta,a}(\varphi;\cdot)\|_{L_\infty} \le a, \quad \varphi \in \Phi_m, \tag{3.3}$$

and

$$\dim_{ps} H_{\delta,n,a} \le \dim_{ps} H_{\delta,n} \le \dim_{ps} H_n \le n. \tag{3.4}$$

We will prove that

$$\delta > 2^{-2-1/q}(2^q-1)^{1/q}\varepsilon. \tag{3.5}$$

Assume to the contrary that

$$\delta \le 2^{-2-1/q}(2^q-1)^{1/q}\varepsilon, \tag{3.6}$$

where δ is defined by (3.1). Then, recalling that $0 < q \le 1$, we have

$$\|h_{\delta,a}(\hat\varphi;\cdot) - h_{\delta,a}(\check\varphi;\cdot))\|_{L_q}^q \ge \|\hat\varphi - \check\varphi\|_{L_q}^q - \|\hat\varphi(\cdot) - h_{\delta,a}(\hat\varphi;\cdot)\|_{L_q}^q - \|\check\varphi(\cdot) - h_{\delta,a}(\check\varphi;\cdot)\|_{L_q}^q. \tag{3.7}$$

Since $|\hat\varphi(t)| \le a$ and $|\check\varphi(t)| \le a$, $t \in I$, (3.2) implies

$$\|\hat\varphi(\cdot) - h_{\delta,a}(\hat\varphi;\cdot)\|_{L_q}^q \le \|\hat\varphi(\cdot) - h_{\delta}(\hat\varphi;\cdot)\|_{L_q}^q \le 2^q\delta^q,$$

and

$$\|\check\varphi(\cdot) - h_{\delta,a}(\check\varphi;\cdot)\|_{L_q}^q \le \|\check\varphi(\cdot) - h_{\delta}(\check\varphi;\cdot)\|_{L_q}^q \le 2^q\delta^q,$$

which, substituting in (3.7), yields

$$\|h_{\delta,a}(\hat\varphi;\cdot) - h_{\delta,a}(\check\varphi;\cdot))\|_{L_q}^q \ge \|\hat\varphi - \check\varphi\|_{L_q}^q - 2^{q+1}\delta^q \ge 2^{-q}\varepsilon^q. \tag{3.8}$$

Setting $\eta := \varepsilon/2$, we see from (3.8) that the function class $H_{\delta,n,a}$ consists of η-distinguishable functions in L_q. Thus, in view of $\|x\|_{L_1} \ge \|x\|_{L_q}$, $0 < q \le 1$, we conclude that the function class $H_{\delta,n,a}$ contains at least $2^{m/16}$ η-distinguishable functions in L_1. On the other hand, by virtue of (3.3), $\|h_{\delta,a}(\phi;\cdot)\|_{L_\infty} \le a$. Hence by Lemma B we have an upper estimate on the η-packing number $M_\eta(H_{\delta,n,a})_{L_1}$ of the function class $H_{\delta,n,a}$, namely,

$$\begin{aligned} M_\eta(H_{\delta,n,a})_{L_1} &\le e(n+1)(4ea/\eta)^n = e(n+1)\big(4e2a/\varepsilon\big)^n \\ &< 2^{3n}\big(2^5a/\varepsilon\big)^n = 2^{(8+\log_2(a/\varepsilon))n}. \end{aligned}$$

Since $m \ge 16(8+\log_2(a/\varepsilon))n$, it follows that

$$2^{(8+\log_2(a/\varepsilon))n} \le M_\eta(H_{\delta,n,a})_{L_1} < 2^{(8+\log_2(a/\varepsilon))n},$$

a contradiction. Thus (3.6) is contradicted and (3.5) is valid. Hence for any subset $H_n \in L_q$ with $\dim_{ps} H_n \le n$, we have

$$E(\Phi_m, H_n)_{L_q} > 2^{-2-1/q}(2^q-1)^{1/q}\varepsilon,$$

and in turn

$$d_n(\Phi_m)_{L_q}^{psd} \ge 2^{-2-1/q}(2^q-1)^{1/q}\varepsilon.$$

This completes the proof of Lemma 1. □

Lemma 2. *Let $0 < p < 1$, and for $b_i > 0$, $i = 1, \dots, n$, let*

$$\delta_{p,i} := \left(\sum_{j=i}^{n} b_j^p\right)^{1/p} - \left(\sum_{j=i+1}^{n} b_j^p\right)^{1/p}, \quad 1 \le i \le n-1, \quad \delta_{p,n} := b_n.$$

Denote

$$T_{p,n} := \left\{ t := (t_1, \dots, t_n) \mid 0 \le t_1 \le \dots \le t_n, \sum_{i=1}^{n} (b_i t_i)^p \le 1 \right\},$$

and

$$S_{p,n} := \left\{ t := (t_1, \dots, t_n) \mid 0 \le t_1 \le \dots \le t_n, \sum_{i=1}^{n} \delta_{p,i} t_i \le 1 \right\}.$$

If

$$l_{p,n}(t) := \sum_{i=1}^{n} \delta_{p,i} t_i, \quad t \in \mathbf{R}^n,$$

then

$$\max_{t \in T_{p,n}} l_{p,n}(t) = 1, \tag{3.9}$$

and consequently $T_{p,n} \subseteq S_{p,n}$.

Proof. We consider the extremal problem

$$l_{p,n}^p(t) = \left(\sum_{i=1}^{n} \delta_{p,i} t_i\right)^p \longrightarrow \sup; \quad 0 \le t_1 \le \dots \le t_n, \quad \sum_{i=1}^{n} (b_i t_i)^p \le 1.$$

Denote $\tau_i := t_i^p, i = 1, \dots, n$, and let $\tau := (\tau_1, \dots, \tau_n)$. Then we get an equivalent extremal problem,

$$f_{p,n}(\tau) := \left(\sum_{i=1}^{n} \delta_{p,i} \tau_i^{1/p}\right)^p \longrightarrow \sup; \quad 0 \le \tau_1 \le \dots \le \tau_n, \quad \sum_{i=1}^{n} b_i^p \tau_i \le 1.$$

By Minkowski's inequality it is easy to verify that $f_{p,n}$ is convex. Therefore it achieves its maximum on the vertices of

$$Q_{p,n} := \left\{ \tau \,\middle|\, 0 \le \tau_1 \le \dots \le \tau_n, \sum_{i=1}^{n} b_i^p \tau_i \le 1 \right\}.$$

If $e^{(0)} := (0,\dots,0)$, $e^{(1)} := (1,1,\dots,1)$, $e^{(2)} := (0,1,\dots,1),\dots,e^{(n)} := (0,\dots,0,1)$, then these vertices are

$$\tau^{(0)} = e^{(0)}, \quad \tau^{(k)} := \left(\sum_{j=k}^{n} b_j^p\right)^{-1} e^{(k)}, \quad k = 1,\dots,n.$$

Since

$$f_{p,n}(\tau^{(0)}) = 0, \quad f_{p,n}(\tau^{(k)}) = 1, \quad k = 1,\dots,n,$$

we conclude that

$$\max_{\tau\in Q_{p,n}} f_{p,n}(\tau) = \max_{t\in T_{p,n}} l_{p,n}(t) = 1.$$

This completes the proof. □

Lemma 3. *Let* $0 < p, q < 1$ *and* $b_i > 0$, $i = 1,\dots,n$. *Denote*

$$\Theta_{p,n} := \left\{\theta := (\theta_1,\dots,\theta_n) \,\Big|\, \theta_i \ge 0, 1 \le i \le n, \sum_{i=1}^{n}\left(b_i \sum_{j=1}^{i}\theta_j\right)^p \le 1\right\}.$$

For $a_i \ge 0$, $1 \le i \le n$, *let*

$$f_{q,n}(\theta) := \left(\sum_{i=1}^{n}(a_i\theta_i)^q\right)^{1/q}, \quad \theta \in \mathbf{R}_+^n.$$

Then

$$\max_{\theta\in\Theta_{p,n}} f_{q,n}(\theta) \le n^{1/q-1} \max_{1\le i\le n} a_i \left(\sum_{j=i}^{n} b_j^p\right)^{-1/p}.$$

Proof. The inequality

$$\left(\sum_{i=1}^{n}(a_i\theta_i)^q\right)^{1/q} \le n^{1/q-1}\sum_{i=1}^{n} a_i\theta_i =: g_{q,n}(\theta), \quad \theta \in \Theta_{p,n},$$

follows by the concavity of u^q. Set

$$t_i := \sum_{j=1}^{i}\theta_i, \quad i = 1,\dots,n.$$

Then

$$\theta_1 = t_1, \quad \theta_i = t_i - t_{i-1}, \quad i = 2, \dots, n,$$

and

$$g_{q,n}(\theta) = n^{1/q-1}\biggl(a_1 t_1 + \sum_{i=2}^{n} a_i (t_i - t_{i-1})\biggr) =: h_{q,n}(t).$$

Hence, by Lemma 2,

$$\max_{\theta\in\Theta_{p,n}} g_{q,n}(\theta) = \max_{t\in T_{p,n}} h_{q,n}(t) \le \max_{t\in S_{p,n}} h_{q,n}(t),$$

where $T_{p,n}$ and $S_{p,n}$ were defined in Lemma 2. The function $h_{q,n}$ is linear, thus it achieves its maximum at one of the vertices of the simplex $S_{p,n}$, that is, at $t^{(k)}$, $1 \le k \le n$, where $t^{(0)} := (0, \dots, 0)$, and

$$t^{(k)} := \biggl(\sum_{j=k}^{n} b_j^p\biggr)^{-1/p} e^{(k)}, \quad k = 1 \dots, n.$$

Now $h_{q,n}(\tau^{(0)}) = 0$, and for $k \ge 1$,

$$\tau_i^{(k)} - \tau_{i-1}^{(k)} = \begin{cases} 0 & i \ne k \\ \Bigl(\sum_{j=k}^{n} b_j^p\Bigr)^{-1/p} & i = k, \end{cases}$$

where we take $\tau_0^{(k)} = 0$, $1 \le k \le n$. Hence

$$\max_{t\in S_{p,n}} h_{q,n}(t) = n^{1/q-1} \max_{1\le k\le n} \biggl\{ a_k \biggl(\sum_{j=k}^{n} b_j^p\biggr)^{-1/p} \biggr\}. \qquad \square$$

We need a well-known relation between various quasi-norms of polynomials, see, e.g., [**2**, Chapter 4, Theorem 2.7].

Lemma C. *Let π_{r-1} be a polynomial of degree $\le r-1$, $r \in \mathbf{N}$, and $p, q \ge p_0$. Then there exists a constant $c = c(r, p_0)$ such that for any finite interval J,*

$$\|\pi_{r-1}\|_{L_q(J)} \le c\,|J|^{1/q-1/p} \|\pi_{r-1}\|_{L_p(J)}.$$

Finally, in the proof of (2.9), we use the following relation between the degrees of rational approximation and those of free-knots splines, due to Pekarskii [**10**] and Petrushev [**11**], see also [**6**, Chapter 10, Theorem 6.2].

Lemma D. *Let* $r \in \mathbf{N}$*,* $0 < p < \infty$*,* $\lambda > 0$*,* $\gamma = \min\{1, p\}$*, and* $x \in L_p$*. Then*

$$E(x, R_n)_{L_p} \le cn^{-\lambda}\left(\sum_{k=1}^{n} k^{-1}\big(k^{\lambda}E\big(x, \Sigma_{r,k}\big)_{L_p}\big)^{\gamma}\right)^{1/\gamma},$$

where $c = c(r, p, \lambda)$*.*

4. Proof of Theorem 1. The upper bound in (2.6) is trivial. Thus, we prove the lower bounds. To this end, we are going to construct extremal functions.

Let I be the generic interval $(0, 1)$, and fix $r, m \in \mathbf{N}$, and $0 < p < 1$. Let

(4.1) $$\varepsilon_s := \varepsilon_s(p, r, m) := m^{-(1-p)^{s-r}}, \quad s = 0, 1, \dots, r,$$

and set

$$\tau_s := \tau_s(p, r, m) := \sum_{k=0}^{s-1} 2^{s-2-k}\varepsilon_k + \varepsilon_s/2, \quad s = 1, \dots, r.$$

Define

(4.2) $$\phi_0(t) := \phi_0(t; p, r, m) := \begin{cases} m^{(1-(1-p)^r)/p(1-p)^r} & t \in (-\varepsilon_0/2, \varepsilon_0/2), \\ 0 & t \notin (-\varepsilon_0/2, \varepsilon_0/2), \end{cases}$$

and

$$\begin{aligned} \phi_s(t) := \phi_s(t; p, r, m) &:= \int_{-\infty}^{t} (\phi_{s-1}(\tau + \tau_s) - \phi_{s-1}(\tau - \tau_s))\, d\tau \\ &= \int_{t-\tau_s}^{t+\tau_s} \phi_{s-1}(\tau)\, d\tau, \quad t \in \mathbf{R}, \quad s = 1, \dots, r. \end{aligned}$$

It is easy to see that

$$\operatorname{supp}\phi_s = \left[-\sum_{k=0}^{s} 2^{s-1-k}\varepsilon_k, \sum_{k=0}^{s} 2^{s-1-k}\varepsilon_k\right], \quad s = 0,1,\dots,r, \tag{4.3}$$

hence

$$\operatorname{supp}\phi_0 \subset \operatorname{supp}\phi_1 \subset \cdots \subset \operatorname{supp}\phi_r.$$

Since by (4.1) we have $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_r$, it follows from (4.3) that

$$\varepsilon_s \le |\operatorname{supp}\phi_s| \le 2^{s+1}\varepsilon_s, \quad s = 0,1,\dots,r. \tag{4.4}$$

Also, we have

$$\phi_s(t) = \phi_s(-t) \ge 0, \quad t \in \mathbf{R}, \quad s = 0,1,\dots,r, \tag{4.5}$$

and

$$\phi_s(t) \equiv \|\phi_s\|_{L_\infty(\mathbf{R})}, \quad t \in (-\varepsilon_s/2, \varepsilon_s/2), \quad s = 0,1,\dots,r. \tag{4.6}$$

By virtue of (4.4) and (4.6), we obtain

$$\begin{gathered}\|\phi_0\|_{L_\infty(\mathbf{R})} \prod_{k=0}^{s-1}\varepsilon_k \le \|\phi_s\|_{L_\infty(\mathbf{R})} \le 2^{s(s+1)/2}\|\phi_0\|_{L_\infty(\mathbf{R})} \prod_{k=0}^{s-1}\varepsilon_k, \\ s = 0,1,\dots,r.\end{gathered} \tag{4.7}$$

Hence, combining (4.4) through (4.7) we conclude that

$$\begin{aligned}\|\phi_s\|^p_{L_p(\mathbf{R})} &= \int_{\operatorname{supp}\phi_s} |\phi_s(t)|^p\,dt \\ &\le \int_0^{2^{s+1}\varepsilon_s} \|\phi_s\|^p_{L_\infty(\mathbf{R})} dt \\ &\le 2^{s+1}\varepsilon_s 2^{ps(s+1)/2}\|\phi_0\|^p_{L_\infty(\mathbf{R})}\left(\prod_{k=0}^{s-1}\varepsilon_k\right)^p \\ &\le 2^{(s+1)(s+2)/2}\|\phi_0(\cdot)\|^p_{L_\infty(\mathbf{R})}\varepsilon_s\left(\prod_{k=0}^{s-1}\varepsilon_k\right)^p.\end{aligned} \tag{4.8}$$

Now by (4.1) and (4.2)

$$\begin{aligned}\|\phi_0(\cdot)\|^p_{L_\infty(\mathbf{R})}\varepsilon_s\Big(\prod_{k=0}^{s-1}\varepsilon_k\Big)^p &= m^{p(1-(1-p)^r)/p(1-p)^r}m^{-(1-p)^{s-r}}\prod_{k=0}^{s-1}m^{-p(1-p)^{k-r}}\\ &= m^{(1-(1-p)^r)/(1-p)^r}m^{-(1-p)^s/(1-p)^r}m^{-(1-(1-p)^s)/(1-p)^r}\\ &= m^{-1},\end{aligned}$$

which substituting in (4.8), yields

$$\|\phi_s(\cdot)\|^p_{L_p(\mathbf{R})} \le 2^{(s+1)(s+2)/2}m^{-1}, \quad s = 0, 1, \dots, r. \tag{4.9}$$

By virtue of (4.7) and (4.2), we obtain

$$\begin{aligned}\|\phi_r\|_{L_\infty(\mathbf{R})} &\ge \|\phi_0\|_{L_\infty(\mathbf{R})}\prod_{k=0}^{r-1}\varepsilon_k\\ &= m^{(1-(1-p)^r)/p(1-p)^r}m^{-(1-(1-p)^r)/p(1-p)^r}\\ &= 1,\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\|\phi_r(\cdot)\|_{L_\infty(\mathbf{R})} &\le 2^{r(r+1)/2}\|\phi_0(\cdot)\|_{L_\infty(\mathbf{R})}\prod_{k=0}^{r-1}\varepsilon_k\\ &= 2^{r(r+1)/2}m^{(1-(1-p)^r)/p(1-p)^r}m^{-(1-(1-p)^r)/p(1-p)^r}\\ &= 2^{r(r+1)/2}.\end{aligned} \tag{4.11}$$

In turn (4.10) combined with (4.1), (4.5) and (4.6), implies

$$\phi_r(t) \ge 1, \quad t \in \big[-(2m)^{-1}, (2m)^{-1}\big]. \tag{4.12}$$

Finally, (4.1), (4.4) and (4.5) yield

$$|\operatorname{supp}\phi_r| \le 2^{r+1}m^{-1},$$

and

$$\operatorname{supp}\phi_r \subset \left[-2^r m^{-1}, 2^r m^{-1}\right]. \tag{4.13}$$

Next, set

$$\varphi_r(t) := (r+1)^{-1}2^{-(3r(r+1))/(2p)}\phi_r\big(2^{r+1}t\big), \quad t\in \mathbf{R}.$$

Then it follows from (4.13) that

$$\operatorname{supp}\varphi_r \subset \left[-(2m)^{-1}, (2m)^{-1}\right], \tag{4.14}$$

and by (4.11) we have

$$\begin{aligned}\|\varphi_r\|_{L_\infty(\mathbf{R})} &\le (r+1)^{-1}2^{-3r(r+1)/(2p)}\,2^{r(r+1)/2}\\ &< (r+1)^{-1}2^{-(3-1)r(r+1)/2}\\ &= (r+1)^{-1}2^{-r(r+1)}.\end{aligned} \tag{4.15}$$

Finally, (4.12) implies

(4.16)
$$\varphi_r(t) \ge (r+1)^{-1}2^{-3r(r+1)/(2p)}, \quad t\in(-2^{-r-2}m^{-1}, 2^{-r-2}m^{-1}).$$

Direct calculations using (4.9) yield, for $s=0,1,\dots,r$,

(4.17)
$$\begin{aligned}\|\varphi_r^{(s)}\|_{L_p(\mathbf{R})}^p &= \int_{\mathbf{R}}\left|(r+1)^{-1}2^{-(3r(r+1))/2p}2^{(r+1)s}\phi_r^{(s)}\big(2^{r+1}t\big)\right|^p dt\\ &= (r+1)^{-p}2^{-(3r(r+1))/2}2^{(r+1)sp}2^s\int_{\mathbf{R}}\left|\phi_{r-s}\big(2^{r+1}t\big)\right|^p dt\\ &\le (r+1)^{-p}2^{-(3r(r+1))/2}2^{(r+2)s}2^{-(r+1)}\|\phi_{r-s}\|_{L_p(\mathbf{R})}^p\\ &\le (r+1)^{-p}2^{-(3r(r+1))/2}2^{(r+2)s}\\ &\quad\times 2^{-(r+1)}2^{(r-s+1)(r-s+2)/2}m^{-1}\\ &\le (r+1)^{-p}2^{-r(r+1)/2}m^{-1}\\ &\le (r+1)^{-p}m^{-1}.\end{aligned}$$

Let $t_{m,i} := i/m$, $i=0,1,\dots,m$, and set $I_{m,i} := [t_{m,i-1}, t_{m,i}]$, $i=1,\dots,m$. Denote $\bar t_{m,i} := (t_{m,i-1}+t_{m,i})/2$, $i=1,\dots,m$, and set

$$\varphi_{p,r,m,i}(t) := \varphi_r(t-\bar t_{m,i}), \quad t\in\mathbf{R}, \quad i=1,\dots,m.$$

It follows by (4.14) and (4.16) that

$$\operatorname{supp}\varphi_{p,r,m,i} \subset I_{m,i}, \quad i = 1, \dots, m, \tag{4.18}$$

and

$$\begin{gathered}\varphi_{p,r,m,i}(t) \geq (r+1)^{-1}2^{-(3r(r+1))/(2p)}, \\ t \in (\bar{t}_{m,i} - 2^{-r-2}m^{-1}, \bar{t}_{m,i} + 2^{-r-2}m^{-1}), \\ i = 1, \dots, m.\end{gathered} \tag{4.19}$$

While (4.15) and (4.17) yield

$$\|\varphi_{p,r,m,i}(\cdot)\|_{L_\infty} \leq (r+1)^{-1}2^{-r(r+1)}, \quad i = 1, \dots, m, \tag{4.20}$$

and

$$\|\varphi^{(s)}_{p,r,m,i}(\cdot)\|^p_{L_p} \leq (r+1)^{-p}m^{-1}, \quad i = 1, \dots, m. \tag{4.21}$$

Write

$$\Phi_{p,r,m} := \Phi_{p,r,m}(I) := \left\{\varphi \mid \varphi := \sum_{i=1}^m v_i\varphi_{p,r,m,i}, v := (v_1, \dots, v_m) \in F_m\right\},$$

where F_m is the class of sign-vectors defined in Lemma A. Then by Lemma A

$$\operatorname{card}\Phi_{p,r,m} \geq 2^{m/16}. \tag{4.22}$$

Let $\varphi \in \Phi_{p,r,m}$. Then, by virtue of (4.18) and (4.21) we obtain for any $0 \leq s \leq r$,

$$\begin{aligned}\|\varphi^{(s)}\|_{L_p(I)} &= \left(\int_I |\varphi^{(s)}(t)|^p\,dt\right)^{1/p} \\ &= \left(\sum_{i=1}^m |v_i|^p \int_{I_{m,i}} |\varphi^{(s)}_{p,r,m,i}(t)|^p dt\right)^{1/p} \\ &= \left(\sum_{i=1}^m \|\varphi^{(s)}_{p,r,m,i}(\cdot)\|^p_{L_p}\right)^{1/p} \\ &\leq \left(\sum_{i=1}^m (r+1)^{-p}m^{-1}\right)^{1/p} \\ &= (r+1)^{-1},\end{aligned}$$

so that

$$\sum_{s=0}^{r} \|\varphi^{(s)}\|_{L_p} \le 1, \quad \varphi \in \Phi_{p,r,m}.$$

It also follows from (4.18) and (4.20) that

$$\begin{aligned}\|\varphi\|_{L_\infty} &= \Big\| \sum_{i=1}^{m} v_i \varphi_{p,r,m} \Big\|_{L_\infty} \\ &= \max_{1\le i\le m} \big\{ |v_i| \|\varphi_{p,r,m}(\cdot)\|_{L_\infty} \big\} \\ &\le (r+1)^{-1} 2^{-r(r+1)} \le 1.\end{aligned}$$

Hence, we conclude that

$$\Phi_{p,r,m} \subset W^r_{p,\infty}, \quad 0<p<1, \quad r,m \in \mathbf{N}. \tag{4.23}$$

For any two different vectors $\hat{v} := (\hat{v}_1, \dots, \hat{v}_m)$ and $\check{v} := (\check{v}_1, \dots, \check{v}_m)$, in F_m, let

$$\hat{\phi} := \sum_{i=1}^{m} \hat{v}_i \varphi_{p,r,m,i} \quad \text{and} \quad \check{\phi} := \sum_{i=1}^{m} \check{v}_i \varphi_{p,r,m,i},$$

be the associated functions, respectively. If $\|\hat{v} - \check{v}\|_{l_1^m} \ge m/2$, then, evidently, there exist indices $i_1, \dots, i_{\lceil m/4 \rceil}$ such that $\hat{v}_{i_k} = -\check{v}_{i_k}$, $k = 1, \dots, \lceil m/4 \rceil$. Therefore, by (4.18) and (4.19) we get for $0<q<1$,

$$\begin{aligned}\|\hat{\varphi}(\cdot) - \check{\varphi}(\cdot)\|^q_{L_q(I)} &= \int_I \Big| \sum_{i=1}^{m} (\hat{v}_i - \check{v}_i) \varphi_{p,r,m,i}(t) \Big|^q \, dt \\ &= \sum_{i=1}^{m} \int_{I_{m,i}} |\hat{v}_i - \check{v}_i|^q |\varphi_{p,r,m,i}(t)|^q \, dt \\ &\ge \sum_{i=1}^{m} |\hat{v}_i - \check{v}_i|^q \int_{\bar{t}_{m,i} - 2^{-r-2} m^{-1}}^{\bar{t}_{m,i} + 2^{-r-2} m^{-1}} |\varphi_{p,r,m,i}(t)|^q \, dt \\ &\ge \sum_{i=1}^{m} |\hat{v}_i - \check{v}_i|^q 2^{-r-1} m^{-1} (r+1)^{-q} 2^{-(3r(r+1)q)/(2p)} \\ &\ge 2^{-r-1} m^{-1} (r+1)^{-q} 2^{-(3r(r+1)q)/(2p)} \sum_{i=1}^{\lceil m/4 \rceil} 2^q \\ &\ge 2^{-r-1} m^{-1} (r+1)^{-q} 2^{-(3r(r+1)q)/(2p)} 2^q 2^{-2} m \\ &= (r+1)^{-q} 2^{q-(r+3)-(3r(r+1)q)/(2p)}.\end{aligned}$$

Thus, for

$$\varepsilon := (r+1)^{-1}2^{1-(r+3)/q-(3r(r+1))/(2p)}.$$

we have

$$\|\hat{\varphi}(\cdot) - \check{\varphi}(\cdot)\|_{L_q(I)} \geq \varepsilon, \quad \hat{\varphi} \neq \check{\varphi}, \quad \hat{\varphi}, \check{\varphi} \in \Phi_{p,r,m}.$$

If we set

$$a := (r+1)^{-1}2^{-r(r+1)},$$

then by (4.20) we have

$$\|\varphi_{p,r,m,i}\|_{L_\infty(\mathbf{R})} \leq a, \quad \varphi \in \Phi_{p,r,m}.$$

Therefore for

$$m := \lceil 16(8 + \log_2(a/\varepsilon)) \rceil n, \quad n \in \mathbf{N},$$

it follows by virtue of (4.22) and Lemma 1, that

$$d_n(\Phi_{p,r,m})^{psd}_{L_q(I)} \geq 2^{-2-1/q}(2^q - 1)^{1/q}\varepsilon =: c,$$

where $c = c(r,p,q)$. This, by (4.23), in turn implies

$$d_n\big(W^r_{p,\infty}\big)^{psd}_{L_q(I)} \geq c,$$

where $c = c(r,p,q)$. The lower bounds

$$d_n(W^r_{p,\infty})^{lin}_{L_q} \geq d_n(W^r_{p,\infty})^{kol}_{L_q} \geq c,$$

and

$$E\big(W^r_{p,\infty}, \Sigma_{r,n}\big)_{L_q} \geq c,$$
$$E(W^r_{p,\infty}, R_n)_{L_q} \geq c,$$

where $c = c(r,p,q)$, now follow readily from (2.3) through (2.5). This completes the proof of Theorem 1. □

5. Proof of Theorem 2 (Upper bounds). Here it is more convenient to take $I := (-1,1)$. Fix $n \in \mathbf{N}$ and set

$$\beta := \frac{r-1+1/q}{r-1-1/p+1/q} \geq 1, \tag{5.1}$$

which is well defined since by assumption $r-1-1/p+1/q>0$. We partition I by

$$t_i := t_{\beta,n,i} := \begin{cases} 1-((n-i)/n)^\beta & i=0,1,\dots,n, \\ -1+((n+i)/n)^\beta & i=-1,\dots,-n, \end{cases}$$

and set

$$I_i := I_{\beta,n,i} := \begin{cases} [t_{i-1},t_i) & i=1,\dots,n, \\ (t_i,t_{i+1}] & i=-1,\dots,-n. \end{cases}$$

Given an $x\in V_p^r$, we denote by

$$\pi_{r-1,i}(x;t) := \pi_{r-1}(x;t;t_i) := \sum_{s=0}^{r-1} x^{(s)}(t_i)\frac{(t-t_i)^s}{s!}, \qquad i=0,\pm 1,\dots,\pm(n-1),$$

its Taylor polynomial of the degree $r-1$ about t_i, and define the associated piecewise polynomial

$$\sigma_{r,n}(x;t) := \sigma_{\beta,r,n}(x;t) := \begin{cases} \pi_{r-1,i-1}(x;t) & t\in I_i,\ i=1,\dots,n, \\ \pi_{r-1,i+1}(x;t) & t\in I_i,\ i=-1,\dots,-n. \end{cases}$$

We first assume that $x\in V_p^r$ satisfies in addition

$$x^{(s)}(0)=0, \quad s=0,\dots,r-1.$$

Then

$$x(t)=\frac{1}{(r-1)!}\int_0^t x^{(r)}(\tau)(t-\tau)^{r-1}\,d\tau, \quad t\in I.$$

Set

$$\check{x}(t) := \frac{1}{(r-1)!}\int_0^t |x^{(r)}(\tau)|(t-\tau)^{r-1}\,d\tau, \quad t\in I,$$

and

$$\hat{x}(t) := \frac{1}{(r-1)!}\int_0^t \big(|x^{(r)}(\tau)|-x^{(r)}(\tau)\big)(t-\tau)^{r-1}\,d\tau, \quad t\in I.$$

Clearly, $x=\check{x}-\hat{x}$, and

$$\sigma_{r,n}(x;t)=\sigma_{r,n}(\check{x};t)-\sigma_{r,n}(\hat{x};t), \quad t\in I.$$

It readily follows that

$$\|\check{x}\|_{\mathcal{V}_p^r} \le 1 \quad \text{and} \quad \|\hat{x}\|_{\mathcal{V}_p^r} \le 2.$$

Also, it is easy to see that

$$\check{x}^{(s)}(t) \ge 0, \quad \text{and} \quad \hat{x}^{(s)}(t) \ge 0, \quad s = 0, \dots, r-1, \quad t \in [0,1),$$

and

$$(-1)^{r-s}\check{x}^{(s)}(t) \ge 0, \quad \text{and} \quad (-1)^{r-s}\hat{x}^{(s)}(t) \ge 0,$$
$$s = 0, \dots, r-1, \quad t \in (-1,0].$$

Moreover, for every $s = 0, \dots, r-1$ the functions $\check{x}^{(s)}$ and $\hat{x}^{(s)}$ are nondecreasing in $[0,1)$ because $\check{x}^{(r)}(t) \ge 0$ and $\hat{x}^{(r)}(t) \ge 0$ almost everywhere for $t \in I$. Respectively, the functions $(-1)^{r-s}\check{x}^{(s)}$ and $(-1)^{r-s}\hat{x}^{(s)}$ are nonincreasing in $(-1,0]$ for every $s = 0, \dots, r-1$.

Let $0 < q \le p < 1$. Then it follows immediately from Hölder's inequality that $\check{x} \in L_q$, and we will prove that

$$\|\check{x}(\cdot) - \sigma_{r,n}(\check{x};\cdot)\|_{L_q([0,1))} \le cn^{-r}, \tag{5.2}$$

where $c = c(r,p,q)$. A similar proof yields the same inequality for the norm of $\hat{x}$ in $[0,1)$, and for the norms of $\check{x}$ and $\hat{x}$ in $(-1,0]$.

To this end, we observe that (5.2) is trivial for $n = 1$, so that we may assume $n > 1$.

From the definition of $\pi_{r-1,i-1}$ and by Taylor's expansion, we have

$$\check{x}(t) - \pi_{r-1,i-1}(\check{x};t) = \frac{1}{(r-1)!}\int_{t_{i-1}}^{t} \check{x}^{(r)}(\tau)(t-\tau)^{r-1}\, d\tau,$$
$$i = 1, \dots, n-1.$$

If we denote

$$\theta_i := \theta_{r,i}(\check{x}) := \check{x}^{(r-1)}(t_i) - \check{x}^{(r-1)}(t_{i-1}), \quad i = 1, \dots, n-1,$$

then $\theta_i \ge 0$, $i = 1, \dots, n-1$, since $\check{x}^{(r-1)}$ is nondecreasing in $[0,1)$ and, by the above,

$$|\check{x}(t) - \pi_{r-1,i-1}(\check{x};t)| \le c|I_i|^{r-1}\theta_i, \quad t \in I_i, \quad i = 1, \dots, n-1.$$

Hence

$$\|\breve{x}(\cdot)-\sigma_{r,n}(\breve{x};\cdot)\|_{L_q(I_i)} \le c|I_i|^{r-1+1/q}\theta_i, \quad i=1,\dots,n-1. \tag{5.3}$$

For $i=n$ we get by Hölder's inequality

$$\begin{aligned}
&\|\breve{x}(\cdot)-\pi_{r-1,n-1}(\breve{x};\cdot)\|_{L_q(I_n)}\\
&\qquad = \frac{1}{(r-1)!}\left(\int_{t_{n-1}}^{1}\left|\int_{t_{n-1}}^{t}\breve{x}^{(r)}(\tau)(t-\tau)^{r-1}d\tau\right|^q dt\right)^{1/q}\\
&\qquad \le c\,|I_n|^{r-1}\left(\int_{t_{n-1}}^{1}\left|\int_{t_{n-1}}^{t}|\breve{x}^{(r)}(\tau)|d\tau\right|^q dt\right)^{1/q}\\
&\qquad \le c\,|I_n|^{r-1-1/p+1/q}\left(\int_{t_{n-1}}^{1}\left|\int_{t_{n-1}}^{t}|\breve{x}^{(r)}(\tau)|d\tau\right|^p dt\right)^{1/p}\\
&\qquad \le c\,|I_n|^{r-1-1/p+1/q}\left(\int_{0}^{1}\left|\int_{0}^{t}|\breve{x}^{(r)}(\tau)|d\tau\right|^p dt\right)^{1/p}\\
&\qquad \le c\,|I_n|^{r-1-1/p+1/q}\|\breve{x}\|_{\mathcal{V}_p^r}\\
&\qquad \le c\,|I_n|^{r-1-1/p+1/q}.
\end{aligned}$$

Hence

$$\|\breve{x}(\cdot)-\sigma_{\beta,r,n}(\breve{x};\cdot)\|_{L_q(I_n)} \le c|I_n|^{r-1-1/p+1/q}. \tag{5.4}$$

Since $q<1$, we apply the inequality $a^q+b^q\le 2^{1-q}(a+b)^q$, $a,b\ge 0$, to obtain from (5.3) and (5.4),

(5.5)

$$\begin{aligned}
&\|\breve{x}(\cdot)-\sigma_{\beta,r,n}(\breve{x};\cdot)\|_{L_q([0,1))}\\
&\qquad \le c\left(\sum_{i=1}^{n-1}\left(2^{1/q-1}|I_i|^{r-1+1/q}\theta_i\right)^q\right)^{1/q} + c2^{1/q-1}|I_n|^{r-1-1/p+1/q}.
\end{aligned}$$

Thus we need an estimate on the sum on the righthand side. Observe that, for $t\in I_i$, $2\le i\le n$,

$$\begin{aligned}
\breve{x}^{(r-1)}(t) &= \breve{x}^{(r-1)}(t)-\breve{x}^{(r-1)}(t_{i-1})+\sum_{j=1}^{i-1}\left[\breve{x}^{(r-1)}(t_j)-\breve{x}^{(r-1)}(t_{j-1})\right]\\
&\ge \sum_{j=1}^{i-1}\theta_j \ \ge\ 0.
\end{aligned}$$

Hence

$$\begin{aligned}\|\check{x}^{(r-1)}\|_{L_p([0,1))}^p &= \int_0^1 |\check{x}^{(r-1)}(t)|^p\,dt\\ &= \sum_{i=1}^n \int_{I_i} |\check{x}^{(r-1)}(t)|^p\,dt\\ &\ge \sum_{i=2}^n \int_{I_i} |\check{x}^{(r-1)}(t)|^p\,dt\\ &\ge \sum_{i=2}^n \left(|I_i|^{1/p}\sum_{j=1}^{i-1}\theta_j\right)^p.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\check{x}^{(r-1)}\|_{L_p([0,1))}^p &= \int_0^1 |\check{x}^{(r-1)}(t)|^p\,dt\\ &= \int_0^1 \left|\int_0^t \check{x}^{(r)}(\tau)d\tau\right|^p dt\\ &\le \|\check{x}\|_{\mathcal{V}_p^r}^p \le 1.\end{aligned}$$

Together these two inequalities imply

$$\sum_{i=1}^{n-1}\left(|I_{i+1}|^{1/p}\sum_{j=1}^{i}\theta_j\right)^p \le 1. \tag{5.6}$$

Now, simple calculations show that

$$c_1(n-i+1)^{\beta-1}/n^\beta \le |I_{n,i}| \le c_2(n-i+1)^{\beta-1}/n^\beta, \quad i=1,\dots,n,$$

for some constants $c_1 = c_1(\beta) > 0$ and $c_2 = c_2(\beta)$, which substituting in (5.5) and (5.6) yield, respectively,

(5.7)

$$\begin{aligned}&\|\check{x}(\cdot) - \sigma_{r,n}(\check{x};\cdot)\|_{L_q([0,1))}\\ &\qquad\le \left(\sum_{i=1}^{n-1}\left(\left(\check{c}_1(n-i)^{\beta-1}/n^\beta\right)^{r-1+1/q}\theta_i\right)^q\right)^{1/q} + \check{c}_1 n^{-\beta(r-1-1/p+1/q)},\end{aligned}$$

and

$$\sum_{i=1}^{n-1}\biggl(\bigl(\check{c}_2(n-i)^{\beta-1}/n^{\beta}\bigr)^{1/p}\sum_{j=1}^{i}\theta_j\biggr)^p\le 1,$$

for some constants $\check{c}_1=\check{c}_1(r,p,q)$ and $\check{c}_2=\check{c}_2(r,p,q)$.

Thus with

$$a_i:=\bigl(\check{c}_1(n-i)^{\beta-1}/n^{\beta}\bigr)^{r-1+1/q}$$

and

$$b_i:=\bigl(\check{c}_2(n-i)^{\beta-1}/n^{\beta}\bigr)^{1/p},\quad i=1,\dots,n-1,$$

we have to estimate

$$\biggl(\sum_{i=1}^{n-1}\Bigl(\bigl(\check{c}_1(n-i)^{\beta-1}/n^{\beta}\bigr)^{r-1+1/q}\theta_i\Bigr)^q\biggr)^{1/q}=\biggl(\sum_{i=1}^{n-1}(a_i\theta_i)^q\biggr)^{1/q}$$
$$=: f_{q,n-1}(\theta),$$

under the constraint

$$\theta_i\ge 0,\quad i=1,\dots,n-1,\quad \sum_{i=1}^{n-1}\biggl(b_i\sum_{j=1}^{i}\theta_j\biggr)^p\le 1.$$

This is exactly what Lemma 3 is about, and we conclude by it that

$$f_{q,n-1}(\theta)\le (n-1)^{-1+1/q}\max_{1\le i\le n-1}\biggl\{a_i\biggl(\sum_{j=i}^{n-1}b_j^p\biggr)^{-1/p}\biggr\}, \tag{5.8}$$

where $c=c(r,p,q)$. So all we need is to estimate the righthand side of (5.8).

Straightforward calculations yield

$$\sum_{j=i}^{n-1}b_j^p=\check{c}n^{-\beta}\sum_{j=i}^{n-1}(n-j)^{\beta-1}\ge\tilde{c}n^{-\beta}(n-i)^{\beta},$$

whence,

$$\max_{1\le i\le n-1}\left\{a_i\left(\sum_{j=i}^{n-1} b_j^p\right)^{-1/p}\right\}$$
$$\le c_*\beta^{-1/p} n^{-\beta(r-1-1/p+1/q)} \max_{1\le i\le n-1}(n-i)^{(\beta-1)(r-1+1/q)-\beta/p}$$
$$\le c_* n^{-\beta(r-1-1/p+1/q)}(n-1)^{(\beta-1)(r-1+1/q)-\beta/p} \le cn^{-r+1-1/q},$$

since the choice of β in (5.1) guarantees that

$$\max_{1\le i\le n-1}(n-i)^{(\beta-1)(r-1+1/q)-\beta/p} = (n-1)^{(\beta-1)(r-1+1/q)-\beta/p}.$$

Substituting in (5.8) yields

$$f_{q,n-1}(\theta) \le cn^{-r}, \tag{5.9}$$

where $c = c(r,p,q)$. The choice of β in (5.1) also gives

$$n^{-\beta(r-1-1/p+1/q)} \le n^{-r},$$

which, substituted together with (5.9) into (5.7), yields

$$\|\check{x}(\cdot) - \sigma_{r,n}(\check{x};\cdot)\|_{L_q([0,1))} \le cn^{-r}, \quad n = 1,2,\ldots, \tag{5.10}$$

where $c = c(r,p,q)$. Similarly we obtain

$$\|\hat{x}(\cdot) - \sigma_{r,n}(\hat{x};\cdot)\|_{L_q([0,1))} \le cn^{-r}, \quad n = 1,2,\ldots, \tag{5.11}$$

where $c = c(r,p,q)$.

Combining (5.10) and (5.11) we conclude that for $0 < q \le p < 1$ we have

$$\|x(\cdot) - \sigma_{r,n}(x;\cdot)\|_{L_q([0,1))} \le cn^{-r}, \quad n = 1,2,\ldots, \tag{5.12}$$

where $c = c(r,p,q)$.

If, on the other hand, $0 < p < q < 1$, then in general we can no longer guarantee that $x \in \mathcal{V}_p^r$ necessarily belongs to L_q. We have this because we have assumed that $r - 1 - 1/p + 1/q > 0$. In order to see this we first observe that in this case $r>1$. We will show that if $x\in\mathcal{V}_p^r$, then for all $t\in I$ we have the pointwise convergence,

$$\begin{aligned}x(t)&=\sigma_{r,2^0}(x;t)+\sum_{\nu=1}^{\infty}\big(\sigma_{r,2^\nu}(x;t)-\sigma_{r,2^{\nu-1}}(x;t)\big)\\&=\sigma_{r,2^0}(\check{x};t)+\sum_{\nu=1}^{\infty}\big(\sigma_{r,2^\nu}(\check{x};t)-\sigma_{r,2^{\nu-1}}(\check{x};t)\big)\\&\quad-\sigma_{r,2^0}(\hat{x};t)-\sum_{\nu=1}^{\infty}\big(\sigma_{r,2^\nu}(\hat{x};t)-\sigma_{r,2^{\nu-1}}(\hat{x};t)\big).\end{aligned}$$

In fact we will show more, namely, that

$$\sigma_{q,r}(\check{x};t):=|\sigma_{r,2^0}(\check{x};t)|^q+\sum_{\nu=1}^{\infty}\big|\sigma_{r,2^\nu}(\check{x};t)-\sigma_{r,2^{\nu-1}}(\check{x};t)\big|^q$$

and

$$\sigma_{q,r}(\hat{x};t):=|\sigma_{r,2^0}(\hat{x};t)|^q+\sum_{\nu=1}^{\infty}\big|\sigma_{r,2^\nu}(\hat{x};t)-\sigma_{r,2^{\nu-1}}(\hat{x};t)\big|^q$$

converge pointwise for all $t\in I$ and any $0<q<1$.

Indeed, for a fixed $t\in I$,

$$\begin{aligned}|x(t)-\sigma_{r,2^\nu}(x;t)|&\le\max_{i=1,\dots,2^v}\big|I_{2^\nu,i}\big|^{r-1}\bigg|\int_0^t|x^{(r)}(\tau)|\,d\tau\bigg|\\&\le c2^{-(r-1)\nu}\bigg|\int_0^t|x^{(r)}(\tau)|\,d\tau\bigg|.\end{aligned}$$

Since $r>1$, the above series are dominated by a convergent geometric series.

Now for $\nu\in\mathbf{N}$ and all $1\le i\le 2^{\nu-1}$, we have $I_{2^{\nu-1},i}=I_{2^\nu,2i-1}\cup I_{2^\nu,2i}$. Also,

$$\begin{aligned}\sigma_{r,2^{\nu-1}}(\check{x};t)&=\pi_{r-1}(\check{x};t,t_{2^{\nu-1},i-1})\\&=\pi_{r-1}(\check{x};t,t_{2^\nu,2i-2}),\quad t\in I_{2^{\nu-1},i}\end{aligned}$$

while

$$\sigma_{r,2^\nu}(\check{x};t)=\begin{cases}\pi_{r-1}(\check{x};t,t_{2^\nu,2i-2}) & t\in I_{2^\nu,2i-1},\\ \pi_{r-1}(\check{x};t,t_{2^\nu,2i-1}) & t\in I_{2^\nu,2i}.\end{cases}$$

Hence

$$\sigma_{r,2^\nu}(\check{x};t) - \sigma_{r,2^{\nu-1}}(\check{x};t) = \begin{cases} 0 & t \in I_{2^\nu,2i-1}, \\ \pi_{r-1}(\check{x};t,t_{2^\nu,2i-1}) - \pi_{r-1}(\check{x};t,t_{2^{\nu-1},i-1}), & t \in I_{2^\nu,2i}, \end{cases}$$

so that

$$\|\sigma_{r,2^\nu}(\check{x};\cdot) - \sigma_{r,2^{\nu-1}}(\check{x};\cdot)\|_{L_q(I_{2^{\nu-1},i})} = \|\pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1}) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|_{L_q(I_{2^\nu,2i})}. \tag{5.13}$$

By virtue of Lemma C we have

$$\|\pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1}) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|_{L_q(I_{2^\nu,2i})} \le c\,|I_{2^\nu,2i}|^{1/q-1/p}\|\pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1}) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|_{L_p(I_{2^\nu,2i})}, \tag{5.14}$$

where $c = c(r,p,q)$, and

$$\begin{aligned} \|\pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1}) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|^p_{L_p(I_{2^\nu,2i})} &\le \|\check{x}(\cdot) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|^p_{L_p(I_{2^\nu,2i})} \\ &\quad + \|\check{x}(\cdot) - \pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1})\|^p_{L_p(I_{2^\nu,2i})} \\ &\le \|\check{x}(\cdot) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|^p_{L_p(I_{2^{\nu-1},i})} \\ &\quad + \|\check{x}(\cdot) - \pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1})\|^p_{L_p(I_{2^\nu,2i})}. \end{aligned} \tag{5.15}$$

Substituting (5.14) and (5.15) in (5.13) implies

$$\begin{aligned} \|\sigma_{r,2^\nu}(\check{x};\cdot) - \sigma_{r,2^{\nu-1}}(\check{x};\cdot)\|^q_{L_q(I_{2^{\nu-1},i})} &\le c\,|I_{2^{\nu-1},i}|^{1-q/p}\|\check{x}(\cdot) - \pi_{r-1}(\check{x};\cdot,t_{2^{\nu-1},i-1})\|^q_{L_p(I_{2^{\nu-1},i})} \\ &\quad + c\,|I_{2^\nu,2i}|^{1-q/p}\|\check{x}(\cdot) - \pi_{r-1}(\check{x};\cdot,t_{2^\nu,2i-1})\|^q_{L_p(I_{2^\nu,2i})}, \end{aligned} \tag{5.16}$$

where $c = c(r,p,q)$, and where we used the convexity of the function $u^{q/p}$.

Denoting

$$\begin{aligned} \theta_{2^\nu,i} &:= \theta_{r,2^\nu,i}(\check{x}) \\ &:= \check{x}^{(r-1)}(t_{2^\nu,i}) - \check{x}^{(r-1)}(t_{2^\nu,i-1}), \quad i = 1,\ldots,2^\nu - 1, \end{aligned}$$

similar to (5.3) and (5.4) we obtain

$$\|\breve{x}(\cdot) - \pi_{r-1}(\breve{x}; \cdot, t_{2^\nu,i-1})\|_{L_p(I_{2^\nu,i})} \le |I_{2^\nu,i}|^{r-1+1/p}\theta_{2^\nu,i}, \quad i = 1, \dots, 2^\nu - 1, \tag{5.17}$$

and

$$\|\breve{x}(\cdot) - \pi_{r-1}(\breve{x}; \cdot, t_{2^\nu,2^\nu-1})\|_{L_p(I_{2^\nu,2^\nu})} \le |I_{2^\nu,2^\nu}|^{r-1}. \tag{5.18}$$

Substituting (5.17) and (5.18) in (5.16) yields,

$$\begin{aligned} \|\sigma_{r,2^\nu}(\breve{x};\cdot) - \sigma_{r,2^{\nu-1}}(\breve{x};\cdot)\|_{L_q([0,1)} & \\ \le \breve{c}\bigg(\sum_{i=1}^{2^{\nu-1}-1}\big(|I_{2^{\nu-1},i}|^{r-1+1/q}\theta_{2^{\nu-1},i}\big)^q\bigg)^{1/q} & \\ + \breve{c}\,|I_{2^{\nu-1},2^{\nu-1}}|^{r-1-1/p+1/q} & \\ + \breve{c}\bigg(\sum_{i=1}^{2^{\nu}-1}\big(|I_{2^{\nu},i}|^{r-1+1/q}\theta_{2^{\nu},i}\big)^q\bigg)^{1/q} & \\ + \breve{c}\,|I_{2^{\nu},2^{\nu}}|^{r-1-1/p+1/q}, & \end{aligned} \tag{5.19}$$

with some constant $\breve{c} = \breve{c}(r,p,q)$, and our goal is to estimate the righthand side of (5.19). But we have done just that for β satisfying (5.1). Observe that we have obtained the estimate of the righthand side of (5.7) by Lemma 3, for all $0 < p, q < 1$, provided $r-1-1/p+1/q > 0$. Thus we conclude that for the prescribed β,

$$\|\sigma_{r,2^\nu}(\breve{x};\cdot) - \sigma_{r,2^{\nu-1}}(\breve{x};\cdot)\|_{L_q[0,1)} \le c2^{-\nu r},$$

where $c = c(r,p,q)$. Similarly we have

$$\|\sigma_{r,2^\nu}(\breve{x};\cdot) - \sigma_{r,2^{\nu-1}}(\breve{x};\cdot)\|_{L_q(-1,0]} \le c2^{-\nu r},$$

where $c = c(r,p,q)$. And combined we end up with

$$\|\sigma_{r,2^\nu}(\breve{x};\cdot) - \sigma_{r,2^{\nu-1}}(\breve{x};\cdot)\|_{L_q(I)}^q \le c2^{-\nu rq}, \quad \nu = 1, 2, \dots, \tag{5.20}$$

where $c = c(r,p,q)$, so that the series

$$\sum_{\nu=1}^{\infty}\|\sigma_{r,2^\nu}(\breve{x};\cdot) - \sigma_{r,2^{\nu-1}}(\breve{x};\cdot)\|_{L_q(I)}^q \le \sum_{\nu=1}^{\infty} c2^{-\nu rq} < \infty.$$

It thus follows by Fatou lemma that the function

$$\sigma_{q,r}(\check{x};t) := |\sigma_{r,2^{\nu-1}}(\check{x};t)|^q + \sum_{\nu=1}^{\infty} \big|\sigma_{r,2^{\nu}}(\check{x};t) - \sigma_{r,2^{\nu-1}}(\check{x};t)|^q$$

is integrable in I, and since

$$|\check{x}(t)|^q \le \sigma_{q,r}(\check{x};t), \quad t \in I,$$

we conclude that $\check{x} \in L_q(I)$. Moreover, by virtue of (5.20), we readily get

$$\begin{aligned} \|\check{x}(\cdot) - \sigma_{r,2^n}(\check{x};\cdot)\|_{L_q(I)}^q &\le \sum_{\nu=n+1}^{\infty} \|\sigma_{r,2^{\nu}}(\check{x};\cdot) - \sigma_{r,2^{\nu-1}}(\check{x};\cdot)\|_{L_q(I)}^q \\ &\le \sum_{\nu=n+1}^{\infty} c2^{-\nu r q} \le c2^{-nrq}, \quad n = 0,1,2,\dots, \end{aligned}$$

where $c = c(r,p,q)$. Similarly we obtain the upper bounds

$$\|\hat{x}(\cdot) - \sigma_{r,2^n}(\hat{x};\cdot)\|_{L_q(I)} \le c2^{-nr}, \quad n = 0,1,2,\dots,$$

where $c = c(r,p,q)$, and together we have

$$\|x(\cdot) - \sigma_{r,2^n}(x;\cdot)\|_{L_q(I)} \le c2^{-nr}, \quad n = 0,1,2,\dots, \tag{5.21}$$

where $c = c(r,p,q)$.

Recall that the upper bounds (5.12) and (5.21) have been proved under the additional assumption that

$$x^{(s)}(0) = 0, \quad s = 0,\dots,r-1.$$

If this is not the case, then we let

$$\tilde{x}(t) := x(t) - \sum_{s=0}^{r-1} x^{(s)}(0)\frac{t^s}{s!}, \quad t \in I.$$

Evidently $\tilde{x} \in \mathcal{V}_p^r$, $\|\tilde{x}\|_{\mathcal{V}_p^r} = \|x\|_{\mathcal{V}_p^r}$, and

$$\tilde{x}^{(s)}(0) = 0, \quad s = 0,\dots,r-1.$$

Finally,

$$x(t)-\sigma_{r,n}(x;t)=\tilde{x}(t)-\sigma_{r,n}(\tilde{x};t), \quad t\in I.$$

Thus we conclude that for $x\in V_p^r$,

$$\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I)}\le cn^{-r}, \quad 0<q\le p<1, \quad n=1,2,\dots, \tag{5.22}$$

and

$$\|x(\cdot)-\sigma_{r,2^n}(x;\cdot)\|_{L_q(I)}\le c2^{-nr}, \quad 0<p<q<1, \quad n=0,1,2,\dots, \tag{5.23}$$

where $c=c(r,p,q)$.

Let $\mathcal{S}_r:=\mathcal{S}_{\beta,r}$, be a space of piecewise polynomials of degree $\le r-1$ on each subinterval $I_{r,i}$, $i=\pm1,\dots,\pm n$, and continuous at the point $t=0$. Then $\dim\mathcal{S}_r=2rn-1$, and the mapping defined above $\sigma_{r,n}:\mathcal{V}_p^r\to\mathcal{S}_r$ is linear. Hence it follows immediately by (5.22), and it follows by standard technique from (5.23) that

$$d_n(V_p^r)_{L_q}^{lin}\le cn^{-r}, \quad 0<p,q<1, \quad n=1,2,\dots,$$

where $c=c(r,p,q)$. In view of (2.3) we immediately obtain

$$d_n(V_p^r)_{L_q}^{psd}\le d_n(V_p^r)_{L_q}^{kol}\le cn^{-r}, \quad 0<p,q<1, \quad n=1,2,\dots,$$

where $c=c(r,p,q)$.

Obviously we also have

$$E(V_p^r,\Sigma_{r,n})_{L_q}\le cn^{-r}, \quad 0<p,q<1, \quad n=1,2,\dots,$$

where $c=c(r,p,q)$, and finally applying Lemma D with $\lambda=r+1/q$ and $\gamma=q$, the last inequality yields,

$$E(V_p^r,R_n)_{L_q}\le cn^{-r}, \quad 0<p,q<1, \quad n=1,2,\dots,$$

where $c=c(r,p,q)$. This completes the proof of the upper bounds in Theorem 2. $\square$

6. Proof of Theorem 2 (Lower bounds). The proof follows the same lines as that of the lower bounds in Theorem 1, but it is simpler.

Let $\varphi \in C_0^\infty(\mathbf{R})$ be nonnegative with $\operatorname{supp}\varphi = [0,1] =: I$, $\|\varphi\|_{L_\infty} = 1$, and $\varphi(t) = 1$ if $t \in [1/4, 3/4]$. For $r \in \mathbf{N}$, let

$$\phi_r(t) := \varphi(t)/\|\varphi^{(r)}\|_{L_\infty}, \quad t \in \mathbf{R},$$

and for $m \in \mathbf{N}$ to be prescribed, take $t_i := t_{m,i} := i/m$, $i = 0, 1, \ldots, m$, and $I_i := I_{m,i} := [t_{i-1}, t_i]$, $i = 1, \ldots, m$. Denote

$$\phi_{r,m,i}(t) := m^{-r}\phi_r(m(t - t_{i-1})), \quad t \in \mathbf{R}, \quad i = 1, \ldots, m,$$

Then, $\operatorname{supp}\phi_{r,m,i} = I_i$, $i = 1, \ldots, m$,

$$(6.1) \qquad \|\phi^{(r)}_{r,m,i}\|_{L_\infty} = 1, \quad 0 \le \phi_{r,m,i}(t) \le m^{-r}\|\varphi^{(r)}\|^{-1}_{L_\infty}, \quad t \in I,$$

and

$$(6.2) \quad \phi_{r,m,i}(t) = m^{-r}\|\varphi^{(r)}(\cdot)\|^{-1}_{L_\infty}, \quad t \in [t_{i-1} + 1/(4m), t_i - 1/(4m)].$$

Write

$$\Phi_{r,m} := \Phi_{r,m}(I) := \left\{ \phi \mid \phi := \sum_{i=1}^m v_i \phi_{r,m,i}, \quad v := (v_1, \ldots, v_m) \in F_m \right\},$$

where F_m is the class of sign-vectors defined in Lemma A. Then, by virtue of (6.1), we have

$$\|\phi\|_{L_\infty(I)} \le m^{-r}\|\varphi^{(r)}\|^{-1}_{L_\infty(I)}, \quad \|\phi^{(r)}\|_{L_\infty(I)} \le 1, \quad \phi \in \Phi_{r,m},$$

so that $\Phi_{r,m} \subset V_p^r$. Hence

$$(6.3) \qquad d_n(V_p^r)^{psd}_{L_q} \ge d_n(\Phi_{r,m})^{psd}_{L_q}, \quad 0 < q < 1, \quad n \ge 1.$$

For any two different vectors $\hat{v} := (\hat{v}_1, \ldots, \hat{v}_m)$ and $\check{v} := (\check{v}_1, \ldots, \check{v}_m)$, in F_m, let

$$\hat{\phi} := \sum_{i=1}^m \hat{v}_i \phi_{r,m,i} \quad \text{and} \quad \check{\phi} := \sum_{i=1}^m \check{v}_i \phi_{r,m,i},$$

be the associated functions in $\Phi_{r,m}$. If $\|\hat{v}-\check{v}\|_{l_1^m} \geq m/2$, then there exist $\lceil m/4 \rceil$ indices $i_1, \dots, i_{\lceil m/4 \rceil}$ such that $\hat{v}_{i_k} = -\check{v}_{i_k}$, $k = 1, \dots, \lceil m/4 \rceil$. Hence, by (6.2),

$$\begin{aligned}
\|\hat{\phi} - \check{\phi}\|_{L_q}^q &= \int_I \left| \sum_{i=1}^m (\hat{v}_i - \check{v}_i)\phi_{r,m,i}(t) \right|^q dt \\
&= \sum_{i=1}^m \int_{I_{m,i}} |\hat{v}_i - \check{v}_i|^q (\phi_{r,m,i}(t))^q \, dt \\
&\geq \sum_{k=1}^{\lceil m/4 \rceil} |\hat{v}_{i_k} - \check{v}_{i_k}|^q \int_{t_{m,i_k-1}+(1/4m)}^{t_{m,i_k}-1/4m} m^{-rq} \|\varphi^{(r)}\|_{L_\infty}^{-q} \, dt \\
&= m^{-rq} \|\varphi^{(r)}\|_{L_\infty}^{-q} (2m)^{-1} \sum_{k=1}^{\lceil m/4 \rceil} 2^q \\
&\geq m^{-rq} \|\varphi^{(r)}\|_{L_\infty}^{-q} (2m)^{-1} 2^q \lceil m/4 \rceil \\
&\geq 2^{q-3} \|\varphi^{(r)}\|_{L_\infty}^{-q} m^{-rq} =: \varepsilon^q.
\end{aligned}$$

If we set $a := m^{-r}\|\varphi^{(r)}\|_{L_\infty}^{-1}$, and given $n \in \mathbf{N}$, we take $m = \lceil 80(2^{3/q-1}+1) \rceil n$, then applying Lemma 1, as we did in the proof of Theorem 1, we conclude that

$$d_n(\Phi_{r,m})_{L_q} \geq cn^{-r}, \quad n \in \mathbf{N}, \quad 0 < q < 1,$$

where $c = c(r,q)$. By virtue of (6.3) and (2.3) this implies

$$\begin{gathered}
d_n(V_p^r)_{L_q}^{lin} \geq d_n(V_p^r)_{L_q}^{kol} \geq d_n(V_p^r)_{L_q}^{psd} \geq cn^{-r}, \\
0 < p, q < 1, \quad n = 1, 2, \dots,
\end{gathered}$$

where $c = c(r,q)$. The lower bounds

$$E\big(V_p^r, \Sigma_{r,n}\big)_{L_q} \geq cn^{-r}, \quad 0 < p, q < 1, \quad n = 1, 2, \dots,$$

and

$$E(V_p^r, R_n)_{L_q} \geq cn^{-r}, \quad 0 < p, q < 1, \quad n = 1, 2, \dots,$$

where $c = c(r, q)$, readily follow from (2.4) and(2.5). This completes the proof of the lower bounds in Theorem 2. $\square$

7. Relations between the spaces $\mathcal{W}_p^r$ and $\mathcal{V}_p^r$. Let X and Y be linear spaces equipped with the (quasi-)seminorms $\|x\|_X$ and $\|y\|_Y$, respectively. If $X \subseteq Y$, we say that X is embedded in Y, notation $X \hookrightarrow Y$, if $\|x\|_Y \le c\|x\|_X$ for all $x \in X$. Otherwise we write $X \not\hookrightarrow Y$.

The following relations hold between $\mathcal{W}_p^r$ and $\mathcal{V}_p^r$.

Proposition 1. *For every $r \in \mathbf{N}$, $\mathcal{V}_p^r \not\hookrightarrow \mathcal{W}_p^r$, $0 < p \le \infty$. However, while for $1 \le p \le \infty$, $\mathcal{W}_p^r \hookrightarrow \mathcal{V}_p^r$, if $0 < p < 1$, then $\mathcal{W}_p^r \not\hookrightarrow \mathcal{V}_p^r$.*

Proof. We begin with the easiest part which is to observe that if $1 \le p \le \infty$, then by Hölder inequality,

$$\|x\|_{\mathcal{V}_p^r} \le c\|x\|_{\mathcal{W}_p^r}, \quad \forall\, x \in \mathcal{W}_p^r,$$

where $c := 2^{1/p-1}p^{-1/p}|I|$. Thus, $\mathcal{W}_p^r \hookrightarrow \mathcal{V}_p^r$.

On the other hand, let $0 < p \le \infty$, and take $0 < \varepsilon < |I|$. Recall that t_0 is the midpoint of I, and set

$$x_{\varepsilon,p,0}(t) := \begin{cases} \varepsilon^{-1/p-1} & t \in (-|I|/2 + t_0, -(|I| - \varepsilon)/2 + t_0), \\ 0 & t \in [-(|I| - \varepsilon)/2 + t_0, t_0 + (|I| - \varepsilon)/2], \\ \varepsilon^{-1/p-1} & t \in (t_0 + (|I| - \varepsilon)/2, t_0 + |I|/2), \end{cases}$$

and

$$x_{\varepsilon,p,s}(t) := \int_{t_0}^{t} x_{\varepsilon,p,s-1}(\tau)d\tau, \quad s = 1, \ldots, r, \quad t \in I.$$

Then clearly, $x_{\varepsilon,p,r} \in \mathcal{W}_p^r \cap \mathcal{V}_p^r$, and straightforward calculations yield

$$\|x_{\varepsilon,p,r}\|_{\mathcal{W}_p^r} = \varepsilon^{-1} \quad \text{and} \quad \|x_{\varepsilon,p,r}\|_{\mathcal{V}_p^r} = 2^{-1}(p+1)^{-1/p}.$$

Obviously, there exists no constant $c > 0$ such that

$$\|x_{\varepsilon,p,r}\|_{\mathcal{W}_p^r} \le c\|x_{\varepsilon,p,r}\|_{\mathcal{V}_p^r},$$

for all $\varepsilon \to 0$. Thus $\mathcal{V}_p^r \not\hookrightarrow \mathcal{W}_p^r$.

Finally, let $0 < p < 1$ and take $0 < \varepsilon < |I|$. Set

$$y_{\varepsilon,p,0}(t) := \begin{cases} 0 & t \in [-|I|/2 + t_0, t_0 - \varepsilon/2], \\ \varepsilon^{-1/p} & t \in (-\varepsilon/2 + t_0, t_0 + \varepsilon/2), \\ 0 & t \in (t_0 + \varepsilon/2, t_0 + |I|/2), \end{cases}$$

and

$$y_{\varepsilon,p,s}(t) := \int_{t_0}^{t} y_{\varepsilon,p,s-1}(\tau) d\tau, \quad s = 1, \dots, r, \quad t \in I.$$

Again it is clear that $y_{\varepsilon,p,r} \in \mathcal{W}_p^r \cap \mathcal{V}_p^r$, and again by straightforward calculations,

$$\|y_{\varepsilon,p,r}\|_{\mathcal{W}_p^r(I)} = 1 \quad \text{and} \quad \|y_{\varepsilon,p,r}\|_{\mathcal{V}_p^r(I)} = 2^{-1}(\varepsilon + (p+1)^{-1}\varepsilon + |I|)\varepsilon^{1-1/p}.$$

This time it is clear that there exists no constant $c > 0$ such that

$$\|y_{\varepsilon,p,r}\|_{\mathcal{V}_p^r(I)} \le c \|y_{\varepsilon,p,r}\|_{\mathcal{W}_p^r(I)},$$

for all $\varepsilon \to 0$. Thus $\mathcal{W}_p^r \not\hookrightarrow \mathcal{V}_p^r$. This completes the proof of Proposition 1. $\square$

On the other hand we do have

Proposition 2. *The inclusion* $\mathcal{V}_p^r \subseteq L_p$, *is valid for every* $r \in \mathbf{N}$ *and all* $0 < p \le \infty$.

Proof. For $x \in \mathcal{V}_p^r$, let

$$\pi_{r-1}(x;t;t_0) := \sum_{s=0}^{r-1} x^{(s)}(t_0) \frac{(t-t_0)^s}{s!}$$

denote the Taylor polynomial of x. Then

$$x(t) = \pi_{r-1}(x;t;t_0) + \frac{1}{(r-1)!} \int_{t_0}^{t} x^{(r)}(\tau)(t-\tau)^{r-1} \, d\tau.$$

Now $\pi_{r-1}(x;t;t_0) \in L_p$, $0 < p \le \infty$, so it suffices to prove that the remainder does too.

If $0 < p < \infty$, then

$$\begin{aligned}\left(\int_I\left|\int_{t_0}^t x^{(r)}(\tau)(t-\tau)^{r-1}\,d\tau\right|^p dt\right)^{1/p} \\ &\le 2^{-r+1}|I|^{r-1}\left(\int_I\left|\int_{t_0}^t |x^{(r)}(\tau)|\,d\tau\right|^p dt\right)^{1/p} \\ &= 2^{-r+1}|I|^{r-1}\|x\|_{\mathcal{V}_p^r} < \infty,\end{aligned}$$

and for $p = \infty$,

$$\begin{aligned}\sup_{t\in I}\left|\int_{t_0}^t x^{(r)}(\tau)(t-\tau)^{r-1}d\tau\right| &\le 2^{-r+1}|I|^{r-1}\sup_{t\in I}\left|\int_{t_0}^t |x^{(r)}(\tau)|d\tau\right| \\ &= 2^{-r+1}|I|^{r-1}\|x\|_{\mathcal{V}_\infty^r} < \infty.\end{aligned}$$

Thus the proof is complete. $\square$

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