ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

QUANTUM PROJECTIVE 3-SPACES WHICH EMBED WEIGHTED QUANTUM PLANES

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ABSTRACT. Let A be an Artin-Schelter regular algebra of global dimension 3 having 3 generators of weights $(1, 1, 2)$. All such algebras have been classified. We use these classification results to study some Artin-Schelter regular algebras of global dimension 4 having 4 generators and 6 quadratic defining relations.

To be precise, the 2-Veronese ring $A^{(2)}$ has 4 generators and 7 quadratic defining relations. We study certain Artin-Schelter regular algebras S of global dimension 4 which have $A^{(2)}$ as a graded quotient algebra. Thus, the defining relations of S are obtained by finding appropriate 6-dimensional subspaces of the space of defining relations of $A^{(2)}$. In this article, we focus on the case where A is an Ore extension of the algebra $k_q[x, y] = k\langle x, y \mid yx - qxy\rangle$.

We study the geometry of these regular algebras of dimension 4 by determining the associated varieties of point modules.

1. Introduction. In recent years, a notion of projective geometry for noncommutative graded rings has grown out of work of Artin, Tate and Van den Bergh [**2, 3**]. In the classical, commutative theory of projective geometry, graded polynomial rings $k[x_0, \dots, x_n]$ play an important role as homogeneous coordinate rings of projective linear spaces. In the theory of quantum projective geometry, this role is played by the *Artin-Schelter regular*, or simply *regular*, algebras. An interesting aspect of the associated quantum projective spaces is that they often contain fewer points than their commutative counterparts.

Throughout this paper, the term *algebra* is used to denote an associative algebra with unity over an algebraically closed field k. We work

The research of all three authors was funded by NSF grant DMS-9619402. The research of the third author was supported by National Security Agency grant MDA904-00-1-0017.
Received by the editors on February 17, 1999, and in revised form on December 12, 2002.

with **N**-graded algebras; that is, we assume the algebra A has a k-vector space decomposition $A = \oplus_{i\geq 0} A_i$ such that $A_i A_j \subseteq A_{i+j}$ for all i, j. We work only with algebras which are *locally finite*, meaning that each A_i is a finite dimensional k-vector space, and *connected graded*, meaning that $A_0 = k$.

A *regular algebra* is a graded k-algebra A satisfying a series of properties originally given in [**1**]. In particular, we require A to have finite global (homological) dimension d and polynomial growth and to satisfy the following version of the Gorenstein condition:

$$\underline{\mathrm{Ext}}^i_A({}_A k, A) = \begin{cases} 0 & \text{if } i \neq d \\ k_A(e) \text{ for some } e \in \mathbf{Z} & \text{if } i = d. \end{cases}$$

Here, ${}_A k$, respectively k_A, denotes the field k considered as a left, respectively right, A-module.

If S is a regular algebra of global dimension $n+1$ with $n+1$ generators of degree 1 and $n(n+1)/2$ quadratic defining relations, then it is natural to think of the noncommutative scheme defined by S as a quantum projective space of dimension n, denoted $\mathbf{P}^n_q$. For generalities on quantum schemes defined by noncommutative graded rings, see [**4**].

Regular algebras of global dimension 2 are easy to classify completely, and examples of $\mathbf{P}^1_q$ are well understood. Regular algebras of global dimension 3 that are generated by elements of degree one were classified by Artin and Schelter [**1**], and a corresponding classification for algebras not generated in degree one was completed by Stephenson [**10, 11**]. A regular algebra of global dimension 3 with 3 generators, not all of which are in degree 1, can be thought of as determining a 'weighted quantum plane.'

This paper is concerned with examples of $\mathbf{P}^3_q$ which arise from embeddings of weighted quantum planes. This is meant to mimic the following commutative situation. Consider the commutative polynomial ring $A = k[x, y, z]$ with grading induced by $\deg(x, y, z) = (1, 1, 2)$. The grading on A induces an action of k^* on the open affine $\mathbf{A}^3 \setminus \{\mathbf{0}\}$ defined by $(a_1, a_2, a_3) \mapsto (\lambda a_1, \lambda a_2, \lambda^2 a_3)$, and the resulting quotient space is the weighted projective plane $\mathbf{P}(1, 1, 2)$. The 2-Veronese ring $A^{(2)}$ may be presented with 4 generators ($w_0 = x^2$, $w_1 = xy$, $w_2 = y^2$ and $w_3 = z$) and 7 quadratic relations. There are six relations of the form $w_i w_j = w_j w_i$, and the other relation is $w_1^2 = w_0 w_2$. In this way we see that $\mathbf{P}(1,1,2)$ is embedded in $\mathbf{P}^3 = \mathrm{Proj}k[w_0, w_1, w_2, w_3]$ as the singular quadric cone defined by the vanishing of $w_1^2 - w_0w_2$. The homogeneous coordinate ring of this cone with respect to this embedding is precisely $A^{(2)}$.

We study the analogous process in the noncommutative setting in order to produce regular algebras of global dimension 4 with 4 generators and 6 quadratic relations. Our starting point is a regular algebra A of global dimension 3 with three generators of weights $(1,1,2)$. Such an algebra is known to be an Ore extension $A = R[z;\sigma,\delta]$, where $R = k\langle A_1\rangle$ is a regular algebra of global dimension 2 and $\deg(z) = 2$ [**10**, Proposition 3.10]. Here, σ is a graded automorphism of R and δ is a graded left σ-derivation of degree 2. The 2-Veronese algebra $A^{(2)}$ has 4 generators of degree 1 and 7 quadratic defining relations, and $A^{(2)} = R^{(2)}[z;\sigma,\delta]$. The goal is to find regular algebras of global dimension 4 as quotients of the free algebra on A_2 by factoring out appropriate 6-dimensional subspaces of the space of relations of $A^{(2)}$.

We should remark that this process is similar to that employed in [**5**], in which the authors study 4-dimensional regular algebras mapping onto the 2-Veronese of certain regular algebras of global dimension 3. The main difference is that in [**5**], the authors started with regular algebras which were generated by two elements of degree 1, whereas our starting point is a regular algebra minimally generated by 3 elements in degrees $(1,1,2)$. Intuitively, the regular algebras of dimension 4 studied in [**5**] arise from embeddings of a quantum nonsingular quadric $\mathbf{P}^1\times\mathbf{P}^1$, whereas our regular algebras of dimension 4 arise from embeddings of a (singular) quantum cone.

In Section 2, we introduce the notion of the twist of a graded algebra; see [**17**]. We use this to show that there is a natural choice for a regular algebra C of global dimension 3 with 3 generators of degree 1 and 3 quadratic relations which maps onto $R^{(2)}$.

Let $H_S(t) = \sum_i \dim_k(S_i)t^i$ be the Hilbert series of the graded ring S. In the remainder of the paper, we determine Noetherian regular algebras S of global dimension 4 with 4 generators of degree 1 and 6 quadratic defining relations such that $H_S(t) = (1-t)^{-4}$, C is a graded subring of S, and S maps onto $A^{(2)}$. Thus, we are classifying certain quadratic regular algebras S of global dimension 4 which fit into the following commutative diagram of graded algebras:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, two heads] & S \arrow[d, two heads] \\ R^{(2)} \arrow[r, hook] & A^{(2)} = R^{(2)}[z;\sigma,\delta] \end{tikzcd}$$

Theorem 1.1. *With the above notation and assumptions, the algebra S is Noetherian and regular with $H_S(t) = (1-t)^{-4}$ if and only if S is isomorphic to an Ore extension $S = C[z;\sigma',\delta']$ for some graded automorphism σ' of C and some graded left σ'-derivation δ' on C of degree* 1.

See Theorem 2.9 for the full statement of this result. It is known that, in some cases, there are regular algebras of global dimension 4 mapping onto $A^{(2)}$ which do not have C as a graded subring. The classification of such algebras is not dealt with in this paper but is a valid avenue for further work.

In [**13**], a list of all possible regular algebras of global dimension 3 having weight $(1,1,2)$ is given, and we will use this as our starting point. Due to the scope of this task, in this article we focus only on the case where $R = k_q[x,y] = k\langle x,y \mid yx - qxy\rangle$. We compute both the relations of the ring $A^{(2)}$ and the ring C, and we then use Theorem 1.1 to determine all of the graded rings S.

We end the paper by studying the geometry of the quantum spaces $\mathbf{P}^3_q$ given by the regular algebras S we have defined. As in [**2**], it is natural to think of the points of the quantum space in terms of isomorphism classes of certain graded S-modules called *point modules*. The isomorphism classes of point modules over S are in natural bijection with the closed points of the graph of an automorphism of a subscheme V of $\mathbf{P}^3$, [**16**, Theorem 1.10].

The standard approach is to determine this graph as a subscheme of $\mathbf{P}^3 \times \mathbf{P}^3$ by multilinearizing the defining relations of S, and then to determine V by projecting the graph onto a single copy of $\mathbf{P}^3$. We accomplish this by a process that is easier to handle computationally using the following result.

Theorem 1.2. *Let C be a regular algebra of global dimension 3 which is isomorphic to a graded twist of the polynomial ring $k[w_0, w_1, w_2]$ with the usual gradation. Let σ' be a graded automorphism of C and let δ' be a graded left σ'-derivation on C of degree 1. Then the scheme of points in the noncommutative scheme $\mathbf{P}^3_q$ defined by $S = C[z; \sigma', \delta']$ is the subscheme of $\mathbf{P}^3$ defined by the vanishing of three specific cubic polynomials.*

See Theorem 3.1 for the precise form of the cubic polynomials.

For most of the algebras S in our classification, we describe the varieties, i.e. reduced schemes, given by the associated scheme of points. Because of the number of special cases involved, we leave cases where $q^n = 1$ for $n \in \{1, 2, 3\}$ for future work.

Remark 1.3. We produce quadratic regular algebras of global dimension 4 with the following point varieties:

$\mathbf{P}^3$

quadric cone

quadric cone + 1 line (intersecting either at the vertex or in 2 distinct points)

smooth quadric surface + 1 line

1 plane + n lines, where $n \in \{0, 1, 2, 3\}$

2 planes + 1 line

n lines, where $n \in \{2, 3, 4, 5, 7\}$

The precise results of this analysis are given in a table at the end of the article.

Some of the quantum $\mathbf{P}^3$s we generate appear elsewhere. For example, when the point variety is $\mathbf{P}^3$, the associated algebra S is a graded twist of $k[w_0, w_1, w_2, w_3]$. The algebras S in our work whose associated point variety is a smooth quadric surface union with a line are graded twists of the algebras in [**16**, Proposition 2.10(b)] with $\alpha = -1$. Also, twists of the algebras S appearing here whose point variety is a cone or a cone union with a line appear in [**15**, Proposition 2.6(b),(b″),(c)].

2. An explicit description of the algebras in question. In order to give the required definitions, we introduce several facts about graded modules over a graded k-algebra A. By a graded A-module we will mean a left, respectively right, A-module M with a vector space decomposition $M = \oplus_{i \in \mathbf{Z}} M_i$ such that $A_i M_j \subseteq M_{i+j}$, respectively $M_j A_i \subseteq M_{i+j}$, for all i and j. If M is any graded A-module, we define a *tail* of M to be any graded module of the form $M_{\geq j} = \oplus_{i \geq j} M_i$. For any integer n we define the *shift of M by n* to be the graded module $M(n) = \oplus M(n)_i$ where $M(n)_i = M_{n+i}$. We denote the one-dimensional graded A-bimodule $A/A_{\geq 1}$ by either ${}_A k$ or k_A depending on whether we are considering its left or right A-module structure.

There are two notions of regularity for noncommutative algebras that are applicable in the present situation.

Definition 2.1. A graded algebra A is *Artin-Schelter regular*, or simply *regular*, if:

• A has finite global dimension d, meaning that the projective dimension of the module ${}_A k$ is a natural number d,

• A has polynomial growth, meaning that there exist $r \in \mathbf{N}$ and $c \in \mathbf{R}$ such that $\dim_k A_n \leq cn^r$ for all $n \geq 0$,

• A is Artin-Schelter Gorenstein, meaning that

$$\underline{\mathrm{Ext}}^i_A({}_A k, A) = \begin{cases} 0 & \text{if } i \neq d \\ k_A(e) \text{ for some } e \in \mathbf{Z} & \text{if } i = d. \end{cases}$$

See [**3**] for a discussion of the functors $\underline{\mathrm{Ext}}^i_A(-,-)$ on the category of graded A-modules.

The second notion of regularity can be defined for algebras which are not necessarily graded.

Definition 2.2. A noncommutative ring A of finite global dimension is *Auslander regular* if for every finitely generated A-module M and every submodule $N \subseteq \mathrm{Ext}^i_A(M, A)$ we have $\mathrm{Ext}^j_A(N, A) = 0$ for all $j < i$.

In the graded case every known Artin-Schelter regular algebra is Auslander regular, but it is not known if the two notions are actually identical. It is known that any connected graded, Noetherian, Auslander regular algebra is also Artin-Schelter regular [**8**, Theorem 6.3], [**14**, Theorem 2.4].

Definition 2.3. Let A be a regular algebra of global dimension d which is minimally generated by d elements in degrees $\mathbf{q} = (q_1, \dots, q_d)$. We will refer to A as a *quantum polynomial ring in d variables of weight* $\mathbf{q}$.

Definition 2.4. Let R be a graded ring, and let τ be a graded automorphism of R. Define a new multiplication $*$ on the underlying graded vector space of R in the following way: for all $a \in R_n$ and $b \in R_m$, let $a * b = a \cdot b^{\tau^n}$, where $\cdot$ is the original multiplication on R. This new multiplication defines a graded ring R^τ called a *graded twist of R by τ.*

The noncommutative scheme defined by a twist of a graded algebra is isomorphic to that defined by the original algebra [**17**, Theorem 1.1].

Definition 2.5. Let M be a locally finite, graded module over a graded k-algebra R, such that $M_i = 0$ for $i << 0$. The *Hilbert series* of M is the formal power series $H_M(t) = \sum_i \dim_k(M_i)t^i$.

Definition 2.6. Let R be a k-algebra. Let σ be a k-linear automorphism of R, and let δ be a k-linear mapping from R to R such that $\delta(xy) = \delta(x)y + \sigma(x)\delta(y)$ (δ is called a (*left*) *σ-derivation*). The ring $R[z; \sigma, \delta]$ generated by R and an indeterminate z satisfying $zr = \sigma(r)z + \delta(r)$ for all $r \in R$ is called a *skew polynomial ring* or an *Ore extension.*

In our case, R is a graded ring, and σ and δ are restricted so that they will respect the grading on R, i.e. for all i, $\sigma(R_i) = R_i$, and there exists $j > 0$ such that for all i, $\delta(R_i) \subseteq R_{i+j}$. This implies that $R[z; \sigma, \delta]$ is a graded ring with $\deg(z) = j$. In this case, we will say that δ is a σ-derivation of degree j.

We begin with a quantum polynomial ring A in three variables of weight $(1, 1, 2)$. By [**10**, Proposition 3.10], A is isomorphic to an Ore extension $R[z;\sigma,\delta]$, where R is a regular algebra of global dimension 2, σ is a graded automorphism of R, and δ is a graded σ-derivation of degree 2. As k is algebraically closed, R is isomorphic to either $k_q[x,y] = k\{x,y\}/\langle yx - qxy\rangle$ for some $q \in k^*$ or $k_J[x,y] = k\{x,y\}/\langle yx - xy - x^2\rangle$. In either case, R is a graded twist of $k[x,y]$ by a graded automorphism τ.

Proposition 2.7. *Let $R = k[x,y]^\tau$ be a graded twist of $k[x,y]$ with the usual gradation by a graded automorphism τ. The vector space automorphism τ^2 of R_2 induces a graded automorphism of the symmetric algebra $S(R_2)$ which we will also denote by τ^2. Let $C = S(R_2)^{\tau^2}$.*

(1) *The algebra C is regular of dimension* 3.

(2) *The identification of R_2 with itself induces a graded surjection $g : C \twoheadrightarrow R^{(2)}$.*

(3) $\ker g$ *is the ideal of C generated by one nonzero normal element $f \in C_2$.*

Proof. (1) The algebra C is a regular Noetherian domain by [**17**, Theorem 1.3].

(2) Both C and $R^{(2)}$ are generated as algebras by R_2. Using $*$ to represent the multiplication in C, the defining relations of C can be written as $r * s^{\tau^{-2}} - s * r^{\tau^{-2}}$ for all $r, s \in C_1 = R_2$. Using juxtaposition to denote the multiplication in R, the quantities $rs^{\tau^{-2}} - sr^{\tau^{-2}}$ are zero in $R^{(2)} \subset R = k[x,y]^\tau$, since r and s are of degree 2 in R. Therefore, $R^{(2)}$ and C are generated by the same set, and the relations of C can be identified with a subset of the relations of $R^{(2)}$. The result follows.

(3) Let $K = \ker g$. We have $H_C(t) = (1-t)^{-3}$ and $H_{R^{(2)}}(t) = (1+t)/(1-t)^2$. Thus $H_K(t) = H_C(t) - H_{R^{(2)}}(t) = t^2/(1-t)^3$, showing that $\dim K_2 = 1$. Let f be a basis for K_2. We have $fC \subseteq K$, and since C is a domain, we have $H_{fC}(t) = t^2 H_C(t) = t^2/(1-t)^3$. Therefore, $fC = K$. A similar argument shows $Cf = K$, showing that f is normal. □

Let $A = R[z;\sigma,\delta]$, where $R = k[x,y]^\tau$, σ is a graded automorphism of R, and δ is a graded σ-derivation of degree 2. Let C be as in

Proposition 2.7. We classify graded algebras S such that

- S is Noetherian and regular of dimension 4,
- S can be presented with 4 generators and 6 quadratic relations,
- $H_S(t) = (1-t)^{-4}$
- there is a graded inclusion $i : C \hookrightarrow S$,
- there is a graded surjection $h : S \twoheadrightarrow A^{(2)}$,
- the following diagram commutes:

$$\begin{array}{ccc} C & \xhookrightarrow{i} & S \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ R^{(2)} & \xhookrightarrow{i} & A^{(2)} \end{array} \tag{2.8}$$

Theorem 2.9. *Let R, A and C be as above. Let S be a graded algebra with 4 generators and 6 quadratic relations such that there is a commutative diagram of graded ring homomorphisms of the form (2.8). Then S is a Noetherian regular algebra of global dimension 4 with Hilbert series $H_S(t) = (1-t)^{-4}$ if and only if S is isomorphic to an Ore extension $C[z; \sigma', \delta']$, where σ' is a graded automorphism of C and δ' is a graded left σ'-derivation on C of degree 1.*

Let $f \in C_2$ be the generator of $\ker g$ as in Proposition 2.7. In the case that S is Noetherian and regular with $H_S(t) = (1-t)^{-4}$, $i(f)$ is normal in S and generates the ideal $\ker h$.

Proof. We consider $R^{(2)}$ as a subring of $A^{(2)}$ and C as a subring of S. Assume $S = C[z; \sigma', \delta']$, where σ' and δ' are as in the theorem statement. The algebra C is Noetherian and Auslander regular since it is a graded twist of a polynomial ring [**17**, Theorem 1.3]. This implies S is Noetherian and that S is Auslander regular of global dimension 4 [**9**, Lemma]. We have $H_C(t) = (1-t)^{-3}$, and it follows that $H_S(t) = (1-t)^{-4}$. This implies that S has polynomial growth, and thus S is Artin-Schelter regular [**8**, Theorem 6.3].

Now, assume that S is Artin-Schelter regular of global dimension 4, is Noetherian, and has Hilbert series $(1-t)^{-4}$. Then S is a domain by

[**3**, Theorem 3.9]. A Hilbert series argument similar to that given in the proof of Proposition 2.7 shows that $\ker h$ is the ideal generated by one nonzero normal element $s \in S_2$. Because (2.8) commutes, we may take $s = i(f)$.

A basis for $S_1 = A_2$ can be chosen in the form $\{w_0, w_1, w_2, z\}$, where $\{w_0, w_1, w_2\}$ generates C. Thus, we consider both C and $R^{(2)}$ as quotients of the free algebra $k\{w_0, w_1, w_2\}$ and both $A^{(2)}$ and S as quotients of $k\{w_0, w_1, w_2, z\}$. Let $\{f_1, f_2, f_3\}$ be the defining relations of C. We can take defining relations of $R^{(2)}$ of the form $\{f_1, f_2, f_3, F\}$, where F is a quadratic relation arising from the fact that $g(f) = 0$ in $R^{(2)}$. Defining relations of $A^{(2)}$ may be identified with the 4 defining relations of $R^{(2)}$, along with three relations of the form

$$f_{i+4} = zw_i - w_i^{\sigma} z - w_i^{\delta} \qquad \text{where } i = 0, 1, 2.$$

Because C embeds in S and S maps onto $A^{(2)}$, we can take a set of defining relations of S of the form

$$\{f_1, f_2, f_3, f_4 + \alpha_0 F, f_5 + \alpha_1 F, f_6 + \alpha_2 F\},$$

where $\alpha_i \in k$.

By looking at the relations of S, one can see that $S = \sum_{i \in \mathbf{N}} Cz^i$. Because $H_C(t) = (1-t)^{-3}$ and $H_S(t) = (1-t)^{-4}$, we see that $S = \oplus_{i \in \mathbf{N}} Cz^i$. Thus, every $s \in S$ can be uniquely written as a finite sum $\sum_{i \in \mathbf{N}} c_i z^i$ for some $c_i \in C$. If $s \neq 0$, define

$$\deg_z(s) = \max\{i \mid c_i \neq 0\}.$$

Given nonzero $s, t \in S$, we have $\deg_z(st) = \deg_z(s) + \deg_z(t)$. Hence, S is isomorphic to an Ore extension $C[z; \sigma', \delta']$ by [**6**, Theorem 9.3.1]. $\square$

Let $R = k_q[x, y] = k\{x, y\}/\langle yx - qxy \rangle$. Then R is a twist of $k[x, y]$ by the automorphism τ where $x^\tau = x$ and $y^\tau = q^{-1} y$. For the moment, we use $*$ to denote the multiplication in R and $\cdot$ to denote the multiplication in $k[x, y]$. (Later, multiplication will be denoted by juxtaposition when the ring involved is clear.) The vector space automorphism τ^2 of

$$R_2 = \operatorname{span}\{w_0 = x \cdot x, w_1 = x \cdot y, w_2 = y \cdot y\}$$

gives an algebra automorphism of $k[w_0, w_1, w_2]$ of the form

$$w_0^{\tau^2} = w_0, \quad w_1^{\tau^2} = q^{-2}w_1, \quad w_2^{\tau^2} = q^{-4}w_2.$$

The natural graded homomorphism g from $C = k[w_0, w_1, w_2]^{\tau^2}$ to $R^{(2)}$ is given by

$$\begin{aligned} g(w_0) &= x \cdot x = x * x^{\tau^{-1}}, \\ g(w_1) &= x \cdot y = x * y^{\tau^{-1}}, \\ g(w_2) &= y \cdot y = y * y^{\tau^{-1}}. \end{aligned}$$

The defining relations for C as a quotient of $k\{w_0, w_1, w_2\}$ are $w_i w_j^{\tau^{-2}} = w_j w_i^{\tau^{-2}}$ for $\{i, j\} \subset \{0, 1, 2\}$. The normal element $f \in C_2$ generating $\ker g$, see Proposition 2.7, is given by any nonzero scalar multiple of $w_0 w_2^{\tau^{-2}} - w_1 w_1^{\tau^{-2}}$ because

$$\begin{aligned} g(w_0 w_2^{\tau^{-2}} - w_1 w_1^{\tau^{-2}}) &= x * x^{\tau^{-1}} * y^{\tau^{-2}} * y^{\tau^{-3}} - x * y^{\tau^{-1}} * x^{\tau^{-2}} * y^{\tau^{-3}} \\ &= x * x^{\tau^{-1}} * y^{\tau^{-2}} * y^{\tau^{-3}} - x * x^{\tau^{-1}} * y^{\tau^{-2}} * y^{\tau^{-3}} \\ &= 0. \end{aligned}$$

These computations are summed up in the following result.

Proposition 2.10. *Suppose $R = k_q[x, y]$ for some $q \in k^*$. Then the algebra C given by Proposition* 2.7 *is*

$$C = k\langle w_0, w_1, w_2 \mid w_1 w_0 - q^2 w_0 w_1, w_2 w_0 - q^4 w_0 w_2, w_2 w_1 - q^2 w_1 w_2 \rangle.$$

The mapping $g : C \to R^{(2)}$ is given by $g(w_0) = x^2$, $g(w_1) = qxy$ and $g(w_2) = qy^2$. The normal element f generating $\ker g$ *is $q^2 w_0 w_2 - w_1^2$.*

Remark 2.11. For the remainder of Section 2, we fix the following notation, hypotheses and strategy.

- $R = k_q[x, y]$, with a graded automorphism σ and a graded σ-derivation δ of degree 2.
- C is the graded algebra, with surjective graded homomorphism $g : C \twoheadrightarrow R$, described in Proposition 2.10.

- $\ker g$ is generated by the nonzero normal element $f = q^2 w_0 w_2 - w_1^2 \in C_2$.

- We will use a part of [**3**, Lemma 8.4], stated herein as Lemma 2.12, in order to show that σ gives rise to a unique automorphism σ' of C.

- We classify all algebras $S = C[z'; \sigma', \delta']$ satisfying the equivalent conditions of Theorem 2.9. It suffices to determine all σ'-derivations δ' on C such that $g \circ \delta' = \delta \circ g$.

- Because $R_4 = C_2/(f)$, the map $\delta : R_2 \to R_4$ gives rise to infinitely many linear maps from C_1 to C_2 satisfying $g \circ \delta' = \delta \circ g$. These can be defined in the following way. For $i = 0, 1, 2$, let $\widehat{w_i^\delta}$ be a fixed element of C_2 such that $g(\widehat{w_i^\delta}) = [g(w_i)]^\delta$. Choose scalars $\alpha_0, \alpha_1, \alpha_2 \in k$. Define $w_i^{\delta'} = \widehat{w_i^\delta} + \alpha_i f$ for $i = 0, 1, 2$, and let $\delta' : C_1 \to C_2$ be the linear transformation given by linear extension. Our goal is to determine which, if any, of these maps δ' extend to give σ'-derivations on C.

- δ' will extend to a σ'-derivation on C if and only if δ' sends the quadratic defining relations of C to zero in C. The fact that δ is a σ-derivation on R implies that, for any defining relation f_i of C, the image of $f_i^{\delta'}$ in C is an element of $\ker g$. Thus, the image of $f_i^{\delta'}$ in C is of the form $L_i f$ for some $L_i \in C_1$. The fact that C is a domain implies that δ' extends to a σ'-derivation on C if and only if $L_i = 0$ for $i = 1, 2, 3$.

- We divide the problem into 5 cases determined by the classification of quantum polynomial rings of weight $(1, 1, 2)$. In each case, we give formulas for $w_i^{\sigma'}$ and $\widehat{w_i^\delta}$. We assume that δ' has the form given above, and we compute necessary and sufficient conditions on the scalars α_i for δ' to be a σ'-derivation on C. These computations are detailed in subsections 2.1 through 2.5.

In order to show that the given automorphisms of $R^{(2)}$ give rise to automorphisms of C, we need the following fact from [**3**, Lemma 8.4].

Lemma 2.12. *Let σ and τ be automorphisms of a vector space V. Let $S[V]$ be the symmetric algebra of V. If $\sigma\tau = \tau\sigma$, then σ extends to an automorphism of $S[V]^\tau$.*

We now give the details of the classification of quantum polynomial rings of weight $(1,1,2)$ that are Ore extensions of $R = k_q[x,y]$. The article [**13**] gives a complete list of all possible defining relations, and [**12**] gives a list of the isomorphism classes of these algebras. The classification of these algebras can be divided into the 5 cases below. The study of the cases in which $R = k_J[x,y] = k\langle x,y \mid yx - xy - x^2\rangle$ requires a lengthy analysis in itself, and we leave this for future work. The 5 cases in the theorem below can be further broken down into isomorphism classes, but it is more convenient to deal with them in the more general form at present. Each algebra can be represented as $k_q[x,y][z;\sigma,\delta]$, and we give the possible forms of σ and δ below. The matrix for σ gives its action on the basis $\{x,y\}$ of the 2-dimensional space $k_q[x,y]_1$, and the 2×4 matrix for δ gives the map $k_q[x,y]_1 \to k_q[x,y]_3$ for the respective bases $\{x,y\}$ and $\{x^3, x^2y, xy^2, y^3\}$.

Theorem 2.13 [**12**], [**13**, Propositions 1.3, 1.4, 1.7]. *Let A be a quantum polynomial ring of weight $(1,1,2)$ that can be written as $k_q[x,y][z;\sigma,\delta]$. Then, by a change of generators, we may assume σ and δ satisfy one of the following*:

Case 1.
$$\sigma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \delta = \begin{pmatrix} p_3 & p_2 & p_1 & p_0 \\ t_3 & t_2 & t_1 & t_0 \end{pmatrix},$$
where $q, a, b \in k^*$ and $p_i, t_i \in k$ satisfy
$$\begin{aligned} &(q-b)p_0 = 0 \\ &(q^{2-i} - a)t_i = (1 - bq^i)p_{i+1} \quad \text{for } i = 0,1,2 \\ &(q^{-1} - a)t_3 = 0. \end{aligned}$$

Case 2. $q = -1$,
$$\sigma = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
where $b \in k^*$.

Case 3. $q = -1$,
$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

Case 4. $q = 1$,

$$\sigma = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $b \in k^*$.

Case 5. $q = 1$,

$$\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 2.14. *Let σ be as in Theorem* 2.13, *and let C be as in Proposition* 2.10. *The action of σ on $R_2 = C_1$ extends to a graded automorphism σ' of C.*

Proof. We have $C = k[w_0, w_1, w_2]^{\tau^2}$, where $\tau = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$. One can check that τ^2 commutes with σ in each of the 5 cases. Apply Lemma 2.12. $\square$

The key question in each case is whether the given σ-derivation δ of $R^{(2)}$ gives rise to one or more σ'-derivations δ' on C. We address that problem in the next series of results.

2.1 The algebras in Theorem 2.13, Case 1. Let $R = k_q[x, y]$, and let σ and δ be as in Theorem 2.13, Case 1. From the action of σ and δ on R, we compute the following:

$$\begin{aligned}
&w_0^{\sigma'} = a^2 w_0, \quad w_1^{\sigma'} = abw_1, \quad w_2^{\sigma'} = b^2 w_2 \\
&\widehat{w_0^{\delta}} = p_0(q + aq^{-2})w_1w_2 + p_1(q + aq^{-1})w_0w_2 + p_2(1 + aq^{-1})w_0w_1 \\
&\qquad + p_3(1 + a)w_0^2 \\
&\widehat{w_1^{\delta}} = q^{-1}p_0w_2^2 + q^{-1}(p_1 + at_0)w_1w_2 + q^{-2}(p_2 + at_1)w_1^2 \\
&\qquad + (p_3 + at_2)w_0w_1 + aqt_3w_0^2 \\
&\widehat{w_2^{\delta}} = t_0(q^{-1} + q^{-1}b)w_2^2 + t_1(q^{-1} + b)w_1w_2 + t_2(1 + q^2b)w_0w_2 \\
&\qquad + t_3(1 + q^3b)w_0w_1.
\end{aligned}$$

Theorem 2.1.1. *Let $q \in k^*$, and let σ and δ be as in Theorem* 2.13, *Case* 1. *Retain the notation and hypothesis given in Remark* 2.11. *Then there exists an Ore extension $S = C[z; \sigma', \delta']$ such that* (2.8) *commutes if and only if one of the four conditions below is satisfied.*

(1) $x^\delta \in xR$ *and* $y^\delta \in yR$

(2) $x^\delta \notin xR$, $y^\delta \in yR$, $b = q$, $a = \pm q^3$

(3) $x^\delta \in xR$, $y^\delta \notin yR$, $a = q^{-1}$, $b = \pm q^{-3}$

(4) $x^\delta \notin xR$, $y^\delta \notin yR$, $a = q^{-1}$, $b = q$, $q^8 = 1$.

In the case that such an Ore extension exists, δ' is defined by $w_i^{\delta'} = \widehat{w_i^\delta} + \alpha_i f$ for the expressions given above, but the following conditions hold:

$$
\begin{array}{lll}
\alpha_0 = 0 & \textit{or} & (a,b) \in \{(q^2,1),(-q^2,-1)\} \\
\alpha_1 = 0 & \textit{or} & (a,b) \in \{(q,q^{-1}),(q,-q^{-1}),(-q,q^{-1}),(-q,-q^{-1})\} \\
\alpha_2 = 0 & \textit{or} & (a,b) \in \{(1,q^{-2}),(-1,-q^{-2})\}
\end{array}
$$

Proof. As outlined above, we set $w_i^{\delta'} = \widehat{w_i^\delta} + \alpha_i f$, where $f = q^2 w_0 w_2 - w_1^2$. We compute the value of δ' on the defining relations of C given in Proposition 2.10.

$$
\begin{aligned}
(w_1w_0 - q^2w_0w_1)^{\delta'} &= w_1^{\delta'}w_0 + w_1^{\sigma'}w_0^{\delta'} - q^2w_0^{\delta'}w_1 - q^2w_0^{\sigma'}w_1^{\delta'} \\
&= [\alpha_1(q^4 - a^2q^2)w_0 + \alpha_0(ab - q^2)w_1 \\
&\quad + p_0(q - q^{-5}a^2)w_2]f. \\
(w_2w_0 - q^4w_0w_2)^{\delta'} &= w_2^{\delta'}w_0 + w_2^{\sigma'}w_0^{\delta'} - q^4w_0^{\delta'}w_2 - q^4w_0^{\sigma'}w_2^{\delta'} \\
&= [\alpha_2q^4(1 - a^2)w_0 + \alpha_0(b^2 - 1)w_2]f. \\
(w_2w_1 - q^2w_1w_2)^{\delta'} &= w_2^{\delta'}w_1 + w_2^{\sigma'}w_1^{\delta'} - q^2w_1^{\delta'}w_2 - q^2w_1^{\sigma'}w_2^{\delta'} \\
&= [t_3(b^2q^6 - 1)w_0 + \alpha_2(1 - q^2ab)w_1 \\
&\quad + \alpha_1(b^2 - q^{-2})w_2]f.
\end{aligned}
$$

In order for δ' to extend to a σ'-derivation on C, all of the linear terms that are multiplied by f in the expressions above must vanish. This forces $p_0(q - q^{-5}a^2) = t_3(b^2q^6 - 1) = 0$, leading to the 4 conditions in the theorem statement. The remaining equations imply:

$$\begin{aligned}\alpha_1(q^2-a^2) &= \alpha_0(ab-q^2) = \alpha_2(1-a^2) = \alpha_0(b^2-1) = \alpha_2(1-q^2ab)\\ &= \alpha_1(b^2-q^{-2}) = 0.\end{aligned}$$

These force the conditions given on the scalars α_i in the theorem statement. □

2.2 The algebras in Theorem 2.13, Case 2. Let $R = k_q[x,y]$ where $q=-1$, and let σ and δ be as in Theorem 2.13, Case 2. From the action of σ and δ on R, we compute the following:

$$\begin{gathered} w_0^{\sigma'} = -b^2 w_2, \quad w_1^{\sigma'} = -bw_1, \quad w_2^{\sigma'} = -w_0\\ \widehat{w_0^{\delta}} = 0, \quad \widehat{w_1^{\delta}} = 0, \quad \widehat{w_2^{\delta}} = 0.\end{gathered}$$

The relations of C are $w_iw_j = w_jw_i$, and the normal element mapping to zero in $R^{(2)}$ is $f = w_0w_2 - w_1^2$.

Theorem 2.2.1. *Let $R = k_q[x,y]$ where $q=-1$, and let σ and δ be as in Theorem* 2.13, *Case* 2. *Then there exists an Ore extension $S = C[z;\sigma',\delta']$ such that* (2.8) *commutes.*

In this case, δ' is defined by $w_i^{\delta'} = \widehat{w_i^{\delta}} + \alpha_i f$ for some $\alpha_0,\alpha_1,\alpha_2 \in k$. For δ' to be a σ'-derivation, it is necessary and sufficient that $\alpha_1 = (b+1)\alpha_0 = \alpha_2 - \alpha_0 = 0$.

Proof. We check the value of δ' on the defining relations of C.

$$\begin{aligned}(w_1w_0 - w_0w_1)^{\delta'} &= w_1^{\delta'}w_0 + w_1^{\sigma'}w_0^{\delta'} - w_0^{\delta'}w_1 - w_0^{\sigma'}w_1^{\delta'}\\ &= [\alpha_1 w_0 - \alpha_0(b+1)w_1 + b^2\alpha_1 w_2]f.\\ (w_2w_0 - w_0w_2)^{\delta'} &= w_2^{\delta'}w_0 + w_2^{\sigma'}w_0^{\delta'} - w_0^{\delta'}w_2 - w_0^{\sigma'}w_2^{\delta'}\\ &= [(\alpha_2-\alpha_0)w_0 + (b^2\alpha_2 - \alpha_0)w_2]f.\\ (w_2w_1 - w_1w_2)^{\delta'} &= w_2^{\delta'}w_1 + w_2^{\sigma'}w_1^{\delta'} - w_1^{\delta'}w_2 - w_1^{\sigma'}w_2^{\delta'}\\ &= [-\alpha_1 w_0 + \alpha_2(b+1)w_1 - \alpha_1 w_2]f.\end{aligned}$$

The map δ' extends to a σ'-derivation on C if and only if all of the coefficients on the linear terms above vanish. This proves the result. □

2.3 The algebras in Theorem 2.13, Case 3. Let $R = k_q[x,y]$ where $q = -1$, and let σ and δ be as in Theorem 2.13, Case 3. From the action of σ and δ on R, we compute the following:

$$w_0^{\sigma'} = -w_2, \quad w_1^{\sigma'} = w_1, \quad w_2^{\sigma'} = -w_0$$
$$\widehat{w_0^{\delta}} = -w_0^2 + w_0w_1, \quad \widehat{w_1^{\delta}} = 0, \quad \widehat{w_2^{\delta}} = -w_0^2 + w_0w_1$$

The relations of C are $w_iw_j = w_jw_i$, and the normal element mapping to zero in $R^{(2)}$ is $f = w_0w_2 - w_1^2$.

Theorem 2.3.1. *Let $R = k_q[x,y]$ where $q = -1$, and let σ and δ be as in Theorem 2.13, Case 3. Then there exists an Ore extension $S = C[z;\sigma',\delta']$ such that (2.8) commutes.*

In this case, δ' is defined by $w_i^{\delta'} = \widehat{w_i^{\delta}} + \alpha_i f$ for some $\alpha_0, \alpha_1, \alpha_2 \in k$. For δ' to be a σ'-derivation, it is necessary and sufficient that $\alpha_1 = \alpha_0 - \alpha_2 = 0$.

Proof. We check the value of δ' on the defining relations of C.

$$\begin{aligned}
(w_1w_0 - w_0w_1)^{\delta'} &= w_1^{\delta'}w_0 + w_1^{\sigma'}w_0^{\delta'} - w_0^{\delta'}w_1 - w_0^{\sigma'}w_1^{\delta'} \\
&= [\alpha_1 w_0 + \alpha_1 w_2]f. \\
(w_2w_0 - w_0w_2)^{\delta'} &= w_2^{\delta'}w_0 + w_2^{\sigma'}w_0^{\delta'} - w_0^{\delta'}w_2 - w_0^{\sigma'}w_2^{\delta'} \\
&= [(\alpha_2 - \alpha_0)w_0 + (\alpha_2 - \alpha_0)w_2]f. \\
(w_2w_1 - w_1w_2)^{\delta'} &= w_2^{\delta'}w_1 + w_2^{\sigma'}w_1^{\delta'} - w_1^{\delta'}w_2 - w_1^{\sigma'}w_2^{\delta'} \\
&= [-\alpha_1 w_0 - \alpha_1 w_2]f.
\end{aligned}$$

The map δ' extends to a σ'-derivation on C if and only if all of the coefficients on the linear terms above vanish. This proves the result. $\square$

2.4. The algebras in Theorem 2.13, Case 4. Let $R = k[x,y]$, and let σ and δ be as in Theorem 2.13, Case 4. From the action of σ and δ on R, we compute the following:

$$w_0^{\sigma'} = b^2w_0 + 2bw_1 + w_2, \quad w_1^{\sigma'} = b^2w_1 + bw_2, \quad w_2^{\sigma'} = b^2w_2$$
$$\widehat{w_0^{\delta}} = 0, \quad \widehat{w_1^{\delta}} = 0, \quad \widehat{w_2^{\delta}} = 0.$$

The relations of C are $w_iw_j = w_jw_i$, and the normal element mapping to zero in $R^{(2)}$ is $F = w_0w_2 - w_1^2$.

Theorem 2.4.1. *Let $R = k[x,y]$, and let σ and δ be as in Theorem 2.13, Case 4. Then there exists an Ore extension $S = C[z;\sigma',\delta']$ such that (2.8) commutes.*

In this case, δ' is defined by $w_i^{\delta'} = \widehat{w_i^{\delta}} + \alpha_i f$ for some $\alpha_0, \alpha_1, \alpha_2 \in k$. For δ' to be a σ'-derivation, it is necessary and sufficient that $\alpha_0 = \alpha_1 = \alpha_2 = 0$.

Proof. We check the value of δ' on the defining relations of C.

$$\begin{aligned}(w_1w_0 - w_0w_1)^{\delta'} &= w_1^{\delta'}w_0 + w_1^{\sigma'}w_0^{\delta'} - w_0^{\delta'}w_1 - w_0^{\sigma'}w_1^{\delta'}\\ &= [\alpha_1(1-b^2)w_0 + (\alpha_0(b^2-1) - 2b\alpha_1)w_1\\ &\quad + (b\alpha_0 - \alpha_1)w_2]f.\\ (w_2w_0 - w_0w_2)^{\delta'} &= w_2^{\delta'}w_0 + w_2^{\sigma'}w_0^{\delta'} - w_0^{\delta'}w_2 - w_0^{\sigma'}w_2^{\delta'}\\ &= [\alpha_2(1-b^2)w_0 - 2b\alpha_2w_1 + (\alpha_0(b^2-1) - \alpha_2)w_2]f.\\ (w_2w_1 - w_1w_2)^{\delta'} &= w_2^{\delta'}w_1 + w_2^{\sigma'}w_1^{\delta'} - w_1^{\delta'}w_2 - w_1^{\sigma'}w_2^{\delta'}\\ &= [\alpha_2(1-b^2)w_1 + (\alpha_1(b^2-1) - b\alpha_2)w_2]f.\end{aligned}$$

The map δ' extends to a σ'-derivation on C if and only if all of the coefficients on the linear terms above vanish. This proves the result. $\square$

2.5 The algebras in Theorem 2.13, Case 5. Let $R = k[x,y]$, and let σ and δ be as in Theorem 2.13, Case 5. From the action of σ and δ on R, we compute the following:

$$w_0^{\sigma'} = w_0 + 2w_1 + w_2, \quad w_1^{\sigma'} = w_1 + w_2, \quad w_2^{\sigma'} = w_2$$
$$\widehat{w_0^{\delta}} = 2w_0^2 + w_0w_1, \quad \widehat{w_1^{\delta}} = w_0w_1, \quad \widehat{w_2^{\delta}} = 0.$$

The relations of C are $w_iw_j = w_jw_i$, and the normal element mapping to zero in $R^{(2)}$ is $f = w_0w_2 - w_1^2$.

Theorem 2.5.1. *Let $R = k[x,y]$, and let σ and δ be as in Theorem 2.13, Case 5. Then there exists no Ore extension $S = C[z;\sigma',\delta']$ such that (2.8) commutes.*

Proof. We define $\delta'(w_i) = \widehat{w_i^\delta} + \alpha_i f$ for some scalars $\alpha_0, \alpha_1, \alpha_2$. We check the value of δ' on the relation $w_1 w_0 - w_0 w_1$.

$$\begin{aligned}(w_1 w_0 - w_0 w_1)^{\delta'} &= w_1^{\delta'} w_0 + w_1^{\sigma'} w_0^{\delta'} - w_0^{\delta'} w_1 - w_0^{\sigma'} w_1^{\delta'} \\ &= [2w_0 - 2\alpha_1 w_1 + (\alpha_0 - \alpha_1) w_2] f.\end{aligned}$$

The linear expression above cannot vanish because of the $2w_0$ term. This proves the result. □

3. Points of Quantum $\mathbf{P}^3$. In this section, we determine the point varieties for some of the regular algebras of dimension 4 defined in Section 2. Because of the number of special cases involved, we include only the cases where $q^n \neq 1$ for $n \in \{1, 2, 3\}$.

Let S be a Noetherian regular algebra of global dimension 4 defined by 4 generators and 6 quadratic relations with Hilbert series $H_S(t) = (1-t)^{-4}$. Let $\operatorname{GrMod} S$ be the category of graded S-modules, and let $\operatorname{Fdim} S$ be the full subcategory consisting of direct limits of finite dimensional modules. The quotient category $\operatorname{Tails} S = \operatorname{GrMod} S / \operatorname{Fdim} S$ can be considered as the space of quasi-coherent sheaves on a noncommutative projective 3-space $\mathbf{P}^3_q$. There is not an actual geometric object defined as $\mathbf{P}^3_q$, but rather, we define $\mathbf{P}^3_q$ intuitively by the convention $\operatorname{mod} \mathbf{P}^3_q = \operatorname{Tails} S$, and we study geometry in terms of the category $\operatorname{Tails} S$; see [**4**], for example.

The points of $\mathbf{P}^3_q$ are defined in terms of isomorphism classes of certain graded S-modules. A graded S-module $M = \oplus_{i \in \mathbf{Z}} M_i$ is a *point module* if M is generated by M_0 and $H_M(t) = (1-t)^{-1}$. As in [**2**], each isomorphism class of point modules M defines a point $p(M)$ of the infinite product $\mathbb{P} = \mathbf{P}^3 \times \mathbf{P}^3 \times \cdots$, and thus the set of all point modules over S becomes a subscheme W of $\mathbb{P}$ defined by the vanishing of multilinearized forms of the relations of S. By [**16**, Theorem 1.10], W is isomorphic to its projection onto the first two copies of $\mathbf{P}^3$, and thus the set of isomorphism classes of point modules over S can be identified with a subscheme $W' \subseteq \mathbf{P}^3 \times \mathbf{P}^3$. (The points of W' parameterize isomorphism classes of truncated point modules of length two; see [**2**].) The 'shifted truncation' $M(1)_{\geq 0}$ of a point module is again a point module, uniquely determined by M up to isomorphism. Thus, W' takes on the structure of the graph of an automorphism τ of a subscheme $V \subseteq \mathbf{P}^3$.

In the remainder of this article, we determine the variety, i.e. reduced scheme, associated to the scheme V. The standard way to determine the subscheme of $\mathbf{P}^3$ parameterizing the point modules of S is to "multilinearize" the relations of S by adding a subscript of 0 or 1 to each variable depending on whether it occurs in the first or second position, respectively. The vanishing of these multilinearized relations defines W' as a subscheme of $\mathbf{P}^3 \times \mathbf{P}^3$. Then, V is defined as a subscheme of $\mathbf{P}^3$ by the vanishing of 15 quartic polynomials, which are the 4×4 minors of a 6×4 matrix of linear forms. See [**16**] for a full description of this process.

For the algebras dealt with in this article, we may simplify this process considerably using the following result.

Theorem 3.1. *Let C be a twist of $k[w_0, w_1, w_2]$ by a graded automorphism ρ. Let $S = C[w_3; \sigma', \delta']$ be an Ore extension of C by a graded automorphism σ' of C and a graded σ'-derivation δ' on C of degree 1. Write $w_i^{\delta'} = \sum_v d_{iv} w_v$, where $d_{iv} \in C_1$.*

Let V be the subscheme of $\mathbf{P}^3$ parameterizing the isomorphism classes of point modules over S. Then V is defined as a subscheme of $\mathbf{P}^3 = \operatorname{Proj} k[w_0, w_1, w_2, w_3]$ by the vanishing of the 3 cubic forms

$$r_{ij} = w_i^{\sigma'} \left(w_3 w_j^{\rho} - \sum_v d_{jv} w_v^{\rho} \right) - w_j^{\sigma'} \left(w_3 w_i^{\rho} - \sum_v d_{iv} w_v^{\rho} \right)$$

where $ij \in \{01, 02, 12\}$.

Proof. The defining relations of S as a quotient of $k\{w_0, w_1, w_2, w_3\}$ may be written as

$$\begin{aligned}
w_1^{\rho} w_0 &= w_0^{\rho} w_1 \\
w_2^{\rho} w_0 &= w_0^{\rho} w_2 \\
w_2^{\rho} w_1 &= w_1^{\rho} w_2 \\
w_3 w_0 &= w_0^{\sigma} w_3 + \sum_v d_{0v} w_v \\
w_3 w_1 &= w_1^{\sigma} w_3 + \sum_v d_{1v} w_v \\
w_3 w_2 &= w_2^{\sigma} w_3 + \sum_v d_{2v} w_v.
\end{aligned}$$

Multilinearizing, and writing these relations in the form $M \begin{pmatrix} w_{01} \\ w_{11} \\ w_{21} \\ w_{31} \end{pmatrix} = 0$ yields a matrix M of the form

$$M = \begin{pmatrix} w_1^\rho & -w_0^\rho & 0 & 0 \\ w_2^\rho & 0 & -w_0^\rho & 0 \\ 0 & w_2^\rho & -w_1^\rho & 0 \\ d_{00} - w_3 & d_{01} & d_{02} & w_0^\sigma \\ d_{10} & d_{11} - w_3 & d_{12} & w_1^\sigma \\ d_{20} & d_{21} & d_{22} - w_3 & w_2^\sigma \end{pmatrix}$$

where we have replaced w_{i0} by w_i to avoid excessive notation.

Three of the 15 quartic 4×4 minors of M are identically zero. Let J be the ideal of $P = k[w_0, w_1, w_2, w_3]$ generated by the remaining 12 4×4 minors of M. Let I be the ideal of P generated by $\{r_{01}, r_{02}, r_{12}\}$. By [**7**, Exercise II.5.10], I and J define the same subscheme of $\mathbf{P}^3$ if and only if $I^{\text{sat}} = J^{\text{sat}}$, where I^{sat} and J^{sat} are the saturations of I and J.

Direct computation shows that the generators of J are

$$\begin{gathered} w_0^\rho r_{01},\ w_0^\rho r_{02},\ w_0^\rho r_{12},\ w_1^\rho r_{01},\ w_1^\rho r_{02},\ w_1^\rho r_{12},\ w_2^\rho r_{01},\ w_2^\rho r_{02},\ w_2^\rho r_{12} \\ w_3 r_{01} - d_{22} r_{01} + d_{12} r_{02} - d_{02} r_{12}, \\ w_3 r_{02} + d_{21} r_{01} - d_{11} r_{02} + d_{01} r_{12}, \\ w_3 r_{12} - d_{20} r_{01} + d_{10} r_{02} - d_{00} r_{12}. \end{gathered}$$

It is evident that $J \subseteq I$, showing $J^{\text{sat}} \subseteq I^{\text{sat}}$.

The set $\{w_0^\rho, w_1^\rho, w_2^\rho\}$ is a basis for the span of $\{w_0, w_1, w_2\}$, and thus we see that $w_k r_{ij} \in J$ for all ij and all $k \leq 2$. This implies that $d_{uv} r_{ij} \in J$ for all uv and ij. From the last three generators of J, we can now see that $w_3 r_{ij} \in J$ for all ij. This shows that $P_1 r_{ij} \subseteq J$, and thus $r_{ij} \in J^{\text{sat}}$ for all ij. Therefore, $I \subseteq J^{\text{sat}}$, and thus $I^{\text{sat}} = J^{\text{sat}}$. □

We now give the varieties of points for many of the regular algebras S classified in Section 2. Because of the complexity of the relations, it is useful to use a refined version of Theorem 2.13, Case 1; see [**12**].

Theorem 3.2. *Let A be one of the algebras described in Theorem* 2.13. *Suppose that $q^n \neq 1$ for $n \in \{1,2,3\}$. Then A is isomorphic to an algebra $R[z;\sigma,\delta]$, where σ and δ are one of the following*:

(1) $\sigma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\quad \delta = 0$, *where* $a, b \in k^*$,

(2) $\sigma = \begin{pmatrix} a & 0 \\ 0 & q \end{pmatrix}$, $\quad \delta = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$, *where* $a \in k^*$,

(3) $\sigma = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$, $\delta = \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}$,

(4) $\sigma = \begin{pmatrix} q^2 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad \delta = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&\lambda \end{pmatrix}$, *where* $\lambda \in k$,

(5) $\sigma = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, $\delta = \begin{pmatrix} 0&1&0&0 \\ 0&0&\lambda&0 \end{pmatrix}$, *where* $\lambda \in k$,

(6) $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & q^{-2} \end{pmatrix}$, $\delta = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}$.

In the subsections that follow, we describe the point variety V_S of S (for the algebras S determined in Section 2, except in the cases $q^n = 1$ for $n \in \{1,2,3\}$). We also give the point variety V_A of $A^{(2)}$. These results are summarized in a table at the end of this article. We should remark that the varieties V_A are essentially the same as those given in [**13**, Section 5].

Theorem 3.2. *Case* 1. The 3 cubic polynomials defining the point scheme of S are

$$\begin{aligned} r_{01} &= (q^2b - a)w_0w_1w_3 + (a\alpha_1 w_0 - b\alpha_0 w_1)(w_0w_2 - w_1^2) \\ r_{02} &= q^{-2}(q^2b - a)(q^2b + a)w_0w_2w_3 + (a^2\alpha_2 w_0 - b^2\alpha_0 w_2)(w_0w_2 - w_1^2) \\ r_{12} &= q^{-2}(q^2b - a)w_1w_2w_3 + (a\alpha_2 w_1 - b\alpha_1 w_2)(w_0w_2 - w_1^2). \end{aligned}$$

If $a \neq \pm q^2 b$, all α_i must be zero, and V_S is the union of a plane and 3 lines:

$$V_S = \mathbf{V}(w_3) \cup \mathbf{V}(w_0, w_1) \cup \mathbf{V}(w_0, w_2) \cup \mathbf{V}(w_1, w_2).$$

The three lines intersect in a single point not on the plane.

If $a = -q^2 b$ and all $\alpha_i = 0$, then

$$V_S = \mathbf{V}(w_3) \cup \mathbf{V}(w_1) \cup \mathbf{V}(w_0, w_2),$$

two planes union with a line. The line is disjoint from the intersection of the two planes.

If $a = q^2 b$ and all $\alpha_i = 0$, $V_S = \mathbf{P}^3$.

If $a = q^2 b$ and at least one $\alpha_i \neq 0$, we have,

$$V_S = \mathbf{V}(w_0 w_2 - w_1^2) \cup \mathbf{V}(q^2 \alpha_1 w_0 - \alpha_0 w_1, q^4 \alpha_2 w_0 - \alpha_0 w_2, q^2 \alpha_2 w_1 - \alpha_1 w_2),$$

a quadric cone and a line. When $\alpha_1^2 \neq \alpha_0 \alpha_2$, the line meets the quadric cone only at the vertex $(0 : 0 : 0 : 1)$. In the case where $\alpha_1^2 = \alpha_0 \alpha_2$, the line is embedded in the cone (and, hence, from the perspective of varieties, the point variety is simply the quadric cone).

If $a = \pm q$, $b = \mp q^{-1}$, $\alpha_0 = \alpha_2 = 0$, and $\alpha_1 \neq 0$,

$$V_S = \mathbf{V}(2 w_1 w_3 - \alpha_1 (w_0 w_2 - w_1^2)) \cup \mathbf{V}(w_0, w_2),$$

a smooth quadric surface union with a line which meets the quadric in two distinct points.

The point scheme of $A^{(2)}$ is the subscheme of $\mathbf{P}^3$ given by the three cubic polynomials r_{ij} listed above, together with

$$w_0 Q, \quad w_1 Q, \quad w_2 Q, \quad w_3^2 (a w_1^2 - b q^2 w_0 w_2)$$

where $Q = w_1^2 - w_0 w_2$. The variety V_A associated to this scheme is defined by $\mathbf{V}(Q)$ (the quadric cone) when $a = q^2 b$, and by

$$V_A = \mathbf{V}(w_0, w_1) \cup \mathbf{V}(w_1, w_2) \cup \mathbf{V}(w_3, w_1^2 - w_0 w_2)$$

when $a \neq q^2 b$ (two lines and a smooth curve).

Theorem 3.2. *Case* 2. In this case, $b = q$ and $a = \pm q^3$ by Theorem 2.1.1. If $a = q^3$, the three cubic forms defining the point scheme of S are

$$\begin{aligned}
r_{01} &= w_2 (w_0 w_2 - 2 w_1^2) + (q^3 \alpha_1 w_0 - q \alpha_0 w_1)(w_0 w_2 - w_1^2) \\
r_{02} &= -2 w_1 w_2^2 + (q^5 \alpha_2 w_0 - q \alpha_0 w_2)(w_0 w_2 - w_1^2) \\
r_{12} &= w_2^3 + (q^3 \alpha_1 w_2 - q^5 \alpha_2 w_1)(w_0 w_2 - w_1^2).
\end{aligned}$$

If $a = q^3$ and all $\alpha_i = 0$, the variety V_S is the plane $\mathbf{V}(w_2)$.

If $a = q^3$ and $q^2 = -1$, α_1 can be nonzero. If $\alpha_1 \neq 0$, then V_S is the subvariety of $\mathbf{P}^3$ defined by the vanishing of the three polynomials

$$w_2(w_0w_2-2w_1^2)-q\alpha_1w_0(w_0w_2-w_1^2), \quad w_1w_2^2, \quad w_2(w_2^2-q\alpha_1(w_0w_2-w_1^2)).$$

Thus V_S is three lines, intersecting in a point.

If $a = q^3$ and $q^3 = -1$, α_2 can be nonzero. If $\alpha_2 \neq 0$, then V_S is defined by

$$w_2(w_0w_2-2w_1^2),\ 2w_1w_2^2+q^2\alpha_2w_0(w_0w_2-w_1^2),\ w_2^3+q^2\alpha_2w_1(w_0w_2-w_1^2).$$

The resulting variety V_S is 4 lines, intersecting in a point.

When $a = q^3$, the point variety V_A of $A^{(2)}$ consists of one line.

If $a = -q^3$, the three cubic forms are

$$\begin{aligned}
r_{01} &= w_0(w_2^2 - 2q^3w_1w_3) + (q\alpha_0w_1 + q^3\alpha_1w_0)(w_0w_2 - w_1^2)\\
r_{02} &= (q^4\alpha_2w_0 - \alpha_0w_2)(w_0w_2 - w_1^2)\\
r_{12} &= w_2(w_2^2 - 2q^3w_1w_3) + (q^3\alpha_1w_2 + q^5\alpha_2w_1)(w_0w_2 - w_1^2).
\end{aligned}$$

If $a = -q^3$ and all $\alpha_i = 0$, the variety V_S is defined by

$$w_0(w_2^2 - 2q^3w_1w_3), \quad w_2(w_2^2 - 2q^3w_1w_3)$$

and so is the union of a line and a quadric cone. The line meets the cone in 2 distinct points.

If $a = -q^3$ and $q^2 = -1$, α_1 can be nonzero. In this case V_S is defined by

$$\begin{aligned}
&w_0(w_2^2 - 2q^3w_1w_3 - q\alpha_1(w_0w_2 - w_1^2))\\
&w_2(w_2^2 - 2q^3w_1w_3 - q\alpha_1(w_0w_2 - w_1^2)).
\end{aligned}$$

Thus, V_S is a line union with a nonsingular quadric surface. The line meets the surface in 2 distinct points.

When $a = -q^3$, the point variety V_A is given by a line union with a smooth cubic curve. The line and the curve meet (transversely) in one point.

Theorem 3.2. *Case* 3. In this case, $q^4 = \pm 1$ by Theorem 2.1.1. Since we are assuming that $q^i \neq 1$ for $i \in \{1, 2, 3\}$, q must be a primitive fourth or eighth root of 1, and also $\alpha_0 = \alpha_2 = 0$ (again by Theorem 2.1.1). If $q^4 = -1$, we also must have $\alpha_1 = 0$.

If $q^4 = -1$, the three cubic polynomials defining V_S are

$$\begin{aligned} r_{01} &= w_0(-2q^3 w_1 w_3 + w_2^2 - q w_0^2) \\ r_{02} &= 0 \\ r_{12} &= w_2(-2q^3 w_1 w_3 + w_2^2 - q w_0^2). \end{aligned}$$

Here,

$$V_S = \mathbf{V}(w_0, w_2) \cup \mathbf{V}(2w_1 w_2 - q^2 w_0^2 + q w_2^2)$$

the union of a nonsingular quadric surface and a line, meeting in two distinct points. In this case, the points of $A^{(2)}$ form a singular, irreducible curve.

If $q^2 = -1$, V_S is defined by the cubic forms

$$\begin{aligned} r_{01} &= -2w_1^2 w_2 + w_0(w_2^2 + q w_0^2 - q\alpha_1(w_0 w_2 - w_1^2)) \\ r_{02} &= w_1(w_0^2 + q w_2^2) \\ r_{12} &= 2w_0 w_1^2 + w_2(q w_2^2 - w_0^2 + \alpha_1(w_0 w_2 - w_1^2)). \end{aligned}$$

If $\alpha_1^2 \neq -4q$, V_S is the union of 7 lines, all meeting in a common point. If $\alpha_1^2 = -4q$, V_S is a plane union with 2 lines. The two lines intersect at a point on the plane.

In either case, the variety V_A consists of 4 distinct lines.

Theorem 3.2. *Case* 4. We note that the restrictions we have placed on q force $\alpha_1 = 0$ and either $\alpha_2 = 0$ or $q^2 = -1$. The variety V_S is defined by the polynomials

$$\begin{aligned} r_{01} &= w_1((q^2\lambda - 1)w_0 w_2 - \alpha_0 q(w_0 w_2 - w_1^2)) \\ r_{02} &= (2q^2\lambda - 2)w_0 w_2^2 + (\alpha_2 q^5 w_0 - \alpha_0 q w_2)(w_0 w_2 - w_1^2) \\ r_{12} &= w_1((q^2\lambda - 1)w_2^2 + \alpha_2 q^5(w_0 w_2 - w_1^2)). \end{aligned}$$

If $\lambda = q^{-2}$ and $\alpha_0 = \alpha_2 = 0$, then $V_S = \mathbf{P}^3$.

If $\lambda = q^{-2}$ and at least one α_i is nonzero, then $V_S = \mathbf{V}(w_1, (\alpha_2 q^5 w_0 - \alpha_0 q w_2)) \cup \mathbf{V}(w_0 w_2 - w_1^2)$. If $\alpha_0\alpha_2 = 0$, then V_S is just the cone

$\mathbf{V}(w_0w_2 - w_1^2)$. If $\alpha_0\alpha_2 \neq 0$, then V_S is the cone union with a line intersecting it in the vertex.

If $\lambda \neq q^{-2}$, there are several special cases where V_S can be a plane, a plane union with a line, or a union of between 2 and 5 distinct lines. These results are summarized in the table at the end of the paper. When V_S is a union of several lines, all of the lines intersect in one common point.

The variety V_A is the cone $\mathbf{V}(w_0w_2 - w_1^2)$ in the case $\lambda = q^{-2}$, and consists of 2 lines otherwise.

Theorem 3.2. *Case* 5. Due to the restrictions on q, we must have $\alpha_0 = \alpha_2 = 0$. The three cubic polynomials defining V_S are

$$\begin{aligned} r_{01} &= w_0((q\lambda - 1)w_1^2 + \alpha_1 q^2(w_0w_2 - w_1^2)) \\ r_{02} &= (q\lambda - 1)w_0w_1w_2 \\ r_{12} &= w_2((q\lambda - 1)w_1^2 - \alpha_1 q^2(w_0w_2 - w_1^2)). \end{aligned}$$

If $\lambda = q^{-1}$ and $\alpha_1 = 0$, then V_S is $\mathbf{P}^3$. If $\lambda = q^{-1}$ and $\alpha_1 \neq 0$, then

$$V_S = \mathbf{V}(w_0, w_2) \cup \mathbf{V}(w_0w_2 - w_1^2),$$

a quadric cone with a line intersecting it at the vertex. In both cases, $V_A = \mathbf{V}(w_0w_2 - w_1^2)$.

If $\lambda \neq q^{-1}$, then V_S is generically 3 lines. However, if $\alpha_1 \in \{0, \pm(q^{-2} - \lambda q^{-1})\}$, then V_S is a plane union with a line. In both cases, $V_A = \mathbf{V}(w_1, w_0) \cup \mathbf{V}(w_1, w_2)$ (two lines).

Theorem 3.2. *Case* 6. In this case, we must have $\alpha_1 = 0$, and either $\alpha_0 = 0$ or $q^2 = -1$. The cubic polynomials defining V_S are

$$\begin{aligned} r_{01} &= w_1(q^2w_0^2 + \alpha_0(w_0w_2 - w_1^2)) \\ r_{02} &= -2q^2w_0^2w_2 + (\alpha_2 q^4 w_0 - \alpha_0 w_2)(w_0w_2 - w_1^2) \\ r_{12} &= w_1(-w_0w_2 + \alpha_2 q^2(w_0w_2 - w_1^2)) \end{aligned}$$

If $\alpha_0 = 0$, then V_S is generically two lines

$$V_S = \mathbf{V}(w_0, w_1) \cup \mathbf{V}(w_1, w_2).$$

However, if $\alpha_2 = 0$, then V_S is a plane union with a line, and if $\alpha_2 = 2q^{-2}$, then V_S is just a plane.

If $\alpha_0 \neq 0$ and $q^{-2} = -1$, then V_S consists of 3, 4 or 5 lines, all meeting in a common point.

In all of these cases, V_A consists of the two lines $\mathbf{V}(w_0, w_1)$ and $\mathbf{V}(w_1, w_2)$.

We summarize the results in the following table. We use the following notation:

$Q_s = \mathbf{V}(w_0w_2 - w_1^2)$ (quadric cone)

Q_s' = a quadric cone with a different defining polynomial

Q_n = a nonsingular quadric surface

q_s = nodal curve

$q_n(d)$ = smooth curve of degree d (isomorphic to $\mathbf{P}^1$)

3.2 Case	**point variety of V_S**	**point variety of $A^{(2)}$**	**Condition**
1	1 plane $\cup$ 3 lines	2 lines $\cup\, q_n(2)$	$a \neq \pm q^2 b$, $\alpha_i = 0$
	2 planes $\cup$ 1 line	2 lines $\cup\, q_n(2)$	$a = -q^2 b$, $\alpha_i = 0$
	$Q_n \cup 1$ line	2 lines$\cup\, q_n(2)$	$a = -b^{-1} = \pm q$, $\alpha_1 \neq 0$, $\alpha_0 = \alpha_2 = 0$
	$\mathbf{P}^3$	Q_s	$a = q^2 b$, $\alpha_i = 0$
	$Q_s \cup 1$ line	Q_s	$a = q^2 b$, $\alpha_0\alpha_2 \neq \alpha_1^2$
	Q_s	Q_s	$a = q^2 b$, $\alpha_0\alpha_2 = \alpha_1^2$
2	1 plane	1 line	$a = q^3$, $\alpha_i = 0$
	3 lines	1 line	$q^2 = -1$, $a = -q$, $\alpha_1 \neq 0$, $\alpha_0 = \alpha_2 = 0$
	4 lines	1 line	$q^3 = a = -1$, $\alpha_2 \neq 0$, $\alpha_0 = \alpha_1 = 0$
	$Q_s' \cup 1$ line	$q_n(3) \cup 1$ line	$a = -q^3$, $\alpha_i = 0$
	$Q_n \cup 1$ line	$q_n(3) \cup 1$ line	$q^2 = -1$, $a = q$, $\alpha_1 \neq 0$, $\alpha_0 = \alpha_2 = 0$
3	$Q_n \cup 1$ line	q_s	$q^4 = -1$, $\alpha_i = 0$
	7 lines	4 lines	$q^2 = -1$, $\alpha_1^2 \neq -4q$, $\alpha_0 = \alpha_2 = 0$
	1 plane $\cup\, 2$ lines	4 lines	$q^2 = -1$, $\alpha_1^2 = -4q$, $\alpha_0 = \alpha_2 = 0$

3.2 Case	**point variety of V_S**	**point variety of $A^{(2)}$**	**Condition**
4	$\mathbf{P}^3$	Q_s	$\lambda = q^{-2}$, $\alpha_i = 0$
	$Q_s \cup 1$ line	Q_s	$\lambda = q^{-2}$, $\alpha_1 = 0$, $\alpha_0\alpha_2 \neq 0$
	Q_s	Q_s	$\lambda = q^{-2}$, $\alpha_1 = 0$, $\alpha_0\alpha_2 = 0$ (α_0 and α_2 not both zero)
	1 plane	2 lines	$\lambda \neq q^{-2}$, $2q\lambda - 2q^{-1} = \alpha_0$, $\alpha_1 = \alpha_2 = 0$
	2 lines	2 lines	$\lambda \neq q^{-2}$, $2q\lambda - 2q^{-1} \neq \alpha_0$, $\alpha_1 = \alpha_2 = 0$
	3 lines	2 lines	$q^2 = -1$, $\lambda \neq -1$, $\alpha_2 \neq 0$, $\alpha_0 = q(\lambda + 1)$, $\alpha_1 = 0$
	4 lines	2 lines	$q^2 = -1$, $\lambda \neq -1$, $\alpha_2 \neq 0$, $\alpha_0 = 2q(\lambda + 1)$, $\alpha_1 = 0$
	5 lines	2 lines	$q^2 = -1$, $\lambda \neq -1$, $\alpha_1 = 0$, $\alpha_2 \neq 0$ general
	1 plane $\cup$ 1 line	2 lines	$\lambda \neq q^{-2}$, $\alpha_i = 0$
5	$\mathbf{P}^3$	Q_s	$\lambda = q^{-1}$, $\alpha_i = 0$
	$Q_s \cup 1$ line	Q_s	$\lambda = q^{-1}$, $\alpha_1 \neq 0$, $\alpha_0 = \alpha_2 = 0$
	1 plane $\cup$ 1 line	2 lines	$\lambda \neq q^{-1}$, $\alpha_1 \in \{0, \pm(q^{-2} - \lambda q^{-1})\}$, $\alpha_0 = \alpha_2 = 0$
	3 lines	2 lines	$\lambda \neq q^{-1}$, $\alpha_1 \notin \{0, \pm(q^{-2} - \lambda q^{-1})\}$, $\alpha_0 = \alpha_2 = 0$
6	1 plane	2 lines	$\alpha_0 = \alpha_1 = 0$, $\alpha_2 = 2q^{-2}$
	1 plane $\cup$ 1 line	2 lines	$\alpha_i = 0$
	2 lines	2 lines	$\alpha_0 = \alpha_1 = 0$, $\alpha_2 \notin \{0, 2q^{-2}\}$
	3 lines	2 lines	$q^2 = -1$, $\alpha_0 \neq 0$, $\alpha_1 = 0$, $\alpha_2 = -1$
	4 lines	2 lines	$q^2 = -1$, $\alpha_0 \neq 0$, $\alpha_1 = 0$, $\alpha_2 = -2$
	5 lines	2 lines	$q^2 = -1$, $\alpha_0 \neq 0$, $\alpha_1 = 0$, $\alpha_2 \notin \{-1, -2\}$

Acknowledgments. Discussions with Donatella Delfino and David Hahn at Hope College were helpful in the early stages of this project. Furthermore, we thank Timothy J. Pennings, Hope College, and the National Science Foundation for the opportunity to perform this work.

R. Higginbottom and G. Dietz thank the Mathematics Departments at Bucknell University and the University of Dayton for guidance and encouragement during their undergraduate mathematics education. The authors also thank the referee, who provided many suggestions and corrections which significantly improved this paper.

A portion of the computation for this article was carried out using the computer algebra system *Maple 6*. We have also relied heavily on Schelter's AFFINE subroutines for Maxima, and we are grateful to the author for making this program available to the mathematical community.

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