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SHARP INEQUALITIES FOR THE HURWITZ ZETA FUNCTION

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ABSTRACT. We prove the following double-inequality for the Hurwitz zeta function $\zeta(p,a)=\sum_{\nu=0}^{\infty}(\nu+a)^{-p}$.

Let m and n be integers with $m > n \geq 0$ and let a be a positive real number. Then we have for all real numbers $p > 1$:

$$\frac{m+1+a}{n+1+a} < \left(\frac{\zeta(p,a)-\sum_{\nu=0}^{n}(\nu+a)^{-p}}{\zeta(p,a)-\sum_{\nu=0}^{m}(\nu+a)^{-p}}\right)^{1/(p-1)}$$
$$< \exp\left(\sum_{\nu=n+1}^{m}\frac{1}{\nu+a}\right).$$

Both bounds are best possible.

Our theorem extends and refines a result of Bennett [**2**].

1. Introduction. In order to prove a sharp lower bound for the Cesàro matrix, Bennett [**2**] applied the following inequality for the "tail" of the series representation of the classical Riemann zeta function:

$$f_p(n) < f_p(n+1), \quad n = 1, 2, \dots ,$$

where

$$f_p(n) = n^{p-1}\sum_{\nu=n+1}^{\infty}\nu^{-p}, \quad p > 1.$$

The monotonicity of f_p provides an interesting upper bound for the ratio $\left(\sum_{\nu=n+1}^{\infty}\nu^{-p}/\sum_{\nu=m+1}^{\infty}\nu^{-p}\right)^{1/(p-1)}$, which does not depend on p:

$$\left(\frac{\zeta(p)-\sum_{\nu=1}^{n}\nu^{-p}}{\zeta(p)-\sum_{\nu=1}^{m}\nu^{-p}}\right)^{1/(p-1)} < \frac{m}{n}, \quad p > 1;\ m > n \geq 1. \tag{1.1}$$

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Another application of an inequality for $\zeta(p)-\sum_{\nu=1}^{n}\nu^{-p}$ was given by Cochran and Lee [**3**]. They established an upper bound for $\sum_{\nu=n+1}^{\infty}\nu^{-p}$ and used their result to prove a striking companion of the well-known Carleman inequality for infinite series.

The function

$$\zeta(p,a)=\sum_{\nu=0}^{\infty}(\nu+a)^{-p},$$

introduced by Hurwitz in 1882 for complex numbers p with $\Re p>1$ and real numbers $a>0$, is known in the literature as the Hurwitz zeta function. The function $\zeta(p,a)$ plays an important role in Analytic Number Theory. Its main properties can be found, for instance, in the monograph [**1**] and in the recently published article [**5**]. A probabilistic interpretation of the Hurwitz zeta function is given in [**4**].

In view of (1.1) it is natural to look for a corresponding inequality for $\zeta(p,a)$. More precisely, we ask: let

$$Q_{n,m}(p,a)=\left(\frac{\zeta(p,a)-\sum_{\nu=0}^{n}(\nu+a)^{-p}}{\zeta(p,a)-\sum_{\nu=0}^{m}(\nu+a)^{-p}}\right)^{1/(p-1)}, \tag{1.2}$$

where m and n are fixed integers with $m>n\geq 0$ and $a>0$ is a fixed real number. What is the greatest number $\alpha_{n,m}(a)$ and what is the smallest number $\beta_{n,m}(a)$ such that the double-inequality

$$\alpha_{n,m}(a)\leq Q_{n,m}(p,a)\leq\beta_{n,m}(a)$$

holds for all $p>1$? It is the aim of this note to answer this question. In particular, we show that the upper bound in (1.1) can be improved.

2. Main result. The following theorem provides an extension and a refinement of Bennett's inequality (1.1).

Theorem. *Let m and n be integers with $m>n\geq 0$ and let a be a positive real number. Then we have for all real numbers $p>1$:*

(2.1)

$$\frac{m+1+a}{n+1+a}<\left(\frac{\zeta(p,a)-\sum_{\nu=0}^{n}(\nu+a)^{-p}}{\zeta(p,a)-\sum_{\nu=0}^{m}(\nu+a)^{-p}}\right)^{1/(p-1)}<\exp\Big(\sum_{\nu=n+1}^{m}\frac{1}{\nu+a}\Big).$$

Both bounds are best possible.

Proof. First we establish the lefthand side of (2.1) for $m = n+1$. A simple calculation yields

$$\sum_{\nu=n+1}^{\infty}(\nu+a)^{-p}\Big/\sum_{\nu=n+2}^{\infty}(\nu+a)^{-p}-\Big(\frac{n+2+a}{n+1+a}\Big)^{p-1} \tag{2.2}$$
$$=(n+1+a)^{1-p}\Big(\sum_{\nu=n+2}^{\infty}(\nu+a)^{-p}\Big)^{-1}$$
$$\times\big[(n+2+a)^{p-1}-(n+1+a)^{p-1}\big]S_n(p,a),$$

where

$$S_n(p,a) \tag{2.3}$$
$$=(n+1+a)^{-1}\big[(n+2+a)^{p-1}-(n+1+a)^{p-1}\big]^{-1}-\sum_{\nu=n+2}^{\infty}(\nu+a)^{-p}.$$

Let $x = n+1+a > 1$. Then we have

$$x\big[(x+2)^{p-1}-(x+1)^{p-1}\big]\big(S_n(p,a)-S_{n+1}(p,a)\big) \tag{2.4}$$
$$=\frac{(x+2)^{p-1}-(x+1)^{p-1}}{(x+1)^{p-1}-x^{p-1}}-\frac{x(x+2)^{p-1}}{(x+1)^{p}}.$$

We consider two cases.

Case 1. $1 < p < 2$. Cauchy's mean value theorem gives

$$\frac{(x+2)^{p-1}-(x+1)^{p-1}}{(x+1)^{p-1}-x^{p-1}}>\Big(\frac{x+1}{x}\Big)^{p-2} \tag{2.5}$$
$$=\frac{(x^2+2x+1)^{p-1}-(x^2+2x)^{p-1}}{x^{p-2}(x+1)^{p}}+\frac{x(x+2)^{p-1}}{(x+1)^{p}}.$$

Case 2. $p \ge 2$. Then we get

$$\frac{(x+2)^{p-1}-(x+1)^{p-1}}{(x+1)^{p-1}-x^{p-1}}\ge\Big(\frac{x+2}{x+1}\Big)^{p-2} \tag{2.6}$$
$$=\frac{(x+2)^{p-2}}{(x+1)^{p}}+\frac{x(x+2)^{p-1}}{(x+1)^{p}}.$$

From (2.4), (2.5), and (2.6) we conclude that $n \mapsto S_n(p,a)$, $n = 0,1,2,\dots$, is strictly decreasing. Since, by the mean value theorem,

$$x\left[(x+1)^{p-1} - x^{p-1}\right] \geq (p-1)\min\left(x^{p-1}, \frac{x}{x+1}(x+1)^{p-1}\right), \quad x > 0,$$

we obtain from (2.3) that

$$\lim_{n\to\infty} S_n(p,a) = 0.$$

Thus, we get

$$S_n(p,a) > 0 \quad \text{for} \quad n = 0,1,2,\dots. \tag{2.7}$$

The validity of the lefthand inequality of (2.1) with $m = n+1$ follows from (2.2) and (2.7).

Now, we prove the righthand side of (2.1) for $m = n+1$. We have

$$\begin{aligned} \exp\frac{p-1}{n+1+a} - \sum_{\nu=n+1}^{\infty}(\nu+a)^{-p}\Big/\sum_{\nu=n+2}^{\infty}(\nu+a)^{-p} \\ = \left(\exp\frac{p-1}{n+1+a} - 1\right)\Big(\sum_{\nu=n+2}^{\infty}(\nu+a)^{-p}\Big)^{-1} T_n(p,a), \end{aligned} \tag{2.8}$$

where

$$T_n(p,a) = \sum_{\nu=n+2}^{\infty}(\nu+a)^{-p} - (n+1+a)^{-p}\left(\exp\frac{p-1}{n+1+a} - 1\right)^{-1}. \tag{2.9}$$

Let $\Delta(x) = x/(e^x - 1)$. Since

$$(n+1+a)^{-p}\left(\exp\frac{p-1}{n+1+a} - 1\right)^{-1} = \Delta\Big(\frac{p-1}{n+1+a}\Big)\frac{1}{(p-1)(n+1+a)^{p-1}},$$

we obtain

$$\lim_{n\to\infty}(n+1+a)^{-p}\left(\exp\frac{p-1}{n+1+a} - 1\right)^{-1} = 0,$$

so that (2.9) implies

$$\lim_{n\to\infty} T_n(p,a) = 0. \tag{2.10}$$

Next, we show that $T_n(p,a)$ is strictly decreasing with respect to n. Let $x = n+1+a > 1$ and let $L(A,B) = (A-B)/(\log A - \log B)$ be the logarithmic mean of

$$A = x^p \exp\Big(\frac{p-1}{x+1}\Big)\Big(\exp\frac{p-1}{x} - 1\Big)$$

and

$$B = (x+1)^p\Big(\exp\frac{p-1}{x+1} - 1\Big).$$

Then we get

$$\begin{aligned}(2.11)\quad \frac{AB}{L(A,B)}\Big(\exp\frac{1-p}{x+1}\Big)&\big(T_n(p,a) - T_{n+1}(p,a)\big)\\ &= \frac{p-1}{x+1} - p\log\frac{x+1}{x} + \log\Big(\exp\frac{p-1}{x} - 1\Big)\\ &\quad - \log\Big(\exp\frac{p-1}{x+1} - 1\Big) = u(p), \quad \text{say.}\end{aligned}$$

Differentiation gives

$$(2.12)\qquad (p-1)^2 u''(p) = v\Big(\frac{p-1}{x+1}\Big) - v\Big(\frac{p-1}{x}\Big),$$

where

$$v(t) = \left(\frac{t/2}{\sinh(t/2)}\right)^2.$$

Since v is strictly decreasing on $(0,\infty)$, we conclude from (2.12) that $u''(p) > 0$ for $p > 1$. This implies

$$(2.13)\qquad u'(p) > u'(1^+) = \frac{2x+1}{2x(x+1)} - \log\frac{x+1}{x} = w(x), \quad \text{say.}$$

Here, we have used l'Hôpital's rule twice in evaluating $u'(1^+)$. A short computation yields $w'(x) = -[x(x+1)]^{-2}/2$ and $w(x) > \lim_{t\to\infty} w(t) = 0$. Hence, we get from (2.13) that $u(p) > u(1^+) = 0$. Thus, (2.11) implies that $n \mapsto T_n(p,a)$, $n = 0,1,2\ldots$, is strictly decreasing, so that (2.10) yields

$$(2.14)\qquad T_n(p,a) > 0 \quad \text{for} \quad n = 0,1,2,\ldots.$$

From (2.8) and (2.14) we conclude that the righthand inequality of (2.1) holds with $m = n + 1$.

Now, we prove that (2.1) is valid for $m = n + k$ with $k \geq 1$. Let $Q_{n,m}(p, a)$ be defined as in (1.2). Since

$$Q_{n,n+k}(p,a) = \prod_{j=1}^{k} Q_{n+j-1,n+j}(p,a), \tag{2.15}$$

we conclude from (2.1) with $m = n + 1$:

$$\frac{n+k+1+a}{n+1+a} = \prod_{j=1}^{k} \frac{n+j+1+a}{n+j+a} < Q_{n,n+k}(p,a)$$
$$< \prod_{j=1}^{k} \exp \frac{1}{n+j+a} = \exp\Big(\sum_{\nu=n+1}^{n+k} \frac{1}{\nu+a}\Big).$$

It remains to show that the bounds given in (2.1) are sharp. We prove

$$\lim_{p\to 1} Q_{n,n+1}(p,a) = \exp \frac{1}{n+1+a} \tag{2.16}$$

and

$$\lim_{p\to\infty} Q_{n,n+1}(p,a) = \frac{n+2+a}{n+1+a}, \tag{2.17}$$

so that (2.15), (2.16), and (2.17) imply

$$\lim_{p\to 1} Q_{n,n+k}(p,a) = \exp \sum_{\nu=n+1}^{n+k} \frac{1}{\nu+a} \tag{2.18}$$

and

$$\lim_{p\to\infty} Q_{n,n+k}(p,a) = \frac{n+k+1+a}{n+1+a}. \tag{2.19}$$

The limit relations (2.18) and (2.19) reveal that both bounds in (2.1) are best possible.

To prove (2.16) we use the fact that the function $p \mapsto \zeta(p,a)$ is holomorphic in $\mathbf{C} - \{1\}$ with a simple pole at $p = 1$ with residue 1, see [**1**, p. 255]. This leads to the representations

$$(p-1)\,\zeta\,(p,a) = 1 + \sum_{\nu=0}^{\infty} \gamma_\nu(a)(p-1)^{\nu+1} \tag{2.20}$$

and

$$(p-1)^2\,\frac{\partial\,\zeta\,(p,a)}{\partial p} = -(p-1)\,\zeta\,(p,a) + \sum_{\nu=0}^{\infty} \gamma_\nu(a)(\nu+1)(p-1)^{\nu+1}. \tag{2.21}$$

We have

$$\log Q_{n,n+1}(p,a) = \frac{-\log\left(1 - R_n(p,a)\right)}{p-1},$$

where

$$R_n(p,a) = (n+1+a)^{-p}\Big[\zeta\,(p,a) - \sum_{\nu=0}^{n}(\nu+a)^{-p}\Big]^{-1}.$$

Using l'Hôpital's rule and $R_n(p,a)|_{p=1} = 0$, we obtain

$$\lim_{p\to 1} \log Q_{n,n+1}(p,a) = \frac{\partial R_n(p,a)}{\partial p}\Big|_{p=1}. \tag{2.22}$$

Partial differentiation yields

$$\begin{aligned} (n+1+a)^p\,\frac{\partial R_n(p,a)}{\partial p} &= -(n+1+a)^p\,\big(\log\,(n+1+a)\big)R_n(p,a) \\ &\quad - \frac{(p-1)^2(\partial/\partial p)\,\zeta\,(p,a) + (p-1)^2\sum_{\nu=0}^{n}\big((\nu+a)^{-p}\log\,(\nu+a)\big)}{\Big[(p-1)\,\zeta\,(p,a) - (p-1)\sum_{\nu=0}^{n}(\nu+a)^{-p}\Big]^2}. \end{aligned} \tag{2.23}$$

From (2.20), (2.21), and (2.23) we get

$$\frac{\partial R_n(p,a)}{\partial p}\Big|_{p=1} = \frac{1}{n+1+a},$$

so that (2.22) implies (2.16).

Finally, we prove (2.17). Applying (2.3) and (2.7) we obtain

$$0 < (n+1+a)^p \Big[\zeta\,(p,a) - \sum_{\nu=0}^{n} (\nu+a)^{-p}\Big] - 1 < \Big[\Big(\frac{n+2+a}{n+1+a}\Big)^{p-1} - 1\Big]^{-1}.$$

This leads to

$$\lim_{p\to\infty} (n+1+a)^p \Big[\zeta\,(p,a) - \sum_{\nu=0}^{n} (\nu+a)^{-p}\Big] = 1. \tag{2.24}$$

Using (2.24) and

$$Q_{n,n+1}(p,a) = \left(\frac{(n+1+a)^p\Big[\zeta\,(p,a) - \sum_{\nu=0}^{n}(\nu+a)^{-p}\Big]}{(n+2+a)^p\Big[\zeta\,(p,a) - \sum_{\nu=0}^{n+1}(\nu+a)^{-p}\Big]}\right)^{1/(p-1)} \times \Big(\frac{n+2+a}{n+1+a}\Big)^{p/(p-1)},$$

we get (2.17). This completes the proof of the Theorem. □

Remark. If we set $a = 1$ in (2.1), then we obtain a double-inequality for the Riemann zeta function, which sharpens and complements inequality (1.1):

Let m and n be integers with $m > n \geq 1$. Then we have for all real numbers $p > 1$:

$$\frac{m+1}{n+1} < \left(\frac{\zeta(p) - \sum_{\nu=1}^{n} \nu^{-p}}{\zeta(p) - \sum_{\nu=1}^{m} \nu^{-p}}\right)^{1/(p-1)} < \exp\Big(\sum_{\nu=n+1}^{m} \frac{1}{\nu}\Big).$$

Both bounds are best possible.

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