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MATRICES DEFINING GORENSTEIN LATTICE IDEALS

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ABSTRACT. We study a class of integer matrices that define Gorenstein lattice ideals. We call them Gorenstein matrices. We give a combinatorial characterization of those which are of size $(n+1) \times n$ and we relate them to the Frobenius problem in integer programming theory. We also give a necessary and sufficient condition for Gorensteinness of generic matrices which are defined in integer programming theory.

1. Introduction. Let $S = k[\mathbf{x}] := k[x_1, \dots, x_n]$ be a polynomial ring over a fixed field k. A monomial $x_1^{u_1} \cdots x_n^{u_n}$ in S is denoted by $\mathbf{x}^u$, where $u = (u_1, \dots, u_n) \in \mathrm{N}^n$. A vector $u \in \mathrm{Z}^n$ can be written uniquely as $u = u^+ - u^-$, where u^+ and u^- are positive and negative parts of u, respectively. Let $B = (b_{ij})$ be an integer $n \times d$-matrix of rank d whose columns are vectors $b_1, \dots, b_d$ in Z^n. For the lattice $\mathcal{L}_B$ in Z^n which is spanned by the columns of B, the corresponding lattice ideal in S is the binomial ideal

$$I_{\mathcal{L}_B} := \langle \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \mid u \in \mathcal{L}_B \rangle.$$

The matrix B is called a defining matrix of $I_{\mathcal{L}_B}$. Such a matrix is of course not unique, but one can see easily that it is unique up to action of $SL_d(\mathrm{Z})$, that is, if B' is a second integer $n \times d$-matrix of rank d, then $I_{\mathcal{L}_B} = I_{\mathcal{L}_{B'}}$ if and only if for a unimodular matrix $T \in SL_d(\mathrm{Z})$, we have $B' = BT$.

The relationships between the matrix B and the lattice ideal $I_{\mathcal{L}_B}$ have been studied by many authors [**5, 7, 10, 15**] and [**16**]. It is well known that some numerical invariants and some algebraic properties of the lattice ideal $I_{\mathcal{L}_B}$ can be read off directly from the matrix B.

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For example, the codimension of $I_{\mathcal{L}_B}$ is equal to rank(B) and $I_{\mathcal{L}_B}$ is a prime ideal if and only if B has content (the gcd of all $d \times d$ minors of B) equal to 1 [**7**]. In general we can say that the matrix B has property $\wp$ (such as primeness, radicalness and so on) over a field k, if the lattice ideal $I_{\mathcal{L}_B} \subset k[\mathbf{x}]$ has this property. Note that the statement "over a field k" in this definition is crucial, since a property $\wp$ may or may not depend upon k. For example, primeness is independent of k, but radicalness depends upon k [**4**, Lemma 2.2]. The major motivation of the study of matrices defining lattice ideals is that, not only a lot of information of the lattice ideals is encoded in these matrices, but also they may have interesting properties by themselves even when they do not involve integrality [**6**].

In [**7**] the authors show that if the matrix B is mixed, then it is a complete intersection if and only if there exists a unimodular $d \times d$-matrix T such that the transposed matrix of $B' = BT$ is dominating. We recall from [**7**] that an $r \times s$-matrix M is called mixed if every row of M has both a positive and a negative entry. The matrix M is called dominating if it does not contain a square mixed submatrix. Also in [**5**] it has been shown that a mixed dominating matrix of an arbitrary size decomposes and will have a special format.

In this paper we will study the class of Gorenstein matrices which is more general than the class of complete intersection matrices.

In Section 3, first we show that the Cohen-Macaulay type of an integer $(n+1)\times n$-matrix B is equal to the number of maximal lattice point free polytopes of fibers, Theorem 3.2. Then as a consequence we will give a combinatorial characterization of Gorenstein matrices of size $(n+1)\times n$ in terms of maximal lattice point free polytopes, Corollary 3.3. The geometric significance of this class of matrices is due to affine monomial curves. This is because these matrices define monomial curves when they are toric (prime) matrices [**20**, Chapter 10]. As an application we use this characterization to solve some special cases of the Frobenius Problem, Problem 3.7. The Frobenius number is used to define a symmetric semigroup which is a good criterion for establishing that a monomial curve is Gorenstein [**3, 8, 12**] and [**20**, Chapter 10].

In Section 4, we will study Gorensteinness of a generic matrix which is defined in integer programming theory [**1**]. We will show that

Gorenstein generic matrices are precisely column matrices with some special properties, Theorem 4.7.

2. Grading and fibers. In this section we will show that the lattice ideal $I_{\mathcal{L}_B}$ has a bigraded structure. The following Proposition characterizes those matrices which are homogeneous with respect to a strictly positive integer weight vector $w = (w_1, \dots, w_n)$.

Proposition 2.1. *Let B be an integer $n \times d$-matrix of rank d and $\mathcal{L}_B$ its corresponding lattice in Z^n. Then the following conditions are equivalent*:

(1) *There exists a strictly positive integer vector $w = (w_1, \dots, w_n)$, such that $wB = 0$.*

(2) *$\mathcal{L}_B$ contains no non-negative vectors, i.e., $\mathcal{L}_B \cap \mathrm{N}^n = \{0\}$.*

(3) *For every $u \in \mathrm{R}^n$, the polyhedron $P_u := \{v \in \mathrm{R}^d : Bv \leq u\}$ is a polytope, that is, bounded.*

(4) *The origin is in the interior of the convex hull of the rows of B.*

(5) *S is a Z-graded ring with respect to* $\deg(x_i) = w_i$ *and $I_{\mathcal{L}_B}$ is a homogeneous ideal of S.*

(6) *$I_{\mathcal{L}_B} \subset \langle x_1, \dots, x_n \rangle$.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (5), (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (6) are trivial. (2) $\Rightarrow$ (1). Follows from Stiemke's theorem [**18**, Corollary 7.1k].

(5) $\Rightarrow$ (1). We can find a set of binomial generators $\{\mathbf{x}^{u_1^+} - \mathbf{x}^{u_1^-}, \dots, \mathbf{x}^{u_s^+} - \mathbf{x}^{u_s^-}\}$ such that each binomial of it is homogeneous with respect to the grading given by $\deg(x_i) = w_i$. Thus, $w \cdot u_i = 0$ for $i = 1, \dots, s$. Let $u \in \mathcal{L}_B$ be an arbitrary element. Then by definition of $I_{\mathcal{L}_B}$ we have $\mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in I_{\mathcal{L}_B}$. This implies that u is in the lattice spanned by the vectors $u_1, \dots, u_s$ and therefore $w \cdot u = 0$.

(1) $\Rightarrow$ (3). By [**18**, Corollary 7.1b], for each u we can find a polytope Q and a polyhedral cone C such that $P_u = Q + C$. Clearly $\{x : Bx \leq 0\} = \{0\}$ which implies that $C = \{0\}$. Thus, $P_u = Q$ is a polytope.

(3) $\Rightarrow$ (2). The polyhedral cone P_0 is a polytope implies that $P_0 = \{0\}$. This gives the result. $\square$

All lattices in this paper satisfy the equivalent conditions of Proposition 2.1.

Now abbreviate $\Gamma := \mathrm{Z}^n/\mathcal{L}_B$. For any monomial $\mathbf{x}^u \in S$ we define $\deg(\mathbf{x}^u) := u + \mathcal{L}_B$. Then S is graded by the Abelian group Γ and $I_{\mathcal{L}_B}$ is a homogeneous ideal of S with respect to this grading. The set of all monomials of a fixed degree $C \in \Gamma$ is called a fiber. Equivalently, the fibers are the congruence classes of N^n modulo $\mathcal{L}_B$. Thus, the set of all fibers is $\mathrm{N}^n/\mathcal{L}_B$.

Construction 2.2 (The polytope of a fiber [**15**] and [**16**]). The fiber containing a particular monomial $\mathbf{x}^u$ can be identified with the lattice points in the polyhedron

$$P_u := \{v \in \mathrm{R}^d : Bv \leq u\},$$

via the map

$$\begin{array}{ccccc} P_u \cap \mathrm{Z}^d & \longrightarrow & \mathrm{N}^n & \longrightarrow & S \\ v & \longmapsto & u - Bv & \longmapsto & \mathbf{x}^{u-Bv} \end{array}$$

Since, by Proposition 2.1, P_u is a polytope, each fiber C is a finite set. Two polytopes P_u and $P_{u'}$ are lattice translates of each other if $u - u' \in \mathcal{L}_B$. Disregarding lattice equivalence we write $P_C := P_u$ for all $\mathbf{x}^u \in C$. P_C is called *the polytope of fiber* C.

We say that P_C is a *maximal lattice point free polytope* if P_C contains no lattice points in its interior, but every facet of it contains at least one lattice point in its relative interior. We set

$$T(B) := \{P_C : C \in \mathrm{N}^n/\mathcal{L}_B,\ P_C \text{ is a maximal lattice point free polytope}\}.$$

By the above two graded structures, one can see that the ring S is *local with maximal homogeneous ideal $\langle x_1, \dots, x_n\rangle$ [**2**, Definition 1.5.13]. Thus one can consider the multi-graded minimal free resolution of $S/I_{\mathcal{L}_B}$ over S. The multi-graded Betti number $\beta_{i,C}$ of this definition is the k-dimension of the degree C piece of a Tor module:

$$\beta_{i,C} = \dim_k \operatorname{Tor}_i^S(S/I_{\mathcal{L}_B}, k)_C,$$

this counts the minimal i-syzygies of $S/I_{\mathcal{L}_B}$ having degree C.

Now we will use simplicial complex to compute the Betti numbers $\beta_{i,C}$ as follows:

Lemma 2.3 [**15, 16**] and [**13**, Theorem 9.2]. *Let C be a fiber of monomials of fixed degree C. Let Δ_C be the simplicial complex generated by the supports of all monomials in C. (In other words, Δ_C is a simplicial complex on the set $\{1,\dots,n\}$ and a subset F of $\{1,\dots,n\}$ is a face of Δ_C if and only if C contains a monomial $\mathbf{x}^u = x_1^{u_1}\cdots x_n^{u_n}$ whose* $\operatorname{supp}(u) = \{i : u_i \neq 0\}$ *contains F.) Then*

$$\beta_{i+1,C} = \operatorname{rank} \tilde{H}_i(\Delta_C; k)$$

where $\tilde{H}_i(\Delta_C; k)$ is the ith reduced homology group of Δ_C with coefficients in k.

3. Gorenstein integer matrices of size $(n+1)\times n$ and the Frobenius problem. Let I be a homogeneous ideal in S. Then the k-algebra S/I is Artinian if $\dim_k(S/I)$ is finite. The graded Artinian k-algebra S/I is Gorenstein if $\dim_k Soc(S/I) = 1$, where $\operatorname{Soc}(S/I) = \{\bar{f} \in S/I : \bar{f}\cdot\langle x_1,\dots,x_n\rangle = 0\}$. In general, the k-algebra $R := S/I$ is said to be Gorenstein if R/xR is Gorenstein for some homogeneous nonzero divisor $x \in R$. If R is not Artinian, then by finite iteration of the division operation we give an Artinian ring. The number of necessary divisions is the dimension of S/I. The ideal I is called Gorenstein if S/I is a Gorenstein ring [**20**, Section 1.3].

Lemma 3.1 [**9**, Theorem 3.5, Proposition 3.11] and [**20**, Corollary 4.3.5]. (1) *Let $I \subseteq S$ be a homogeneous ideal of codimension l. Then S/I is Cohen-Macaulay if and only if the graded minimal free resolution of S/I over S has length l.*

(2) *S/I is Gorenstein if and only if S/I is Cohen-Macaulay and the last Betti number in the graded minimal free resolution of S/I over S is equal to* 1.

Theorem 3.2. *Let $B = (b_{ij})$ be an integer $(n+1) \times n$-matrix of rank n. Then B is a Cohen-Macaulay matrix if and only if $\#T(B) \geq 1$. If B is a Cohen-Macaulay matrix, then the Cohen-Maculay type of B, i.e., the last Betti number in the graded minimal free resolution of $S/I_{\mathcal{L}_B}$ over S, is equal to $\#T(B)$.*

Proof. Since Δ_C has at most $n + 1$ vertices, it has no reduced homology if it has an n-dimensional face. (This is because if it has an n-dimensional face, it is a full simplex and therefore it is contractible.) This fact implies that $\tilde{H}_i(\Delta_C; k) = 0$ for all $i \geq n$. Since rank $(\mathcal{L}_B) = n$ and $\beta_{i+1,C} = \operatorname{rank} \tilde{H}_i(\Delta_C; k)$ by Lemma 2.3, we conclude that $\beta_{i,C} = 0$ for all $i > \operatorname{rank}(\mathcal{L}_B)$.

We also note that if C is a fiber such that $P_C \in T(B)$, then Δ_C is the boundary of a n-simplex and therefore it is homologous to the $(n-1)$-sphere. Consequently, $\beta_{i,C} = 1$ if $i = \operatorname{rank}(\mathcal{L}_B)$ and $\beta_{i,C} = 0$ if $i \neq \operatorname{rank}(\mathcal{L}_B)$.

The above discussion together with Lemma 3.1 shows that if $\#T(B) \geq 1$, then B is a Cohen-Macaulay matrix.

Now suppose that B is a Cohen-Macaulay matrix. Then by Lemma 3.1, $S/I_{\mathcal{L}_B}$ has a graded minimal free resolution over S with length $\operatorname{rank}(\mathcal{L}_B) = \operatorname{codim}(I_{\mathcal{L}_B})$. We claim that the last term of this resolution is equal to $\oplus_{P_C \in T(B)} S(-C)$. Suppose C is a fiber such that $S(-C)^{\beta_{n,C}}$ is a direct summand of the last term of the resolution. Since Δ_C is a simplicial complex on the vertex set $\{1, \ldots, n+1\}$, we must have $\beta_{n,C} = 1$. Also the minimality of the resolution implies that $\beta_{i,C} = 0$ for all $i \neq n$. Thus Δ_C is homologous to the $(n-1)$-sphere and therefore it is the boundary of a n-simplex. So, by the definition of P_C we have $P_C \in T(B)$.

The above proof also shows that the Cohen-Macaulay type of a Cohen-Macaulay matrix B is equal to $\#T(B)$. □

Corollary 3.3. *Let B be an integer $(n+1) \times n$-matrix of rank n. Then B is Gorenstein if and only if there exists a unique fiber $C \in \mathrm{N}^{n+1}/\mathcal{L}_B$ such that P_C is a maximal lattice point free polytope.*

Proof. Follows immediately from Lemma 3.1 and Theorem 3.2. □

Example 3.4. If $B = \begin{pmatrix} 2 & -1 \\ 2 & 3 \\ -4 & -2 \end{pmatrix}$, then B is not Gorenstein because the two polytopes $P_{(2,3,0)}$ and $P_{(3,2,0)}$ are maximal lattice point free polytopes which are not lattice translates of each other.

Remark 3.5. If B is an integer $n \times 2$-matrix of rank 2, then Peeva and Sturmfels [**16**] proved that, for the matrix B, being complete intersection is equivalent to being Gorenstein, and they also show that in this case B must be imbalanced (up to the action of $SL_2(\mathrm{Z})$), i.e., $b_{i1} = 0$ or $b_{i2} \leq 0$. Using this observation one can see that, up to a permutation of the rows and columns, a Gorenstein 3×2-matrix must be of the form $B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$, where $b_{21}b_{31} < 0$, $b_{12} > 0$, b_{22} and b_{32} are not positive and at least one of them is not zero. Lemma 3.6 below gives the unique fiber $C \in \mathrm{N}^3/\mathcal{L}_B$ in Corollary 3.3 such that P_C is a maximal lattice point free polytope.

Lemma 3.6. *Let B be a complete intersection $(n+1) \times n$-matrix such that the transposed matrix of B is a mixed dominating matrix. Let $b_1, \dots, b_n$ be the columns of B. Then the unique fiber C in Corollary* 3.3 *is the fiber containing the monomial* $\mathbf{x}^{b_1^+ + \cdots + b_n^+}$.

Proof. If the hypothesis that the transposed of B is mixed dominating does not hold, then one can multiply matrix B by a unimodular matrix until it holds. Since B is complete intersection, it is Gorenstein and by Corollary 3.3 there exists a unique fiber $C \in \mathrm{N}^{n+1}/\mathcal{L}_B$ such that P_C is a maximal lattice point free polytope. By the proof of Theorem 3.2, the fiber C is one which appears in the last term of the minimal free resolution of $S/I_{\mathcal{L}_B}$ over S. Since B is complete intersection we can find the fiber C using the Koszul complex $\mathbf{K}$ of S with respect to the binomials $\mathbf{x}^{b_1^+} - \mathbf{x}^{b_1^-}, \dots, \mathbf{x}^{b_n^+} - \mathbf{x}^{b_n^-}$. For each $i = 1, \dots, n$, suppose f_{u_i} denotes $\mathbf{x}^{b_i^+} - \mathbf{x}^{b_i^-}$ and e_i denotes the element $(0, \dots, 0, 1, 0, \dots, 0) \in S^n$ which has ith component 1 and all its other components 0. Then $\mathbf{K}$ has the form

$$0 \longrightarrow \bigwedge^{n} S^n \longrightarrow \cdots \longrightarrow \bigwedge^{k} S^n \xrightarrow{\partial_k} \bigwedge^{k-1} S^n \longrightarrow \cdots \longrightarrow \bigwedge^{0} S^n \longrightarrow 0,$$

where ∂_k is defined by

$$\partial_k(e_{i(1)} \wedge \cdots \wedge e_{i(k)}) = \sum_{h=1}^{k} (-1)^{n-1} f_{u_{i(h)}} e_{i(1)} \wedge \cdots \wedge \widehat{e_{i(h)}} \wedge \cdots e_{i(k)},$$

where the $\widehat{e_{i(h)}}$ indicates that $e_{i(h)}$ is omitted. If we assign to e_i the degree $b_i^+ + \mathcal{L}_B$, then $\mathbf{K}$ will become the graded minimal free resolution of $S/I_{\mathcal{L}_B}$ over S. Since $\deg(e_1 \wedge \cdots \wedge e_n) = b_1^+ + \cdots + b_n^+ + \mathcal{L}_B$, we get the result. □

Next we want to relate Gorenstein $(n+1) \times n$-matrices to the Frobenius problem in integer programming theory.

Problem 3.7 (Frobenius problem). Let $w = (w_1, \dots, w_n)$ be a strictly positive integer vector whose greatest common divisor is unity. The problem is to find the largest integer f^* which cannot be written as a non-negative integer combination of w_i.

Theorem 3.8 (Scarf and Shallcross [**17**]). *Let $w = (w_1, \dots, w_{n+1})$ be as in Problem* 3.7. *Let B be an integer $(n+1) \times n$-matrix whose columns generate the lattice $\ker_z(w) := \{x \in \mathrm{Z}^{n+1} : w \cdot x = 0\}$. Then*

$$f^* = \max\{w \cdot u : P_u \textit{is a maximal lattice point free polytope}\} - \sum_{i=1}^{n+1} w_i.$$

Corollary 3.9. *With the notation as in Theorem* 3.8, *let B be a Gorenstein matrix. Then*

$$f^* = w \cdot u - \sum_{i=1}^{n+1} w_i,$$

where u is a vector such that the fiber containing $\mathbf{x}^u$ yields the unique maximal lattice point free polytope P_C in Corollary 3.3. *Consequently, if B is a complete intersection matrix such that its transposed matrix is mixed dominating, then*

$$f^* = w \cdot (b_1^+ + \cdots + b_n^+) - \sum_{i=1}^{n+1} w_i,$$

where, $b_1, \dots, b_n$ are the columns of B.

Proof. Let $u \in \mathbb{Z}^{n+1}$ be such that P_u is a maximal lattice point free polytope and gives the maximum value of $w \cdot u$. We choose a lattice point v_0 of P_u and we consider the polytope $P_u - v_0 = P_{u'}$ where $u' = u - Bv_0$ is a non-negative integer vector. Thus $P_{u'}$ is equal to the maximal lattice point free polytope P_C where C is the fiber containing the monomial $\mathbf{x}^{u'}$. By Corollary 3.3, the fiber C is unique and since lattice translation does not change the value of $w \cdot u$, we get the first result.

The second part of the corollary follows from the first part and Lemma 3.6. $\square$

Example 3.10. Let $w = (w_1, w_2)$ be a strictly positive integer vector with $\gcd(w_1, w_2) = 1$. Then it is easy to see that the column of matrix $B = \binom{-w_2}{w_1}$, generates $\ker_{\mathbb{Z}}(w) \subset \mathbb{Z}^2$. Since B is complete intersection by Corollary 3.9, we have

$$f^* = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} 0 \\ w_1 \end{pmatrix} - w_1 - w_2 = w_1 w_2 - w_1 - w_2.$$

Remark 3.11. Let $w_1, \dots, w_{n+1}$ be an arithmetical sequence whose common difference is $d \geq 1$, and assume that (1) $w_1 = qn + r$, where $r, q \in \mathbb{N}$ and $2 \leq r \leq n + 1 \neq 2$ and (2) $\gcd(w_1, d) = 1$. Then one can use Example 3.10 to show that $f^* = (q+d)w_1 - d$ [**20**, Lemma 10.2.12].

Example 3.12. Let $w = (w_1, w_2, w_3)$ be a strictly positive integer vector with $\gcd(w_1, w_2, w_3) = 1$. If w_2 divides w_3 then $f^* = w_1 w_2 - w_1 - w_2$. For the proof, we note that the columns of the matrix

$$B = \begin{pmatrix} -w_2 & 0 \\ w_1 & -w_3/w_2 \\ 0 & 1 \end{pmatrix}$$

generate the lattice $\ker_{\mathbb{Z}}(w)$. Clearly B is a complete intersection matrix satisfying the conditions of Corollary 3.9. Thus,

$$f^* = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} 0 \\ w_1 \\ 1 \end{pmatrix} - w_1 - w_2 - w_3 = w_1 w_2 - w_1 - w_2.$$

4. Gorenstein generic matrices. In this section we want to show when a generic matrix will be Gorenstein. First we recall some definitions from [**1**].

Let $B = (b_{ij})$ be an integer $n \times d$-matrix with $n > d$ and rank $(B) = d$. Then a lattice point $u \in \mathrm{Z}^d$, $u \neq 0$, is a neighbor of the origin, if the body

$$\langle 0, u \rangle := \{v \in \mathrm{R}^d : Bv \leq (Bu)^+\}$$

contains no lattice points in its interior. The set of all neighbors of the origin is denoted by $N(B)$.

Example 4.1. Let $B = (b_1, \dots, b_n)^t$ be an $n \times 1$-matrix. Let $w = (w_1, \dots, w_n)$ be a strictly positive vector such that $wB = 0$. Then it is easy to see that for every $u \in \mathrm{Z}$, $u \neq 0$, we have $\langle 0, u \rangle$ is equal to $[0, u]$ if $u > 0$ and $\langle 0, u \rangle$ is equal to $[u, 0]$ if $u < 0$, where $[-, -]$ denotes a closed interval in the real line R. Thus, clearly we have $N(B) = \{-1, 1\}$.

Definition 4.2. Let $B = (b_{ij})$ be an integer $n \times d$-matrix, with $n > d$ and rank $(B) = d$. Then B is generic if the following conditions hold:

(1) B is homogeneous with respect to a strictly positive weight vector w.

(2) For every non-negative vector v, $vB = 0$ implies $\#\mathrm{supp}\,(v) \geq d+1$.

(3) For every $u \in N(B)$, $\#\mathrm{supp}\,(Bu) = n$.

Example 4.3 (continued from Example 4.1). If the entries of the $n \times 1$-matrix B are all nonzero, then clearly B is generic.

The term 'generic' is justified by the following theorem in integer programming theory.

Theorem 4.4 (Bárány and Scarf [**1**]). (1) *Generic matrices form a dense set in the collection of all matrices satisfying conditions* (1) *and* (2) *in Definition* 4.2.

(2) *Let B be a generic matrix and B' a matrix satisfying*

$$\text{Sign-Pattern}(Bu) = \text{Sign-Pattern}(B'u)$$

for every $u \in N(B)$. Then B' is also generic.

Following Peeva and Sturmfels [**15**] we call a lattice ideal generic if it is generated by binomials with full support. The following theorem shows why a generic lattice ideal is called generic.

Theorem 4.5 (Peeva and Sturmfels [**15**]). *Let $B = (b_{ij})$ be a generic $n \times d$-matrix. A vector $u \in \mathrm{Z}^d$ is in $N(B)$ if and only if $\{\mathbf{x}^{(Bu)^+}, \mathbf{x}^{(Bu)^-}\}$ is a 2-element fiber. In this case $\mathbf{x}^{(Bu)^+} - \mathbf{x}^{(Bu)^-}$ is a minimal generator of $I_{\mathcal{L}_B}$. Consequently, $I_{\mathcal{L}_B}$ has a unique minimal system of Γ-homogeneous binomial generators, which correspond to the elements in $N(B)$.*

Example 4.6. Let $B = \begin{pmatrix} 4 & -2 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{pmatrix}$. Then B is homogeneous with respect to the weight vector $w = (20, 24, 25, 31)$, and

$$N(B) = \pm \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

We have that B is a generic matrix and, by Theorem 4.5, the unique minimal set of Γ-homogeneous binomial generators of $I_{\mathcal{L}_B} \subset k[x_1, x_2, x_3, x_4]$ is equal to $\{x_1^4 - x_2x_3x_4, x_1^3x_3^2 - x_2^2x_4^2, x_1^2x_2^3 - x_3^2x_4^2, x_1x_2^2x_3 - x_4^3, x_2^4 - x_1^2x_3x_4, x_2^3x_3^2 - x_1^3x_4^2, x_3^3 - x_1x_2x_4\}$. A description of neighbors of the origin for four by three matrices can be found in [**19**] and [**1**].

Theorem 4.7. *Let B be a generic matrix. Then B is a Gorenstein matrix if and only if B is a column matrix.*

Proof. First we assume that $B = (b_1, \ldots, b_n)^t$ is an $n \times 1$-matrix. Set $b := (b_1, \ldots, b_n)$. Then the lattice $\mathcal{L}_B$ is generated by the vector b and it is easy to see that $I_{\mathcal{L}_B} = \langle \mathbf{x}^{b^+} - \mathbf{x}^{b^-} \rangle$. Since $I_{\mathcal{L}_B}$ is generated by one element and the codimension of $I_{\mathcal{L}_B}$ is equal to rank $(\mathcal{L}_B) = 1$, $I_{\mathcal{L}_B}$ is complete intersection and therefore it is Gorenstein.

Conversely, suppose B is a Gorenstein matrix. By Theorem 4.5, we may assume that $\{m_1 - x_n m'_1, \ldots, m_r - x_n m'_r\}$ is a unique minimal set of Γ-homogeneous binomial generators of $I_{\mathcal{L}_B}$ where each monomial m_i is not divisible by x_n and each binomial $m_i - x_n m'_i$ contains all variables. Suppose $w = (w_1, \ldots, w_n)$ is a strictly positive integer vector such that $wB = 0$. Then, by Proposition 2.1, $I_{\mathcal{L}_B}$ is homogeneous with respect to deg $(x_i) = w_i$. We fix a degree reverse lexicographic term order $\prec$ of the monomial relative to this grading on S, that is, $x_1^{p_1} \cdots x_n^{p_n} \prec x_1^{q_1} \cdots x_n^{q_n}$ if either $\sum_{i=1}^n p_i w_i < \sum_{i=1}^n q_i w_i$ or $\sum_{i=1}^n p_i w_i = \sum_{i=1}^n q_i w_i$ and the last nonzero coordinate of $(p_1 - q_1, \ldots, p_n - q_n)$ is positive. We denote by M the initial ideal of $I_{\mathcal{L}_B}$ with respect to the fixed term order $\prec$. Then by [**16**, Lemma 8.4] $\{m_1, \ldots, m_r\}$ is the set of minimal generators of M. Since by [**11**, Corollary 2.9] codim $(I_{\mathcal{L}_B}) =$ codim (M), this implies that $I_{\mathcal{L}_B}$ is a complete intersection if and only if M is complete intersection.

Again the genericity of $I_{\mathcal{L}_B}$ implies that M is a generic monomial ideal, by [**14**, Theorem 3.1]. Thus, by [**14**, Theorem 1.5] the minimal free resolution of S/M is the monomial Scarf complex $\mathbf{F}_M$. Also by [**15**, Theorem 4.2] the minimal free resolution of $S/I_{\mathcal{L}_B}$ is the algebraic Scarf complex $\mathbf{F}_{\mathcal{L}_B}$. Since by [**15**, Corollary 5.5] these two complexes differ only in their differential and recover each other, Lemma 3.1 implies that $I_{\mathcal{L}_B}$ is Gorenstein if and only if M is Gorenstein. Since the minimal free resolution of S/M is of $\mathbf{F}_\Delta$-type, by [**21**, Corollary 2.11] we have M is Gorenstein if and only if M is complete intersection. Thus, for the generic lattice ideal $I_{\mathcal{L}_B}$ being Gorenstein is equivalent to being complete intersection.

Now suppose B is a complete intersection matrix and suppose the contrary that $d \geq 2$. Then there exists a unimodular matrix $T \in SL_d(\mathrm{Z})$ such that the transposed matrix of the matrix $BT = B'$ is mixed dominating. Therefore by [**5**, Theorem 2.2] the matrix B', after rearranging the rows and columns if it is necessary, must be of the form

$$B' = \left(\begin{array}{c|c|c} M & 0 & m \\ \hline 0 & N & n \end{array} \right),$$

where M, N, m and n are matrices with properties described in [**5**, Theorem 2.2]. Since the generators of $I_{\mathcal{L}_B} = I_{\mathcal{L}_{B'}}$ are determined by columns of B' (because $I_{\mathcal{L}_B}$ is a complete intersection) this contradicts the genericity of $I_{\mathcal{L}_B}$ because, by Theorem 4.5, a generic lattice ideal has a unique minimal set of Γ-homogeneous binomial generators in which each binomial is with full support. Thus d must be equal to 1. □

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REFERENCES

1. I. Bárány and H. Scarf, *Matrices with identical sets of neighbors*, Math. Oper. Res. **23** (1998), 863–873.

2. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Univ. Press, Cambridge, 1993.

3. M. Estrada and A. López, *A note on symmetric semigroups and almost arithmetic sequences*, Comm. Algebra **22** (1994), 3903–3905.

4. K. Eto, *Binomial arithmetical rank of lattice ideals*, Manuscripta Math. **109** (2002), 455–463.

5. K. Fischer, W. Morris and J. Shapiro, *Affine semigroup rings that are complete intersections*, Proc. Amer. Math. Soc. **125** (1997), 3137–3145.

6. ———, *Mixed dominating matrices*, Linear Algebra Appl. **270** (1998), 191–214.

7. K. Fischer and J. Shapiro, *Mixed matrices and binomial ideals*, J. Pure Appl. Algebra **113** (1996), 39–54.

8. R. Fröberg, *The Frobenius number of some semigroups*, Comm. Algebra **22** (1994), 6021–6024.

9. M. Hochster, *Cohen-Macaulay rings, combinatorics and simplicial complexes*, in *Ring theory* II (B.R. McDonald and R. Morris, eds.), Lecture Notes in Pure and Appl. Math., vol. 26, Dekker, New York, 1977, pp. 171–223.

10. S. Hosten and J. Shapiro, *Primary decomposition of lattice basis ideals*, J. Symbolic Comput. **29** (2000), 625–639.

11. S. Hosten and R. Thomas, *The associated primes of initial ideals of lattice ideals*, Math. Res. Lett. **6** (1999), 83–97.

12. L. Juan, *Some results about symmetric semigroups*, Comm. Algebra **21** (1993), 3637–3645.

13. E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, vol. 227, Springer-Verlag, Berlin, New York, 2005.

14. E. Miller, B. Sturmfels and K. Yanagawa, *Generic and cogeneric monomial ideals*, J. Symbolic Comput. **29** (2000), 691–708.

15. I. Peeva and B. Sturmfels, *Generic lattice ideals*, J. Amer. Math. Soc. **11** (1998), 363–373.

16. ———, *Syzygies of codimension* 2 *lattice ideals*, Math. Z. **229** (1998), 163–194.

17. H. Scarf and D. Shallcross, *The Frobenius problem and maximal lattice free bodies*, Math. Oper. Res. **18** (1993), 511–515.

18. A. Schrijver, *Theory of linear and integer programming*, Wiley Interscience, Chichester, 1986.

19. D. Shallcross, *Neighbors of the origin for four by three matrices*, Math. Oper. Res. **17** (1992), 608–614.

20. R. Villarreal, *Monomial algebras*, Marcel Dekker, Inc., New York, 2001.

21. K. Yanagawa, $\mathbf{F}_\triangle$-*type free resolutions of monomial ideals*, Proc. Amer. Math. Soc. **127** (1999), 377–383.

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