

SOME NEW IYENGAR TYPE INEQUALITIES

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ABSTRACT. Some new Iyengar type inequalities for an integral are obtained by using the generalized Taylor formula with integral remainder.

1. Introduction. Let $f(x)$ be a differentiable function on a closed interval $[a, b]$ such that $|f'(x)| \leq M$. Then

$$(1) \left| \int_a^b f(x) dx - \frac{(b-a)[f(a) + f(b)]}{2} \right| \leq \frac{(b-a)^2 M}{4} - \frac{[f(b) - f(a)]^2}{4M}.$$

In 1938, Iyengar [16] established inequality (1) by using a geometric approach. So, we call (1) the Iyengar inequality.

Using the Rolle and Lagrange mean value theorems, the following inequalities were obtained naturally and simply in [27], producing a refinement of the Iyengar inequality (1).

Theorem A. *Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b) , and let $m \leq f'(x) \leq M$ for*

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$x \in (a, b)$. If $f(x)$ is not a constant, then we have

$$(2) \quad \frac{mM(b-a)^2 + 2(b-a)(Mf(a) - mf(b)) + (f(a) - f(b))^2}{2(M-m)} \\ \leq \int_a^b f(x) dx \\ \leq -\frac{mM(b-a)^2 + 2(b-a)(mf(a) - Mf(b)) + (f(a) - f(b))^2}{2(M-m)}.$$

For $f'(x)$ integrable on $[a, b]$, the inequalities in (2) were obtained independently by Agarwal and Dragomir in [2] using the Hayashi inequality. They expressed the result (2) as

$$(3) \quad \left| \int_a^b f(t) dt - (b-a) \frac{f(a) + f(b)}{2} \right| \\ \leq \frac{[f(b) - f(a) - m(b-a)][M(b-a) - f(b) + f(a)]}{2(M-m)}.$$

In [3], the inequalities in (2) and in (3) were derived in a different manner and rearranged as

$$(4) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] \right| \leq \frac{(b-a)^2}{2(M-m)} (S-m)(M-S) \\ (5) \quad \leq \frac{M-m}{2} \left(\frac{b-a}{2} \right)^2,$$

where $S = f(b) - f(a)/(b-a)$.

In the papers [28, 29], using the Taylor formula for functions with a single variable or several variables, the Iyengar inequality (1) was generalized as follows.

Theorem B [28]. Let $f(x)$ be a differentiable function of $C^n([a, b])$ satisfying $N \leq f^{(n+1)}(x) \leq M$ for $x \in (a, b)$. Denote

$$(6) \quad S_n(u, v, w) = \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} u^k f^{(k-1)}(v) + (-1)^n \frac{w}{n!} u^n$$

and

$$(7) \quad \frac{\partial^k S_n}{\partial u^k} = S_n^{(k)}(u, v, w).$$

Then, for any $t \in [a, b]$, we have when n is odd,

$$(8) \quad \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, N) - S_{n+2}^{(i)}(b, b, N) \right) t^i \leq \int_a^b f(x) dx \\ \leq \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, M) - S_{n+2}^{(i)}(b, b, M) \right) t^i;$$

and when n is even,

$$(9) \quad \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, N) - S_{n+2}^{(i)}(b, b, M) \right) t^i \leq \int_a^b f(x) dx \\ \leq \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, M) - S_{n+2}^{(i)}(b, b, N) \right) t^i.$$

We may deduce both Theorem A and the following result as specializations of Theorem B.

Theorem C [29]. Let $f(x)$ be differentiable on the closed interval $[a, b]$, and $N \leq f''(x) \leq M$ for $x \in (a, b)$. Then

$$(10) \quad \frac{N(b^3 - a^3)}{6} + \frac{\{f(a) - f(b) + bf'(b) - af'(a) + [N(a^2 - b^2)]/2\}^2}{2[(a - b)N - f'(a) + f'(b)]} \\ \leq \int_a^b f(x) dx - bf(b) + af(a) + \frac{b^2 f'(b) - a^2 f'(a)}{2} \\ \leq \frac{M(b^3 - a^3)}{6} + \frac{\{f(a) - f(b) + bf'(b) - af'(a) + [M(a^2 - b^2)]/2\}^2}{2[(a - b)M - f'(a) + f'(b)]}.$$

In recent years, Qi, Guo, Zhang and Cui, using the mean value theorems for functions with a single variable or several variables,

generalized the Iyengar inequality to those involving bounds of higher order derivatives, with norm bounds, for weighted multiple integrals, and the like. See [8, 12, 13, 15], [27–30] and [32].

Meanwhile, Agarwal, Cerone, Dragomir and Wang proceeded in an alternate direction, utilizing the Hayashi inequality or Steffensen inequality, generalizing Iyengar's inequality to those involving higher derivatives, weighted integrals, and so on. See [2–6] and [10].

The Iyengar inequality (1) has been researched by many mathematicians, and there is much literature devoted to it. Please refer to the references in this paper. Before elaborating on the contributions of the current article, it is useful to present some background notations and definitions.

Definition 1. A sequence of polynomials $\{P_i(t, x)\}_{i=0}^{\infty}$ is called harmonic if it satisfies the following Appell condition

$$(11) \quad P'_i(t) \triangleq \frac{\partial P_i(t, x)}{\partial t} = P_{i-1}(t, x) \triangleq P_{i-1}(t)$$

and $P_0(t, x) = 1$ for all defined (t, x) and $i \in \mathbf{N}$.

It is well known that Bernoulli polynomials $B_i(t)$ can be defined by the following expansion

$$(12) \quad \frac{xe^{tx}}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x| < 2\pi, \quad t \in \mathbf{R},$$

and are uniquely determined by the following formulae

$$(13) \quad B'_i(t) = iB_{i-1}(t), \quad B_0(t) = 1;$$

and

$$(14) \quad B_i(t+1) - B_i(t) = it^{i-1}.$$

Similarly, Euler polynomials can be defined by

$$(15) \quad \frac{2e^{tx}}{e^x + 1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x| < \pi, \quad t \in \mathbf{R},$$

and are uniquely determined by the following properties

$$(16) \quad E'_i(t) = iE_{i-1}(t), \quad E_0(t) = 1;$$

$$(17) \quad E_i(t+1) + E_i(t) = 2t^i.$$

For further details about Bernoulli polynomials and Euler polynomials, please refer to [1, 23.1.5 and 23.1.6] and [34]. Moreover, some new generalizations of Bernoulli numbers and polynomials can be found in [14, 18–20, 31].

There are many examples of harmonic sequences of polynomials. For instance, [3, 7, 21], for i a nonnegative integer, $t, \tau, \theta \in \mathbf{R}$ and $\tau \neq \theta$,

$$(18) \quad P_{i,\lambda}(t) \triangleq P_{i,\lambda}(t; \tau; \theta) = \frac{[t - (\lambda\theta + (1-\lambda)\tau)]^i}{i!},$$

$$(19) \quad P_{i,B}(t) \triangleq P_{i,B}(t; \tau; \theta) = \frac{(\tau - \theta)^i}{i!} B_i\left(\frac{t - \theta}{\tau - \theta}\right),$$

$$(20) \quad P_{i,E}(t) \triangleq P_{i,E}(t; \tau; \theta) = \frac{(\tau - \theta)^i}{i!} E_i\left(\frac{t - \theta}{\tau - \theta}\right).$$

As usual, let $B_i = B_i(0)$, $i \in \mathbf{N}$, denote Bernoulli numbers. From properties (13)–(14) and (16)–(17) of Bernoulli and Euler polynomials, respectively, we can obtain easily that, for $i \geq 1$,

$$(21) \quad B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2},$$

and, for $j \in \mathbf{N}$,

$$(22) \quad E_j(0) = -E_j(1) = -\frac{2}{j+1} (2^{j+1} - 1)B_{j+1}.$$

It is also a well-known fact that $B_{2i+1} = 0$ for all $i \in \mathbf{N}$.

In [21], the following generalized Taylor formula was established.

Theorem D. Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. Further, let $I \subset \mathbf{R}$ be a closed interval and $a \in I$. If $f : I \rightarrow \mathbf{R}$ is any function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbf{N}$, then, for any $x \in I$, we have

(23)

$$f(x) = f(a) + \sum_{k=1}^n (-1)^{k+1} [P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)] + R_n(f; a, x),$$

where

$$(24) \quad R_n(f; a, x) = (-1)^n \int_a^x P_n(t)f^{(n+1)}(t)dt.$$

If we set in (23) and (24) $P_n(t) = (t-x)^n/n!$, then we obtain directly the classical Taylor formula with remainder of integral form. Generalized Taylor formulae from product Appel polynomials were considered in [4] which involve the product of polynomials satisfying the Appel condition (1).

In this article, using the generalized Taylor formula (23), we will derive some new Iyengar type inequalities for an integral of functions with a single variable, which generalize some related known results obtained in [8, 12, 13, 15, 16, 27, 28, 32], for example.

2. Some integral identities. In this section we establish two identities involving integrals which form the basis for the procurement of our main results.

Theorem 1. Let $\{P_i(x)\}_{i=0}^{\infty}$ and $\{Q_i(x)\}_{i=0}^{\infty}$ be harmonic sequences of polynomials. If $f : [a, b] \rightarrow \mathbf{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbf{N}$, then we have the following generalized Taylor identity

$$(25) \quad (n+1) \int_a^b f(x) dx \\ = b \sum_{k=0}^n q(k, k; b) - a \sum_{k=0}^n p(k, k; a) + \sum_{k=1}^n \sum_{i=1}^k [p(i, i-1, a) - q(i, i-1; b)]$$

$$\begin{aligned}
 &+ t \sum_{k=0}^n [p(k, k; a) - q(k, k; b)] + \sum_{k=1}^n \sum_{i=1}^k [q(i, i - 1, t) - p(i, i - 1; t)] \\
 &+ \int_a^t (t - s)p(n, n + 1; s) ds + \int_t^b (t - s)q(n, n + 1; s) ds,
 \end{aligned}$$

where $t \in [a, b]$ and

$$(26) \quad p(l, m; x) = (-1)^l P_l(x) f^{(m)}(x),$$

$$(27) \quad q(l, m; x) = (-1)^l Q_l(x) f^{(m)}(x)$$

for any nonnegative integers $0 \leq l \leq n$ and $0 \leq m \leq n + 1$ and $x \in [a, b]$.

Proof. Let t be a parameter such that $a \leq t \leq b$. Then

$$(28) \quad \int_a^b f(x) dx = \int_a^t f(x) dx + \int_t^b f(x) dx.$$

From the generalized Taylor formula (23), for $x \in [a, b]$, it follows that

$$(29) \quad f(x) = f(a) + \sum_{k=1}^n (-1)^{k+1} [P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)] + R_{n,P}(f; a, x),$$

$$(30) \quad f(x) = f(b) + \sum_{k=1}^n (-1)^{k+1} [Q_k(x) f^{(k)}(x) - Q_k(b) f^{(k)}(b)] + R_{n,Q}(f; b, x).$$

Integration by parts gives us

$$\begin{aligned}
 \int_a^t P_k(x) f^{(k)}(x) dx &= [P_k(t) f^{(k-1)}(t) - P_k(a) f^{(k-1)}(a)] \\
 &\quad - \int_a^t P_{k-1}(x) f^{(k-1)}(x) dx.
 \end{aligned}$$

Clearly, we can apply the same procedure to the term $\int_a^t P_{k-1}(x) \times f^{(k-1)}(x) dx$. So, by successive integration by parts, we obtain

$$\begin{aligned}
 (31) \quad &(-1)^k \int_a^t P_k(x) f^{(k)}(x) dx \\
 &= \sum_{i=1}^k (-1)^i [P_i(t) f^{(i-1)}(t) - P_i(a) f^{(i-1)}(a)] + \int_a^t f(x) dx.
 \end{aligned}$$

Similarly, we have

$$(32) \quad (-1)^k \int_t^b Q_k(x) f^{(k)}(x) dx \\ = \sum_{i=1}^k (-1)^i [Q_i(b) f^{(i-1)}(b) - Q_i(t) f^{(i-1)}(t)] + \int_t^b f(x) dx.$$

Integrating both sides of formula (29) over the interval $[a, t]$ and utilizing identity (31) yields

$$(33) \quad (n+1) \int_a^t f(x) dx = (t-a) \sum_{k=0}^n (-1)^k P_k(a) f^{(k)}(a) \\ + \sum_{k=1}^n \sum_{i=1}^k (-1)^{i+1} [P_i(t) f^{(i-1)}(t) - P_i(a) f^{(i-1)}(a)] \\ + (-1)^n \int_a^t \int_a^x P_n(s) f^{(n+1)}(s) ds dx.$$

Similarly, integrating both sides of the result (30) over the interval $[t, b]$ and using the identity (32) gives

$$(34) \quad (n+1) \int_t^b f(x) dx = (b-t) \sum_{k=0}^n (-1)^k Q_k(b) f^{(k)}(b) \\ + \sum_{k=1}^n \sum_{i=1}^k (-1)^{i+1} [Q_i(b) f^{(i-1)}(b) - Q_i(t) f^{(i-1)}(t)] \\ + (-1)^n \int_t^b \int_b^x Q_n(s) f^{(n+1)}(s) ds dx.$$

Now, combining (33) and (34) and utilizing (28) produces

$$(35) \quad (n+1) \int_a^b f(x) dx = b \sum_{k=0}^n (-1)^k Q_k(b) f^{(k)}(b) - a \sum_{k=0}^n (-1)^k P_k(a) f^{(k)}(a) \\ + \sum_{k=1}^n \sum_{i=1}^k (-1)^{i+1} [Q_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)]$$

$$\begin{aligned}
 &+ t \sum_{k=0}^n (-1)^{k+1} [Q_k(b)f^{(k)}(b) - P_k(a)f^{(k)}(a)] \\
 &+ \sum_{k=1}^n \sum_{i=1}^k (-1)^i [Q_i(t) - P_i(t)] f^{(i-1)}(t) \\
 &+ (-1)^n \left[\int_a^t \int_a^x P_n(s) f^{(n+1)}(s) \, ds \, dx \right. \\
 &\qquad \qquad \qquad \left. + \int_t^b \int_b^x Q_n(s) f^{(n+1)}(s) \, ds \, dx \right].
 \end{aligned}$$

Further, interchanging the order of integration for the last two terms in (35) leads to

$$(36) \quad \int_a^t \int_a^x P_n(s) f^{(n+1)}(s) \, ds \, dx = \int_a^t (t-s) P_n(s) f^{(n+1)}(s) \, ds,$$

$$(37) \quad \int_t^b \int_b^x Q_n(s) f^{(n+1)}(s) \, ds \, dx = \int_t^b (t-s) Q_n(s) f^{(n+1)}(s) \, ds.$$

Substituting (36) and (37) into (35), rearranging and introducing the notation in (26) and (27) produces the identity (25). The proof is complete. \square

Remark 1. If we set

$$(38) \quad P_i(s) = Q_i(s) = \frac{(s-x)^i}{i!}$$

for i a nonnegative natural number in (29) and (30) from the very start, then following the same procedure as in the proof of Theorem 1, we would obtain

$$(39) \quad \int_a^t f(x) \, dx = \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \frac{1}{(n+1)!} \int_a^t (t-s)^{n+1} f^{(n+1)}(s) \, ds,$$

$$(40) \quad \int_t^b f(x) \, dx = - \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} + \frac{1}{(n+1)!} \int_t^b (t-s)^{n+1} f^{(n+1)}(s) \, ds,$$

the combination of which produces

$$(41) \quad \int_a^b f(x) dx = \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} - \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} \\ + \frac{1}{(n+1)!} \int_a^b (t-s)^{n+1} f^{(n+1)}(s) ds.$$

The following result is a particularization of Theorem 1 but, because of its intrinsic importance, is denoted as a theorem in its own right rather than a corollary.

Theorem 2. *Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. If $f : [a, b] \rightarrow \mathbf{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbf{N}$, then we have the following identity*

$$(42) \quad (n+1) \int_a^b f(x) dx \\ = b \sum_{k=0}^n p(k, k; b) - a \sum_{k=0}^n p(k, k; a) - \sum_{k=1}^n \sum_{i=1}^k [p(i, i-1; b) - p(i, i-1; a)] \\ - t \sum_{k=0}^n [p(k, k; b) - p(k, k; a)] + \int_a^b (t-s)p(n, n+1; s) ds,$$

where $t \in [a, b]$ and $p(l, m; x) = (-1)^l P_l(x) f^{(m)}(x)$, $x \in [a, b]$, for any nonnegative integers $0 \leq l \leq n$ and $0 \leq m \leq n+1$.

Proof. This follows from taking $P_i(s) = Q_i(s)$ for $0 \leq i \leq \infty$ and $s \in [a, b]$ in Theorem 1. \square

Remark 2. If we set

$$(43) \quad P_i(s) = \frac{[s - (\lambda b + (1-\lambda)a)]^i}{i!} \quad \text{and} \quad Q_i(s) = \frac{[s - (\mu b + (1-\mu)a)]^i}{i!}$$

in identity (25), then

$$\begin{aligned}
 (44) \quad & (n+1) \int_a^b f(x) \, dx \\
 &= \sum_{k=0}^n \frac{(b-a)^k}{k!} [b(\mu-1)^k f^{(k)}(b) - a\lambda^k f^{(k)}(a)] \\
 &\quad - \sum_{k=1}^n \sum_{i=1}^k \frac{(b-a)^i}{i!} [(\mu-1)^i f^{(i-1)}(b) - \lambda^i f^{(i-1)}(a)] \\
 &\quad - t \sum_{k=0}^n \frac{(b-a)^k}{k!} [(\mu-1)^k f^{(k)}(b) - \lambda^k f^{(k)}(a)] \\
 &\quad + \sum_{k=1}^n \sum_{i=1}^k \frac{(-1)^i}{i!} \{ [t - (\mu b + (1-\mu)a)]^i - [t - (\lambda b + (1-\lambda)a)]^i \} f^{(i-1)}(t) \\
 &\quad + \frac{(-1)^n}{n!} \left(\int_a^t (t-s)[s - (\lambda b + (1-\lambda)a)]^n f^{(n+1)}(s) \, ds \right. \\
 &\quad \quad \left. + \int_t^b (t-s)[s - (\mu b + (1-\mu)a)]^n f^{(n+1)}(s) \, ds \, dx \right).
 \end{aligned}$$

If we set

$$(45) \quad P_i(s) = Q_i(s) = \frac{[s - (\lambda b + (1-\lambda)a)]^i}{i!}$$

in formula (42), then

$$\begin{aligned}
 (46) \quad & (n+1) \int_a^b f(x) \, dx = \sum_{k=0}^n \frac{(b-a)^k}{k!} [b(\lambda-1)^k f^{(k)}(b) - a\lambda^k f^{(k)}(a)] \\
 &\quad - \sum_{k=1}^n \sum_{i=1}^k \frac{(b-a)^i}{i!} [(\lambda-1)^i f^{(i-1)}(b) - \lambda^i f^{(i-1)}(a)] \\
 &\quad - t \sum_{k=0}^n \frac{(b-a)^k}{k!} [(\lambda-1)^k f^{(k)}(b) - \lambda^k f^{(k)}(a)] \\
 &\quad + \frac{(-1)^n}{n!} \int_a^b (t-s)[s - (\lambda b + (1-\lambda)a)]^n f^{(n+1)}(s) \, ds.
 \end{aligned}$$

Taking $\lambda = 0$ or $\lambda = 1$ in (46) yields expansions about b and a , respectively. Namely,

$$(47) \quad \begin{aligned} (n+1) \int_a^b f(x) \, dx &= b \sum_{k=0}^n \frac{(a-b)^k}{k!} f^{(k)}(b) - \sum_{k=1}^n \sum_{i=1}^k \frac{(a-b)^i}{i!} f^{(i-1)}(b) \\ &\quad - t \sum_{k=0}^n \frac{(a-b)^k}{k!} f^{(k)}(b) \\ &\quad + \frac{1}{n!} \int_a^b (t-s)(a-s)^n f^{(n+1)}(s) \, ds. \end{aligned}$$

or

$$(48) \quad \begin{aligned} (n+1) \int_a^b f(x) \, dx &= \sum_{k=1}^n \sum_{i=1}^k \frac{(b-a)^i}{i!} f^{(i-1)}(a) - a \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) \\ &\quad + t \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) \\ &\quad + \frac{1}{n!} \int_a^b (t-s)(b-s)^n f^{(n+1)}(s) \, ds. \end{aligned}$$

Remark 3. If the assumption is made that

$$(49) \quad P_i(s) = P_{i,B}(s; b; a) = \frac{(b-a)^i}{i!} B_i \left(\frac{s-a}{b-a} \right)$$

for nonnegative integer i , then from Theorem 1, straightforward calculations and utilizing the properties (21), (22) together with the fact that $B_{2i+1} = 0$ for $i \in \mathbf{N}$ produces

$$\begin{aligned} (n+1) \int_a^b f(x) \, dx &= [bf(b) - af(a)] - \frac{1}{2}(b-a)[af'(a) + bf'(b)] \\ &\quad + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(b-a)^{2k}}{(2k)!} B_{2k} [bf^{(2k)}(b) - af^{(2k)}(a)] + \frac{n}{2}(b-a)[f(a) + f(b)] \end{aligned}$$

$$\begin{aligned}
 & - \sum_{k=2}^n \sum_{i=1}^{[k/2]} \frac{(b-a)^{2i}}{(2i)!} B_{2i} [f^{(2i-1)}(b) - f^{(2i-1)}(a)] \\
 & - t \left\{ [f(b) - f(a)] - \frac{1}{2} (b-a) [f'(a) + f'(b)] \right. \\
 & \qquad \qquad \qquad \left. + \sum_{k=1}^{[n/2]} \frac{(b-a)^{2k}}{(2k)!} B_{2k} [f^{(2k)}(b) - f^{(2k)}(a)] \right\} \\
 & + \frac{(a-b)^n}{n!} \int_a^b (t-s) B_n \left(\frac{s-a}{b-a} \right) f^{(n+1)}(s) ds,
 \end{aligned}$$

where $[x]$ denotes the Gauss function, whose value is the largest integer not exceeding x .

Remark 4. Suppose that in Theorem 2

$$(51) \quad P_i(s) = P_{i,E}(s; a; b) = \frac{(a-b)^i}{i!} E_i \left(\frac{s-b}{a-b} \right)$$

for nonnegative integer i , then, by direct calculation and utilizing (21), (22) and $B_{2i+1} = 0$ for $i \in \mathbf{N}$, we obtain

$$\begin{aligned}
 (52) \quad (n+1) \int_a^b f(x) dx &= af(a) + bf(b) + \sum_{k=1}^{[(n+1)/2]} \frac{2(1-4^k)(b-a)^{2k-1}}{(2k)!} \\
 & \qquad \qquad \qquad \times B_{2k} [af^{(2k-1)}(a) + bf^{(2k-1)}(b)] \\
 & + \sum_{k=1}^n \sum_{i=1}^{[(k+1)/2]} \frac{2(4^i - 1)(b-a)^{2i-1}}{(2i)!} \\
 & \qquad \qquad \qquad \times B_{2i} [f^{(2(i-1))}(a) + f^{(2(i-1))}(b)] \\
 & - t \left\{ af(a) + bf(b) + \sum_{k=1}^{[(n+1)/2]} \frac{2(4^k - 1)(b-a)^{2k-1}}{(2k)!} \right. \\
 & \qquad \qquad \qquad \left. \times B_{2k} [f^{(2k-1)}(a) + f^{(2k-1)}(b)] \right\} \\
 & + \frac{(b-a)^n}{n!} \int_a^b (t-s) E_n \left(\frac{s-b}{a-b} \right) f^{(n+1)}(s) ds,
 \end{aligned}$$

where $[x]$ also denotes the Gauss function as in Remark 3.

3. Some new Iyengar type inequalities. In the work that follows, we adopt the notation:

$$\begin{aligned}
 \Gamma_P &= \sup_{x \in [a, b]} p(n, n+1; x) = \sup_{x \in [a, b]} \{(-1)^n P_n(x) f^{(n+1)}(x)\}, \\
 \gamma_P &= \inf_{x \in [a, b]} p(n, n+1; x) = \inf_{x \in [a, b]} \{(-1)^n P_n(x) f^{(n+1)}(x)\}, \\
 (53) \quad \Gamma_Q &= \sup_{x \in [a, b]} q(n, n+1; x) = \sup_{x \in [a, b]} \{(-1)^n Q_n(x) f^{(n+1)}(x)\}, \\
 \gamma_Q &= \inf_{x \in [a, b]} q(n, n+1; x) = \inf_{x \in [a, b]} \{(-1)^n Q_n(x) f^{(n+1)}(x)\}, \\
 \Gamma &= \sup_{x \in [a, b]} f^{(n+1)}(x), \quad \gamma = \inf_{x \in [a, b]} f^{(n+1)}(x),
 \end{aligned}$$

where the definitions (26) and (27) have been used.

Suppose $f(x)$ is an n -times differentiable function on the closed interval $[a, b]$ and $f^{(n+1)}(x)$ exists on (a, b) . Let

$$(54) \quad M = \Gamma = \sup_{s \in (a, b)} f^{(n+1)}(s), \quad N = \gamma = \inf_{s \in (a, b)} f^{(n+1)}(s).$$

Then, from (41), we can deduce Theorem B and Theorem C, the results obtained in [28] and [29].

Bounds may also be obtained in terms of the traditional Lebesgue norms for $f^{(n+1)} \in L^p([a, b])$, $1 \leq p < \infty$. Specifically, if $f^{(n)}(x)$ is absolutely continuous on the closed interval $[a, b]$, then, for any $t \in [a, b]$, it follows from (41), that

$$\begin{aligned}
 (55) \quad & \left| \int_a^b f(x) dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} \right| \\
 & \leq \frac{(t-a)^{n+2} + (b-t)^{n+2}}{(n+2)!} \int_a^b |f^{(n+1)}(s)| ds.
 \end{aligned}$$

Also, as a simple consequence of Hölder inequality, if $f^{(n+1)} \in L^p([a, b])$ for a positive number $p > 1$, then, for any $t \in [a, b]$, we

have

$$\begin{aligned}
 (56) \quad & \left| \int_a^b f(x) \, dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} \right| \\
 & \leq \frac{(t-a)^{n+1+(1/q)} + (b-t)^{n+1+(1/q)}}{(n+1)! \sqrt[nq+q+1]} \left(\int_a^b |f^{(n+1)}(s)|^p \, ds \right)^{1/p},
 \end{aligned}$$

where q satisfies $1/p + 1/q = 1$.

Note that inequality (56) has been proved in [12, 13].

Let $\{P_i(x)\}_{i=0}^\infty$ and $\{Q_i(x)\}_{i=0}^\infty$ be two harmonic sequences of polynomials. If $f : [a, b] \rightarrow \mathbf{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbf{N}$, then, from identity (25), it is easy to obtain the expression

$$\begin{aligned}
 (57) \quad & \frac{1}{2} [(a-t)^2 \gamma_Q - (b-t)^2 \Gamma_P] \\
 & \leq (n+1) \int_a^b f(x) \, dx + a \sum_{k=0}^n p(k, k; a) - b \sum_{k=0}^n q(k, k; b) \\
 & \quad - \sum_{k=1}^n \sum_{i=1}^k [p(i, i-1, a) - q(i, i-1; b)] - t \sum_{k=0}^n [p(k, k; a) - q(k, k; b)] \\
 & \quad - \sum_{k=1}^n \sum_{i=1}^k [q(i, i-1, t) - p(i, i-1; t)] \\
 & \leq \frac{1}{2} [(a-t)^2 \Gamma_P - (b-t)^2 \gamma_Q].
 \end{aligned}$$

Let $\{P_i(x)\}_{i=0}^\infty$ be a harmonic sequence of polynomials. If $f : [a, b] \rightarrow \mathbf{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some

$n \in \mathbf{N}$, then we have the following inequalities

$$\begin{aligned}
 (58) \quad & \frac{1}{2} [a^2 \gamma_P - b^2 \Gamma_P] - \frac{\{b \Gamma_P - a \gamma_P - \sum_{k=0}^n [p(k, k; b) - p(k, k; a)]\}^2}{2(\gamma_P - \Gamma_P)} \\
 & \leq (n+1) \int_a^b f(x) dx - b \sum_{k=0}^n p(k, k; b) + a \sum_{k=0}^n p(k, k; a) \\
 & \quad + \sum_{k=1}^n \sum_{i=1}^k [p(i, i-1; b) - p(i, i-1; a)] \\
 & \leq \frac{1}{2} [a^2 \Gamma_P - b^2 \gamma_P] - \frac{\{b \gamma_P - a \Gamma_P - \sum_{k=0}^n [p(k, k; b) - p(k, k; a)]\}^2}{2(\Gamma_P - \gamma_P)}.
 \end{aligned}$$

In fact, (58) follows from the identity (42) and maximizing or minimizing a quadratic polynomial in t .

Let $\{P_i(x)\}_{i=0}^\infty$ be a harmonic sequence of polynomials. If $f : [a, b] \rightarrow \mathbf{R}$ is a function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbf{N}$, and defining

$$\begin{aligned}
 \mathcal{U} &:= \max_{s \in [a, b]} |P_n(s)|, \\
 \mathcal{V} &:= \max_{s \in [a, b]} |f^{(n+1)}(s)|,
 \end{aligned}$$

and

$$\mathcal{W} := \max_{s \in [a, b]} |P_n(s) f^{(n+1)}(s)|,$$

then, from (42), maximizing or minimizing a quadratic polynomial in t yields

$$\begin{aligned}
 (59) \quad & \frac{\{(a+b)\mathcal{U}\mathcal{V} - \sum_{k=0}^n [p(k, k; b) - p(k, k; a)]\}^2}{4\mathcal{U}\mathcal{V}} - \frac{a^2 + b^2}{2} \mathcal{U}\mathcal{V} \\
 & \leq \frac{\{(a+b)\mathcal{W} - \sum_{k=0}^n [p(k, k; b) - p(k, k; a)]\}^2}{4\mathcal{W}} - \frac{a^2 + b^2}{2} \mathcal{W} \\
 & \leq (n+1) \int_a^b f(x) dx - b \sum_{k=0}^n p(k, k; b) + a \sum_{k=0}^n p(k, k; a)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^n \sum_{i=1}^k [p(i, i-1; b) - p(i, i-1; a)] \\
& \leq \frac{a^2 + b^2}{2} \mathcal{W} - \frac{\{(a+b)\mathcal{W} + \sum_{k=0}^n [p(k, k; b) - p(k, k; a)]\}^2}{4\mathcal{W}} \\
& \leq \frac{a^2 + b^2}{2} \mathcal{UV} - \frac{\{(a+b)\mathcal{UV} + \sum_{k=0}^n [p(k, k; b) - p(k, k; a)]\}^2}{4\mathcal{UV}}.
\end{aligned}$$

Remark 5. Now it is clear that, if estimating the following general integral remainders

$$(60) \quad \int_a^t (t-s)p(n, n+1; s)ds + \int_t^b (t-s)q(n, n+1; s) ds,$$

and

$$(61) \quad \int_a^b (t-s)p(n, n+1; s) ds$$

that appear in (25) and (42), then using better and more appropriate techniques than previously gives rise to more accurate Iyengar type inequalities.

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