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IDEAL KAEHLERIAN SLANT SUBMANIFOLDS IN COMPLEX SPACE FORMS

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ABSTRACT. Roughly speaking, an ideal immersion of a Riemannian manifold into a space form is an isometric immersion which produces the least possible amount of tension from the ambient space at each point of the submanifold. Recently, Chen studied Lagrangian submanifolds in complex space forms which are ideal. He proved that such submanifolds are minimal. He also classified ideal Lagrangian submanifolds in complex space forms.

In the present paper, we investigate ideal Kaehlerian slant submanifolds in a complex space form. We prove that such submanifolds are minimal. We also obtain obstructions to ideal slant immersions in complex hyperbolic space.

1. Chen invariants and Chen inequalities. It is well known that Riemannian invariants play the most fundamental role in Riemannian geometry. Riemannian invariants provide the intrinsic characteristics of Riemannian manifolds which affect the behavior in general of the Riemannian manifold.

Let M be an n-dimensional Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{1.1}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane

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section L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{1.2}$$

For an integer $k \geq 0$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \dots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \dots + n_k \leq n$. Then $\mathcal{S}(n)$ is the union $\cup_{k\geq 0} \mathcal{S}(n,k)$.

Chen introduced in [**3, 4**] a new type of curvature invariant $\delta(n_1, \dots, n_k)$ known as Chen invariants, defined as follows.

$$\delta(n_1, \dots, n_k) = \tau - \inf\{\tau(L_1) + \dots + \tau(L_k)\}, \tag{1.3}$$

where, at each point $p \in M$, $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \dots, k$.

He proved in [**4**] an optimal relationship between the Chen invariants $\delta(n_1, \dots, n_k)$ and the squared mean curvature $\|H\|^2$, which we call the Chen inequality, for an arbitrary submanifold in a real space form.

Chen also pointed out in [**4**] that the same result holds for totally real submanifolds in complex space forms.

Let $\widetilde{M}(4c)$ be an m-dimensional complex space form with constant holomorphic sectional curvature $4c$. We denote by J the complex structure of $\widetilde{M}(4c)$. The curvature tensor $\widetilde{R}$ of $\widetilde{M}(4c)$ is given by [**13**]:

$$\begin{aligned} \widetilde{R}(X,Y)Z = c(&g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ), \end{aligned} \tag{1.4}$$

for any tangent vector fields X, Y, Z to $\widetilde{M}(4c)$.

An n-dimensional submanifold M of $\widetilde{M}(4c)$ is said to be a *slant submanifold* [**1**] if, for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between JX and the tangent space T_pM is constant which is denoted by θ. Moreover, if M is neither complex, $\theta = 0$, nor totally real, $\theta = \pi/2$, then M is called a proper slant submanifold.

We denote by h and A the second fundamental form and the shape operator, respectively, of M in $\widetilde{M}(4c)$.

The equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &- g(h(X,Z), h(Y,W)) \\ &+ g(h(X,W), h(Y,Z)), \end{aligned} \tag{1.5}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{1.6}$$

where X, Y, Z, W are tangent vector fields to M, and ∇h is defined by

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

For any vector field X tangent to M, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX. Thus, P is an endomorphism of the tangent bundle TM.

We obtained inequalities satisfied by some Chen invariants $\delta'(n_1, \dots, n_k)$ for slant submanifolds in a complex space form in [**11**].

We consider the Riemannian invariant

$$\delta'(n_1, \dots, n_k) = \tau - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where, at each point $p \in M$, $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM invariant by the endomorphism P, such that $\dim L_j = n_j$, $j = 1, \dots, k$. We set

$$b(n_1, \dots, n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^k n_j)}{2(n + k - \sum_{j=1}^k n_j)},$$

$$d(n_1, \dots, n_k) = \frac{1}{2}\left[n(n-1) - \sum_{j=1}^k n_j(n_j - 1)\right].$$

Theorem 1.1. *Let M be an n-dimensional θ-slant submanifold of an m-dimensional complex space form $\widetilde{M}(4c)$. Then, for any $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we have*

$$\begin{aligned} \delta'(n_1, \dots, n_k) \le{}& b(n_1, \dots, n_k)\|H\|^2 + d(n_1, \dots, n_k)\, c \\ &+ \frac{3}{2}\left(n - \sum_{j=1}^k n_j\right) c \cos^2\theta. \end{aligned} \tag{1.7}$$

Moreover, the equality holds at a point $p \in M$ if and only if there exists a tangent basis $\{e_1, \dots, e_n\} \subset T_pM$ and a normal basis

$\{e_{n+1}, \ldots, e_{2m}\} \subset T_p^{\perp}M$ *such that, for any vector* ξ *normal to* M *at* p*, the shape operator* A_ξ *takes the following form*

$$A_\xi = \begin{pmatrix} A_1^\xi & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & A_k^\xi & \\ & 0 & & \mu_\xi I \end{pmatrix}, \tag{1.8}$$

where I *is an identity matrix and* A_j^ξ *is a symmetric* $n_j \times n_j$ *submatrix satisfying*

$$\operatorname{tr}(A_1^\xi) = \cdots = \operatorname{tr}(A_k^\xi) = \mu_\xi. \tag{1.9}$$

A slant submanifold M of a complex space form $\widetilde{M}(4c)$ is called *ideal* if it satisfies the equality case of the inequality (1.7) identically for some $(n_1, \ldots, n_k) \in \mathcal{S}(n)$.

2. Minimality of ideal submanifolds. In the following sections, we will investigate n-dimensional Kaehlerian slant ideal submanifolds in an n-dimensional complex space form $\widetilde{M}(4c)$.

We recall a proper slant submanifold is *Kaehlerian slant* if the endomorphism P is parallel with respect to the Riemannian connection ∇. It is known in [**1**] that this condition is equivalent to

$$A_{FX}Y = A_{FY}X, \quad \forall\, X, Y \in \Gamma(TM). \tag{2.1}$$

First, we will prove that such submanifolds are minimal.

Theorem 2.1. *Let* M *be an* n*-dimensional Kaehlerian slant submanifold of an* n*-dimensional complex space form* $\widetilde{M}(4c)$*. If* M *is an ideal submanifold, then it is minimal.*

Proof. Let M be a Kaehlerian slant submanifold of a complex space form $\widetilde{M}(4c)$ with $\dim M = n$ and $\dim \widetilde{M}(c) = n$, and let $p \in M$.

We distinguish two cases:

(a) $g(h(u,v),Fw)=0$, for all $u,v,w\in T_pM$.

Obviously, it follows that $H(p)=0$.

(b) $g(h(u,u),Fu)\neq 0$, for some $u\in T_pM$.

We will construct an orthonormal basis $\{\varepsilon_1,\dots,\varepsilon_n\}$ of T_pM such that

$$A_{F\varepsilon_1}\varepsilon_i=\lambda_i\varepsilon_i,\quad i=1,\dots,n, \tag{2.2}$$

where $\lambda_1,\dots,\lambda_n$ satisfy $\lambda_1\geq 2\lambda_j$, $j=2,\dots,n$.

We define a function f_p by

$$f_p: T_p^1M\longrightarrow \mathbf{R};\quad u\longmapsto f_p(u)=g(h(u,u),Fu),$$

where T_p^1M is the unit hypersphere of T_pM consisting of all unit vectors in T_pM. Since T_p^1M is a compact set, there exists a vector v in T_p^1M such that f_p attains an absolute maximum at v. We denote $\lambda_1=f_p(v)>0$. It is easily seen that $A_{Fv}v=\lambda_1 v$. We put $\varepsilon_1=v$ and choose $\varepsilon_2,\dots,\varepsilon_n$ so that $\{\varepsilon_1,\dots,\varepsilon_n\}$ is an orthonormal basis of T_pM and each ε_i is an eigenvector of $A_{F\varepsilon_1}$ with eigenvalue λ_i.

Since f_p attains an absolute maximum at ε_1, the function f_i, $i\in\{2,\dots,n\}$, defined by $f_i(t)=f_p((\cos t)\varepsilon_1+(\sin t)\varepsilon_i)$ has a relative maximum at $t=0$. So, by a straightforward computation, we get

$$0\geq f_i''(0)=-3\lambda_1+6\lambda_i,$$

i.e., $\lambda_1\geq 2\lambda_i$, for all $i\geq 2$.

Since $\lambda_1>0$, we find $\lambda_1>\lambda_i$, for $i\geq 2$. In particular, this implies that the eigenspace of $A_{F\varepsilon_1}$ corresponding to the eigenvalue λ_1 is 1-dimensional.

We assume that M is ideal. Then it satisfies

$$\begin{aligned}\delta'(n_1,\dots,n_k)&=b(n_1,\dots,n_k)\|H\|^2+d(n_1,\dots,n_k)\,c\\&\quad+\frac{3}{2}\left(n-\sum_{j=1}^{k}n_j\right)c\cos^2\theta\end{aligned} \tag{2.3}$$

identically for some k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$.

Thus, according to Theorem 1.1, there is an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM at each $p \in M$ such that, for any normal vector ξ at p, the shape operator A_ξ with respect to $\{e_1, \dots, e_n\}$ takes the form (1.8).

With respect to the orthonormal basis $\{e_1, \dots, e_n\}$ chosen above, we put

$$L_i = \mathrm{Span}\,\{e_\alpha \,|\, \alpha \in I_i\}, \tag{2.4}$$

$$\mathrm{tr}_{\,L_i} h = \sum_{\alpha \in I_i} h(e_\alpha, e_\alpha), \tag{2.5}$$

where $I_i = \{n_1 + \cdots + n_{i-1} + 1, \dots, n_1 + \cdots + n_i\}$, $i = 1, \dots, k$, and

$$L_{k+1} = \mathrm{Span}\,\{e_{n_1+\cdots+n_k+1}, \dots, e_n\}.$$

Obviously, we have $T_pM = L_1 \oplus \cdots \oplus L_{k+1}$. We will show that ε_1 lies in one of $L_1, \dots, L_k$. Let

$$\varepsilon_1 = v_1 + \cdots + v_k + v_{k+1}, \tag{2.6}$$

with $v_1 \in L_1, \dots, v_{k+1} \in L_{k+1}$. Then, we have

$$\lambda_1 \varepsilon_1 = A_{\phi\varepsilon_1}\varepsilon_1 = A_{F\varepsilon_1} v_1 + \cdots + A_{F\varepsilon_1} v_k + A_{F\varepsilon_1} v_{k+1}. \tag{2.7}$$

From (1.8) we have $A_{F\varepsilon_1} v_i \in L_i$. Thus, (2.6) and (2.7) imply

$$A_{F\varepsilon_1} v_1 = \lambda_1 v_1, \dots, A_{F\varepsilon_1} v_{k+1} = \lambda_1 v_{k+1}.$$

Since the eigenspace of $A_{F\varepsilon_1}$ corresponding to the eigenvalue λ_1 is 1-dimensional, the above equation implies that exactly one of $v_1, \dots, v_{k+1}$ does not vanish. If we suppose that $\varepsilon_1 = v_{k+1}$, then $n_1 + \cdots + n_k = n - 1$, because the multiplicity of λ_1 is 1. In this case, $\varepsilon_1 = \pm e_n$. Therefore, we get

$$\pm A_{F\varepsilon_i} e_n = A_{F\varepsilon_i}\varepsilon_1 = A_{F\varepsilon_1}\varepsilon_i = \lambda_i \varepsilon_i \perp e_n = \pm \varepsilon_1,$$

for $i = 2, \dots, n$. Thus, we obtain $\lambda_2 = \cdots = \lambda_n = 0$ by applying (1.8). Hence, by (1.9), one has $\lambda_1 = 0$, which is a contradiction. Consequently, ε_1 must belong to one of $L_1, \dots, L_k$.

Without loss of generality, we may now assume $\varepsilon_1 = e_1$. Moreover, since A_{Fe_1} takes the form (1.8), we may also assume $\varepsilon_2 = e_2, \dots, \varepsilon_n = e_n$.

In other words, we may choose the eigenvectors $\varepsilon_2, \dots, \varepsilon_n$ of A_{Fe_1} to be compatible with the decomposition arisen from (1.8). Therefore, by applying (1.8) and (2.1), we have

$$g(h(X_i, Y_i), FX_j) = g(h(X_i, X_j), FY_i) = 0, \tag{2.8}$$

for vectors $X_i, Y_i \in L_i$, $X_j \in L_j$, $1 \le i \ne j \le k$.

If $n_1 + \cdots + n_k = n$, then $k \ge 2$ by the definition of $\mathcal{S}(n)$. In this case, (1.8), (1.9) and (2.8) imply that the second fundamental form of M satisfies

$$h(L_i, L_i) \subset F(L_i), \quad h(L_i, L_j) = 0, \tag{2.9}$$
$$\operatorname{tr}_{L_i} h = 0, \quad i, j = 1, \dots, k. \tag{2.10}$$

If $n_1 + \cdots + n_k < n$, then (1.8) implies that, for any $t \in \{n_1 + \cdots + n_k + 1, \dots, n\}$ and any $j \ne t$, we have

$$g(A_{Fe_t} e_i, e_j) = g(A_{Fe_i} e_t, e_j) = 0. \tag{2.11}$$

By using (2.11) and (1.9) with $\xi = Fe_t$, we also obtain $g(A_{Fe_t} e_t, e_t) = 0$. Hence, $A_{Fe_t} = 0$ for each $t \in \{n_1 + \cdots + n_k + 1, \dots, n\}$. Thus, we obtain (2.9), (2.10) and

$$h(X, e_{n_1+\cdots+n_k+1}) = \cdots = h(X, e_n) = 0, \quad \forall\, X \in T_pM, \tag{2.12}$$

Therefore, in both cases, $H(p) = 0$. □

3. Obstructions to ideal slant immersions. In this section we will prove the nonexistenceness of n-dimensional ideal Kaehlerian slant submanifolds in an n-dimensional complex hyperbolic space with full first normal bundle.

First, we state the following.

Proposition 3.1. *Every minimal slant submanifold of a hyperbolic complex space form is irreducible.*

Proof. Assume that M is an n-dimensional reducible slant submanifold of an m-dimensional complex space form $\widetilde{M}(4c)$, with $c < 0$. Then, locally, M is the Riemannian product of some Riemannian manifolds, say $M = N_1 \times \cdots \times N_s$, $s \geq 2$. If $\dim N_1 = a$, then we can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ such that $e_1, \ldots, e_a$ are tangent to N_1 and $e_{a+1}, \ldots, e_n$ are tangent to $N_2 \times \cdots \times N_s$. Since M is minimal, the Gauss equation yields

$$\begin{aligned} 0 &= \sum_{i=1}^{a} \sum_{j=a+1}^{n} R(e_i, e_j, e_i, e_j) \\ &= a(n-a)(1+3\cos^2\theta)\, c - \left\| \sum_{i=1}^{a} h(e_i, e_i) \right\|^2 - \sum_{i=1}^{a} \sum_{j=a+1}^{n} ||h(e_i, e_j)||^2, \end{aligned}$$

which is impossible. $\square$

Recall that the first normal space $\operatorname{Im} h_p$ and the relative null space $\operatorname{Ker} h_p$ of a submanifold M at a point $p \in M$ are the vector spaces defined respectively by

$$\begin{aligned} \operatorname{Im} h_p &= \operatorname{sp}\{h(X,Y) \mid X, Y \in T_pM\}, \\ \operatorname{Ker} h_p &= \{Z \in T_pM \mid h(X,Z) = 0,\ \forall\, X \in T_pM\}. \end{aligned}$$

It is easily seen that the first normal space $\operatorname{Im} h_p$ and the relative null space $\operatorname{Ker} h_p$ of a Kaehlerian slant submanifold M in a complex space form $\widetilde{M}(4c)$ are related by $(\operatorname{Im} h_p)^{\perp} = F(\operatorname{Ker} h_p)$.

For an ideal n-dimensional Kaehlerian slant submanifold M in an n-dimensional complex space form $\widetilde{M}(4c)$ satisfying (2.3), we denote by $\mathcal{D}_i$ the distribution generated by L_i, where L_i is defined by (2.4).

The following lemma implies the integrability and the minimality of the distributions $\mathcal{D}_1, \ldots, \mathcal{D}_k$.

Lemma 3.2. *Let M be an n-dimensional ideal Kaehlerian slant submanifold of an n-dimensional complex space form $\widetilde{M}(4c)$ satisfying the equality* (2.3) *identically. If the first normal bundle of M is full, then*

$$n_1 + \cdots + n_k = n, \tag{3.1}$$

$$h(\mathcal{D}_i, \mathcal{D}_i) = F(\mathcal{D}_i), \quad h(\mathcal{D}_i, \mathcal{D}_j) = \{0\}, \quad 1 \le i \ne j \le k, \tag{3.2}$$

$$\nabla_{Y_j} X_i \in \mathcal{D}_i \oplus \mathcal{D}_j, \quad 1 \le i \ne j \le k, \tag{3.3}$$

for vector fields X_i *in* $\mathcal{D}_i$ *and* Y_j *in* $\mathcal{D}_j$*, respectively. Moreover,* $\mathcal{D}_1, \dots, \mathcal{D}_k$ *are completely integrable distributions and the leaves of* $\mathcal{D}_1, \dots, \mathcal{D}_k$ *are totally geodesic submanifolds in* M *and minimal submanifolds in* $\widetilde{M}(4c)$*, respectively.*

This lemma follows from the equation of Codazzi like Lemmas 4 and 11 in [**5**] for ideal Lagrangian submanifolds in complex space forms. For an elegant simple proof of Lemma 11, see also [**6**]. So, we omit the proof of Lemma 3.2.

Using the above results, we will obtain a nonexistence theorem for certain ideal slant submanifolds.

Theorem 3.3. *There do not exist* n*-dimensional ideal Kaehlerian slant submanifolds in an* n*-dimensional complex hyperbolic space whose first normal bundle is full.*

Proof. We assume M is an n-dimensional ideal Kaehlerian slant submanifold in an n-dimensional complex hyperbolic space H^n. Then it satisfies the equality (2.3) for some k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$. If the first normal bundle is full, then $n_1 + \cdots + n_k = n$ and $k \ge 2$. Hence, the tangent bundle TM of M is the direct sum $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_k$. According to Lemma 3.2, each $\mathcal{D}_i$ is an integrable distribution with totally geodesic leaves. Moreover, by the form (1.8) of the shape operators of an ideal submanifold, any sum $\mathcal{D}_{j_1} \oplus \cdots \oplus \mathcal{D}_{j_s}$, $s \in \{2, \dots, k\}$ is also an integrable distribution with totally geodesic leaves. Therefore, de Rham's decomposition theorem implies that M is locally the Riemannian product of k Riemannian manifolds $M_1, \dots, M_k$ of dimensions $n_1, \dots, n_k$, respectively, where M_i is an integral submanifold of $\mathcal{D}_i$. Thus, M is a reducible Riemannian manifold. By applying Theorem 2.1, we know that the submanifold M is minimal. Hence, by using Proposition 3.1, we obtain the desired result. $\square$

On the other hand, there do exist n-dimensional ideal Kaehlerian slant submanifolds in the complex Euclidean space $\mathbf{C}^n$ with full first normal bundle. In fact, we have the following.

Theorem 3.4. *Let M be an n-dimensional Kaehlerian slant submanifold in $\mathbf{C}^n$ with full first normal bundle. Then M is ideal if and only if, locally, M is the Riemannian product of some minimal Kaehlerian slant submanifolds M_j, $j = 1, \ldots, k$, with full first normal bundle.*

Proof. Let $x : M \to \mathbf{C}^n \cong \mathbf{R}^{2n}$ be an ideal Kaehlerian slant immersion with full first normal bundle. Moore's lemma [**10**] implies that x is a product immersion, say

$$x = x_1 \times \cdots \times x_k : M_1 \times \cdots \times M_k \to \mathbf{R}^{N_1} \times \cdots \times \mathbf{R}^{N_k} = \mathbf{R}^{2n},$$

where each $x_j : M_j \to \mathbf{R}^{N_j}$ is an isometric immersion and $\dim M_j = n_j$.

By the first equation (3.2), we have $h(\mathcal{D}_j, \mathcal{D}_j) = F(\mathcal{D}_j)$, for each $j \in \{1, \ldots, k\}$. Then each $\mathbf{R}^{N_j}$ must contain a $2n_j$-dimensional subspace $\mathbf{R}^{2n_j}$ of $\mathbf{R}^{2n}$. Therefore, we have $N_j = 2n_j$, for $j = 1, \ldots, k$. It follows by Lemma 3.2 that the first normal bundle of each M_j is full. Moreover, each M_j is a minimal submanifold. Consequently, the ideal submanifold M is, locally, the Riemannian product of some minimal Kaehlerian slant submanifolds.

The converse statement is clear. □

We state a theorem of characterization of ideal Kaehlerian slant submanifolds in the complex Euclidean space.

Theorem 3.5. *Let M be an n-dimensional Kaehlerian slant submanifold of the complex Euclidean space $\mathbf{C}^n$ such that $Im\, h_p \neq T_p^{\perp} M$, at each point $p \in M$. Then M is ideal if and only if M is a ruled minimal submanifold.*

Proof. Let M be an n-dimensional ideal Kaehlerian slant submanifold in $\mathbf{C}^n$. Then, by Theorem 2.1, M is a minimal submanifold.

Let U_l denote the interior of the subset consisting of points in M such that the relative null space at p has dimension l. Since $\mathrm{Im}\ h_p \neq T_p^{\perp} M$, at each point $p \in M$, it follows that $U_l \neq \varnothing$, for some integer $1 \leq l \leq n$. By applying Codazzi equation, it is easily seen that $\operatorname{Ker} h$ is integrable on U_l and each leaf of $(\operatorname{Ker} h)|_{U_l}$ is an l-dimensional totally geodesic submanifold of $\mathbf{C}^n$. Thus, M contains a geodesic of $\mathbf{C}^n$ through each point $p \in U_l$. Since M is the union of the closure of all U_l, we conclude by continuity that M contains a geodesic of the ambient space through each point in M. Therefore, M is a ruled minimal submanifold.

The converse statement is obvious. □

Note added in proof. After the acceptance for publication of this article, we discovered a very recent paper of I. Salavessa [**12**]. By combining Proposition 1.2 of [**12**] and Theorem 2.1 of this article, we have the following nonexistence result.

Theorem 3.6. *There do not exist n-dimensional ideal Kaehlerian slant submanifolds in the complex projective space $P^n(\mathbf{C})$.*

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