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ON RECTIFICATION OF CIRCLES AND AN EXTENSION OF BELTRAMI'S THEOREM

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"The only perfect geometrical figures are the straight line and the circle."

Plato

ABSTRACT. The goal of this paper is to describe all local diffeomorphisms mapping a family of circles, in an open subset of $\mathbf{R}^3$, into straight lines. This paper contains two main results. The first is a complete description of the rectifiable collection of circles in $\mathbf{R}^3$ passing through one point. It turns out that to be rectifiable all circles need to pass through some other common point. The second main result is a complete description of geometries in $\mathbf{R}^3$ in which all the geodesics are circles. This is a consequence of an extension of Beltrami's theorem by replacing straight lines with circles.

Introduction. The problem that the author solved has its origin in *Nomography*[1]: how to reduce a nomogram of aligned points to a circular nomogram? In more mathematical terms: what are local diffeomorphisms that send germs of lines to germs of circles? This question was initially posed for two-dimensional nomograms by G.S. Khovanskii and solved by A.G. Khovanskii in that case, cf. [**6**]. Our result leads to a solution of the corresponding three-dimensional nomography. On the other hand, it is a continuation of Möbius' classical work that describes all transformations taking lines to lines and circles to circles. It is also related to Beltrami's investigations. By Beltrami's classical theorem, all the geometries whose geodesics are locally straight lines have constant curvature, cf. [**2, 9, 12**]. We prove that all geometries in $\mathbf{R}^3$ whose geodesics are locally circles must also have constant curvature. The similar fact in $\mathbf{R}^4$ is wrong. This was communicated to the author by Timorin, cf. [**11**] for details. We also give a complete description for all the metrics of these geometries.

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The paper consists of three sections. In Section 1, we give the precise definition of the rectification for a collection of circles passing through one point and we prove the fundamental theorem of rectification. This is our first main result. Section 2 is devoted to the classification theorems for the *rich families of circles*, i.e., families that look like the families of geodesics (they are point-wise rectifiable and, for each point on each circle, there is an open cone filled by the tangent lines to other circles). This classification gives rise to the following theorem: up to a projective transformation of the space of the image and a Möbius transformation of the space of the inverse image, there exist exactly three diffeomorphism classes of rectifiable rich family of circles. In Section 3, we discuss some applications of our results in Riemannian geometry. First we show how the rectification problem gives rise to an extension of Beltrami's theorem. More precisely, we will prove that if U is a region in $\mathbf{R}^3$ such that all geodesics with respect to some metric $\mathbf{g}$ are circles, then $\mathbf{g}$ has constant curvature. Then by using this fact, we will calculate all Riemannian metrics in which the geodesics are circles.

1. Rectification of a bundle of circles in $\mathbf{R}^3$. In this section we study the behavior of the curves in a rectifiable bundle near the center of the bundle. Our main result in this section is a theorem which is called *"the* 54*-circles theorem."* In order to prove this theorem, we first show that the coefficients of the Taylor polynomials of the curves in a central bundle are polynomial functions in the direction components of the tangent lines. We also show that these polynomials satisfy some symmetry relations. Assuming that all curves in the rectifiable bundle are circles, we prove that all polynomials are divisible by a specific irreducible polynomial. Using these divisibility conditions together with the above symmetric relations, the desired result can be easily obtained.

Let us start with the following definition and notations.

Definition 1.1 (Rectifiable Bundle of Curves). A family of curves in the three-dimensional space $\mathbf{R}^3$, is called *rectifiable* near a point p if there exists a neighborhood U of p and a diffeomorphism of U taking all the curves in the family (more precisely, the portions of the curves contained in the region U) into straight lines (more precisely, into portions of lines lying in the image of the region U). Any collection of curves passing through the point p is called *a bundle of curves* with center p. A bundle is called *simple* if distinct curves of the bundle have distinct tangent lines.

If a bundle of curves with center at the point p is rectifiable at p, then it is simple. This means that each direction $(1, k, m)$, where $k, m \in \mathbf{R}^* = [-\infty, +\infty]$ corresponds to a unique curve $\alpha_{(k,m)}$, where $\alpha_{(k,m)}$ is the intersection of the two surfaces

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0$$

and $(1, k, m)$ is the direction vector of the tangent line to the curve at the point p. For simplicity, we take the point p to be at the origin. Now we are ready to state the following proposition which is useful in the proof of our main result.

Proposition 1.1. *Suppose that a simple bundle of curves $\alpha_{(k,m)}$ is locally rectifiable near the origin by means of a class C^n diffeomorphism. Then for every i, $1 < i \leq n$, there exist 2-variable polynomials P_i and Q_i of degree at most $2i-1$ such that*

$$y^{(i)}(0) = P_{2i-1}(k, m), \qquad z^{(i)}(0) = Q_{2i-1}(k, m)$$

where y and z are expressed in terms of x on the curve $\alpha_{(k,m)}$.

To prove this proposition, we need a lemma that can be easily verified. It is in fact a sharpening of the *implicit function theorem*, cf. [**10**].

Lemma 1.1. *Consider a curve α given by the equations $F(x, y, z) = G(x, y, z) = 0$, where F and G are smooth functions on $\mathbf{R}^3$ such that*

$$(1) \quad F(0,0,0) = G(0,0,0) = 0, \quad \textit{and} \quad d = \det \left[\frac{\partial(F, G)}{\partial(y, z)} \right]_{(0,0,0)} \neq 0.$$

Then by taking x as a local parameter on α near 0, the Taylor coefficients of y and z, with respect to x are polynomials of degree $2i-1$ in the Taylor coefficients of F and G.

Proof. These can be computed directly by using the implicit function theorem. □

Proof of the proposition. Consider a diffeomorphism Φ rectifying a bundle of curves $\alpha_{(k,m)}(x)$. Let T be an arbitrary nonsingular linear mapping of the space. The diffeomorphism $T \circ \Phi$ also rectifies the bundle $\alpha_{(k,m)}(x)$. By a suitable choice of T we can arrange that the rectifying diffeomorphism has the identity differential at the origin. The rectifying diffeomorphism is now given by the map:

$$T \circ \Phi = (f, g, h) = (u, v, w)$$

where

$$\begin{aligned} f &= x + \cdots , \\ g &= y + \cdots , \\ h &= z + \cdots . \end{aligned} \tag{2}$$

and $\cdots$ denotes the higher terms in the expansion of each function. In the uvw-space the bundle of curves $\alpha_{(k,m)}(x)$ is given by the equations $v = k \cdot u, w = m \cdot u$. Consequently, the curves $\alpha_{(k,m)}(x)$ are given by the equations: $F(x, y, z) = 0$, $G(x, y, z) = 0$, where, $F = g - kf, G = h - mf$. Now the coefficients a_{pqr} and b_{pqr} in the Taylor polynomial of the functions F and G depend linearly on $\{k, a_{010} = 1\}$ and on $\{m, b_{001} = 1\}$ respectively. The assertion now follows from Lemma 1.1. □

The next proposition is the second useful fact concerning our main result.

Proposition 1.2. *Under the same assumptions as of Proposition* 1.1*, there exist* 3×3 *symmetric matrices* A, B, *and* C *such that*

$$y^{(2)}(0) = -\langle (B + kA)\lambda, \lambda\rangle, \qquad z^{(2)}(0) = -\langle (C + mA)\lambda, \lambda\rangle$$

where $\lambda = (1, k, m)$ *is the tangent vector at the origin.*

Proof. In order to prove this, suppose that the curve $\alpha_{(k,m)}(x) = (x, y(x), z(x))$ lies on the surface $F(x, y, z) = 0$, where F is any smooth function. So $F(\alpha_{(k,m)}(x)) = 0$. For simplicity we write α instead of $\alpha_{(k,m)}$. By differentiating both sides of the equation we get

$$\langle \nabla F(\alpha(x)), \alpha'(x) \rangle = 0,$$

which implies that

$$(*) \qquad \left\langle \frac{d}{dx} \nabla F(\alpha(x)), \alpha'(x) \right\rangle + \langle \nabla F(\alpha(x)), \alpha^{(2)}(x) \rangle = 0$$

where

$$\frac{d}{dx} \nabla F(\alpha(x)) = H(\alpha(x))\alpha'(x) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right) \alpha'(x).$$

This expression holds for any surface F containing the graph of the curve $\alpha(x)$, in particular for $F = v - ku = g - kf$, where, f and g are the same functions in (2). Since

$$\nabla F(\alpha(0)) = (-k, 1, 0), \qquad \alpha^{(2)}(0) = (0, y^{(2)}(0), z^{(2)}(0))$$

and $H(\alpha(0)) = B + kA$, where

$$B = \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right)(0) \quad \text{and} \quad A = \left(\frac{\partial^2 (-f)}{\partial x_i \partial x_j} \right)(0).$$

By $(*)$ we get

$$y^{(2)}(0) = \langle (-k, 1, 0), \alpha^{(2)}(0) \rangle = -\lambda H \alpha((0)) \lambda^T,$$

where $\lambda = (1, k, m)$. Hence

$$y^{(2)}(0) = -\langle (B + kA)\lambda, \lambda \rangle.$$

Similarly, for calculating $z^{(2)}(0)$, we let $G = w - m \cdot u = h - m \cdot f$, where f and g are the same functions as in (2). Clearly, $\nabla G_m(\alpha(0)) = (-m, 0, 1)$ and $H(\alpha(0)) = C + mA$, where $C = (\partial^2 h / \partial x_i \partial x_j)(0)$. Hence $z^{(2)}(0) = -\langle (C + mA)\lambda, \lambda \rangle$. $\square$

Remark 1.1. These propositions can be easily extended to arbitrary dimensions, but we don't need to do this.

Before stating the first main result, we give a definition.

Definition 1.2. Let us say that 54 lines passing through the origin are generic if there exists a unique homogeneous cone of degree 9 containing them all. In this case, the corresponding 54 points in $R\mathcal{P}^2$ are called 9-*good*.

One can easily show that for almost all 54 lines passing through the origin, the above condition holds. Having said this, we have the following:

Theorem 1.1 (The 54-circles theorem). *Consider a simple bundle of circles passing through the origin such that the set of tangent lines of the circles contains* 54 *generic lines, then there exists a local diffeomorphism about the origin mapping all the circles into straight lines if and only if all the circles in the bundle pass through one common point distinct from the origin.*

Proof. In one direction the proof is obvious. In fact, if the bundle passes through the second point Q, then we can make an inversion with respect to a sphere centered at this point. But in the opposite direction, it is rather complicated and follows as:

The bundle of circles passing through the origin in $\mathbf{R}^3$ can be written explicitly by a system of equations consisting of two spheres. The simplicity condition easily implies that this system depends only on two parameters, namely the components of tangent vector at the origin. Using this fact, the above system of equations can be expressed in the following form

$$\begin{cases} y = kx + A(x^2 + y^2 + z^2) \\ z = mx + B(x^2 + y^2 + z^2), \end{cases} \tag{3}$$

where $A = A(k, m)$ and $B = B(k, m)$ are some functions of the parameters k and m. We show that the rectifiability of the bundle is equivalent to the linearity of the functions A and B. We wish to solve the equations for the circles in the bundle up to terms of fourth order. By differentiating both equations with respect to x for the Taylor series of $y(x)$ and $z(x)$ we obtain

$$y(x) = kx + \phi_2(k,m)x^2 + \phi_3(k,m)x^3 + \phi_4(k,m)x^4 + \cdots$$
$$z(x) = mx + \psi_2(k,m)x^2 + \psi_3(k,m)x^3 + \psi_4(k,m)x^4 + \cdots$$

where, by letting $f = 1 + k^2 + m^2$, we have

$$\begin{aligned} \phi_2 &= Af \\ \psi_2 &= Bf \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_3 &= 2A(kA + mB)f \\ \psi_3 &= 2B(kA + mB)f \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_4 &= A(A^2 + B^2)f^2 + 4A(kA + mB)^2 f \\ \psi_4 &= B(A^2 + B^2)f^2 + 4B(kA + mB)^2 f \end{aligned} \tag{6}$$

where ϕ_l and ψ_l are polynomials of degrees at most $2l - 1$ in the two variables k and m (Proposition 1.1), and A, B are, at the outset, just rational functions in k and m. We want to show that A and B are polynomials of degree 1.

Multiplying equations (5) by f yields

$$\begin{aligned} f\phi_3 &= 2Af(kA + mB)f = \phi_2(k\phi_2 + m\psi_2) \\ f\psi_3 &= 2Bf(kA + mB)f = \psi_2(k\phi_2 + m\psi_2) \end{aligned} \tag{7}$$

According to Proposition 1.1, the functions $(\phi_2, \psi_2), (\phi_3, \psi_3)$ and (ϕ_4, ψ_4) are polynomials of degree at most 3, 5, and 7 in k, m respectively. Equations (7) are satisfied for all values of $(1, k, m)$ corresponding to circles in the bundle, i.e., by at least 54 directional vectors corresponding to 54 generic lines. Since 2-variable polynomials of degree 5 which coincide at 54 directional vectors, coincide identically, so these equations are in fact identities. These identities imply either: f divides $(k\phi_2 + m\psi_2)$, or: f divides both ϕ_2 and ψ_2. In the latter case equations (4) show that A and B are polynomials of degree 1. So we may assume that $fg = k\phi_2 + m\psi_2$ for some polynomial g.

Now

$$\begin{aligned} fg &= k\phi_2 + m\psi_2 \\ &= kAf + mBf \qquad \text{(by equation (4))} \\ &= (kA + mB)f \end{aligned}$$

Thus $kA + mB = g$ is a polynomial.

Multiplying equations (6) by f yields

$$\begin{aligned} f\phi_4 &= (Af)((Af)^2 + (Bf)^2 + 4(Af)(kA + mB)^2 f \\ &= \phi_2(\phi_2^2 + \psi_2^2) + 4\phi_2 g^2 f. \end{aligned}$$

Similarly

$$f\psi_4 = \psi_2(\phi_2^2 + \psi_2^2) + 4\psi_2 g^2 f.$$

By the same reasoning as before, these equations are again identities. This shows that either: f divides $(\phi_2^2 + \psi_2^2)$, or: f divides both ϕ_2 and ψ_2. In the second case, as before, we are done, so we may assume that $f|(\phi_2^2 + \psi_2^2)$.

Equation (5) gives

$$\begin{aligned} m\phi_3 + k\psi_3 &= 2mA(kA = mB)f + 2kB(kA = mB)f \\ &= 2f(mA + kB)(kA + mB) \\ &= 2f(km(A^2 + B^2) = 2(k^2 + m^2)AB). \end{aligned}$$

So $f(m\phi_3 + k\psi_3) = 2km(\phi_2^2 + \psi_2^2) + 2(k^2 + m^2)\phi_2\psi_2$. Since we already know that $f|(\phi_2^2 + \psi_2^2)$ it follows that $f|\phi_2\psi_2$. Thus we also have $f|(\phi_2 \pm \psi_2)^2$ and thus $f|\phi_2 \pm \psi_2$ and finally $f|\phi_2$ and $f|\psi_2$. Since these polynomials are polynomials of degree 3 in k and m, $A(k,m)$ and $B(k,m)$ would be polynomials of degree 1, in k and m ,i.e.,

$$A(k,m) = ak + bm + c, \qquad B(k,m) = dk + em + f.$$

Now by using the symmetric relations in Proposition 1.2 we see that $b = d = 0$ and $a = e$. Hence the functions A and B are in the form

$$A(k,m) = \alpha k + \beta, \qquad B(k,m) = \alpha m + \gamma.$$

These linear functions show that our rectifiable bundle of circles necessarily has the form

$$\begin{cases} S_1 + kS_2 = 0 \\ S_1 + mS_3 = 0, \end{cases} \tag{8}$$

where $S_1 = 0, S_2 = 0$ and $S_3 = 0$ are the equations for certain non-tangent spheres passing through the point $(0,0,0)$. We denote by Q the second point of intersection of the spheres $S_1 = 0, S_2 = 0$ and $S_3 = 0$. All the circles in the bundle pass through Q. In order to rectify such a bundle of circles, it suffices to map the point Q to infinity via an inversion. Now the proof of theorem is complete. $\square$

Remark 1.2. The analogous result fails in $\mathbf{R}^4$. Suppose that this is not the case. Then for every point p the bundle of all circles (geodesics) passing through the point p, being rectifiable by the exponential map, should pass through some other common point. Now it can be easily shown, as in the proof of Theorem 1.1, that there exists a germ of local diffeomorphism mapping all the circles in a neighborhood into straight lines. This would give rise to a four-dimensional extension of Beltrami's theorem which in turn implies that the corresponding metric in $\mathbf{R}^4$ has constant curvature. But there is a famous example of a metric in $\mathbf{R}^4$, i.e., *Fubini-study* metric which has circle geodesics but non-constant curvature, cf. [**1**] or [**3**] for details.

2. Classification theorems. Consider the space S of equations of spheres in $\mathbf{R}^3$ i.e., the space of non-zero polynomials of the form

$$V = \Big\{ a\Big(\sum_{i=1}^{3} x_i^2\Big) + \langle b, x\rangle + c \mid a, c \in \mathbf{R},\ b \in \mathbf{R}^3 \Big\}.$$

Clearly, every element of this form is defined up to a factor. Thus the space V is isomorphic to the projective space $\mathbf{R}P^4$. A projective subspace L of V of dimension k, $k = 1, 2, 3$, is called a k-dimensional linear system of spheres. Among all different linear systems, there are three systems which are closely related to the three geometries of Lobachevski, Euclid, and Riemann: the linear system of all spheres orthogonal, respectively to a fixed sphere of positive radius: $\sum_{i=1}^3 x_i^2 = 1$, zero radius: $\sum_{i=1}^3 x_i^2 = 0$, and imaginary radius: $\sum_{i=1}^3 x_i^2 = -1$. These three different linear systems can be expressed as:

1. $A(\sum_{i=1}^3 x_i^2) + \langle B, x\rangle + A = 0$.
2. $\langle B, x\rangle + D = 0$.
3. $A(\sum_{i=1}^3 x_i^2) + \langle B, x\rangle - A = 0$.

Definition 2.1. A three-dimensional *net of spheres* is any set of spheres, the equations of which lie in some three-dimensional linear system but not in any two-dimensional linear system.

Definition 2.2 (Characteristic map). A *characteristic map* of three-dimensional net is a map $\Phi : \mathbf{R}^3 \to \mathbf{R}P^3$ defined by

$$\Phi(X) = [S_1(X) : S_2(X) : S_3(X) : S_4(X)]$$

where, S_1, S_2, S_3, S_4 are any 4 independent quadratic polynomials in the space V. A characteristic map Φ depends on the choice of the polynomials S_i and is therefore defined up to a projective transformation.

Definition 2.3 (Degenerate point). The point (x_1, x_2, x_3) of a characteristic map Φ is called a *degenerate point* of three-dimensional net of spheres if Φ has the zero Jacobian at that point. The degenerate points of the three linear systems of spheres indicated above consist, respectively, of the points on the sphere $\sum_{i=1}^3 x_i^2 = 1$, the point $(0,0,0)$, and the empty set.

Definition 2.4 (Rich family of circles in $\mathbf{R}^3$). Let Δ be a family of circles in some domain U. Δ is called a *rich family*, if there exists a subfamily $\Gamma \subseteq \Delta$ such that

1. For each $P \in U$ there exists a circle $\gamma \in \Gamma$ such that $P \in \gamma$.

2. If $\gamma \in \Gamma$ and $P \in \gamma$, then there exists an open cone K_P (it is assumed that the cone depends continuously on the point P) such that the tangent line of γ at the point P lies inside K_P, and any other direction in K_P corresponds to a circle in Γ.

The next two theorems give a complete description of all local diffeomorphisms which rectify a rich family of circles in a domain U.

Theorem 2.1. *A rich family of circles in $\mathbf{R}^3$ in a neighborhood of the point P is rectifiable if and only if there exists a germ of a diffeomorphism $\Phi : (\mathbf{R}^3, P) \to \mathbf{R}\mathcal{P}^3$ given by*

$$\Phi(X) = [S_1(X) : S_2(X) : S_3(X) : S_4(X)].$$

where

$$S_i(X) = a_i(x^2 + y^2 + z^2) + b_i x + c_i y + d_i z + e_i, \quad i = 1, \dots, 4.$$

with a non-zero Jacobian such that every circle in the family is the inverse image of a line under Φ.

Proof. Let us first suppose that the rich family of circles is rectifiable. Let $c = 0$ be the equation for some circle in the family passing through the point P, with a tangent lying inside the cone K_p. Let A and B be two points on the circle $c = 0$ lying close to P but on different sides. The circles in the family passing through the points A and B form rectifiable bundles by our assumption. Hence by Theorem 1.1 they pass through the points C and D respectively distinct from A and B. Through each point Q close to the P, i.e., Q contained in both K_A and K_B, there exist circles in the family of circles passing through the points A, C and in the family of circles passing through the points B, D. Again by Theorem 1.1 all circles in the family passing through the point Q pass a single point S distinct from the point Q. Now suppose that the lines containing the segments $\overline{AC}$ and $\overline{BD}$ intersect at some point O. According to the different positions of the point O, we have the following different cases.

Case (1). The point O lies outside c. By definition of the power of the point O with respect to the circle c, we have: $\overline{OA} \cdot \overline{OC} = \overline{OB} \cdot \overline{OD}$. Let us denote this number by r^2. Let S be a sphere of radius r centered at the point O. It is easy to see that $R^2 + r^2 = \overline{OM}^2$, where R and M are the radius and center of the circle c respectively.

Case (2). The point O is on the circle c. In this case, the sphere S is just a single point O. In other words, we have a sphere of radius zero.

Case (3). The point O lies inside c. In this case, one can easily show that $R^2 - r^2 = \overline{OM}^2$, where r, R, O and M are the same as the case(1).

Now by mapping the spheres in cases (1) and (3) to the unit sphere at the origin, and similarly the point O in case (2) into origin, we can easily see that the equations of circles lie on the following three different spheres:

$$\begin{aligned} Ax + By + Cz + D(1 - x^2 - y^2 - z^3) = 0, \\ Ax + By + Cz + D(0 + x^2 + y^2 + z^2) = 0, \\ Ax + By + Cz + D(1 + x^2 + y^2 + z^2) = 0. \end{aligned}$$

To complete the proof we only need to show that for each rich family of rectifiable circles the characteristic map is the same from one point to another. To prove this last assertion, suppose that $P, Q \in U$ such that $P \neq Q$. Since any curve joining the two points P and Q is a finite curve or a segment, we can cover this curve by a finite number of balls such that the center of each ball lies in the next. Each characteristic map is a analytic function and any two characteristic maps coincide in the intersection of their domain, so they coincide identically by the theorem of analytic continuation.

Thus the equations for all the circles in our rectifiable family lie in two different linear combinations of the equations of spheres (including planes) $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ and $S_4 = 0$. Moreover, we can easily see that near a degenerate point of a rich family there does not exist any rectifiable rich subfamily (this can be verified separately for the three linear systems of circles). Near a nondegenerate point of the family, the family is rectified by the characteristic transformation

$$\Phi(x, y, z) = [S_1(x, y, z) : S_2(x, y, z) : S_3(x, y, z) : S_4(x, y, z)].$$

where

$$S_i(X) = a_i(x^2 + y^2 + z^2) + b_i x + c_i y + d_i z + e_i, \quad i = 1, \dots, 4. \qquad \square$$

It remains to show that, up to a projective transformation, there exists no other rectifying map. This is an immediate consequence of the following lemma.

Lemma 2.1. *A local diffeomorphism of the space which sends a rich family of lines into lines is a projective transformation.*

Proof. It is well known that a homeomorphism which sends all lines into lines is a projective transformation. The proof of this fact based on constructing an everywhere Möbius flat net, cf. [**7**]. To this end, suppose that four lines l_i, $i = 1, \dots, 4$, in a plane Π are in general position. Suppose F is a map which sends these four lines into another four lines $F(l_i)$, $i = 1, \dots, 4$, in general position. Now for any projective transformation T, $T(l_i)$, $i = 1, \dots, 4$, are also in general position. Then there exists a projective transformation U such that U maps $T(l_i)$ into $F(l_i)$. So without loss of generality, we may assume that $T(l_i) = F(l_i)$, $i = 1, \dots, 4$. For each i, $i = 1, \dots, 4$, let us denote this line by m_i. Suppose that A, B, C, D and a, b, c, d are four vertices of the quadrilateral formed by l_i and m_i, $i = 1, \dots, 4$, respectively. Since both maps F and T map the diagonals of the first quadrilateral to the diagonals of the second one, it follows that $F(P) = T(P)$, where P and $F(P)$ are the intersection points of the pairs of the diagonals respectively. Continuing in this way, we get a countable dense subset of some neighborhood such that $F = T$ on this subset. Since both F and T are continuous, we have $F = T$ on this neighborhood. Now the proof of Lemma 2.1 is based on the same argument. If P is a point in the neighborhood U, then we apply the same reasoning for any plane Π passing through the point P. First of all, we can map any four lines in general position into a parallelogram via a projective transformation. Secondly, by definition of a rich family of lines for any point $a \in U$, there exists a line l passing through the point a. Since $a \in l$, there exists a cone K_a such that l lies inside K_a. For any other point $b \in l$, there exists another cone K_b such that l lies inside K_b. Now we can easily construct a parallelogram such that all four sides as well as its diameters contained in our rich family. Finally, we construct a *Möbius flat net* inside this parallelogram, all of whose lines lie in the rich family. This implies that the mapping F is locally projective. The connectedness of the region U now implies that F is projective. $\square$

We proved that

Theorem 2.2. *Up to a projective transformation of the space of the image and a Möbius transformation of the space of the inverse image, there exist exactly three local diffeomorphisms which rectify rich families of circles. They are given by:*

1. $\Phi(x, y, z) = [x : y : z : 1 - x^2 - y^2 - z^2]$

2. $\Phi(x, y, z) = [x : y : z : 1]$
3. $\Phi(x, y, z) = [x : y : z : 1 + x^2 + y^2 + z^2]$.

3. Applications in Riemannian geometry. As is well known, there are three classical geometries in which the geodesics in some local coordinate system are straight lines. Beltrami's theorem together with the Minding-Riemann theorem, both of which are n-dimensional results, ensure that these geometries are the only ones with these properties. Fixing $n = 3$, we will show that there are precisely three classical geometries whose geodesics are circles. This will be a consequence of the three-dimensional extension of Beltrami's theorem which we are going to prove by replacing straight lines with circles. However, this result, unlike the previous statement, does not hold in arbitrary dimension. In fact, there is a very natural metric in $\mathbf{R}^4$ whose geodesics are circles, but it does not have constant curvature and hence it does not coincide with the three classical geometries.

3.1 Riemannian geometry. In this subsection, we give a proof of the three-dimensional extension of the Beltrami theorem. Next, we recall the three metrics in which the geodesics are straight lines. In the end, we use these metrics to calculate the corresponding metrics with circle geodesics.

Theorem 3.1 (Three-dimensional extension of the Beltrami theorem for circles). *Let U be a region in $\mathbf{R}^3$ and g a Riemannian metric on U. If all geodesics in U are circles* (*or parts of circle*), *then g has constant curvature.*

Proof. First of all, note that all geodesics (circles) passing through one point are rectifiable by the exponential map. Secondly, by Theorem 1.1 any rectifiable bundle in $\mathbf{R}^3$ passes through some other common point distinct from the center. On the other hand, all geodesics in a region U form a rich family of circles. Now by using Theorem 2.1 together with Theorem 2.2 we get the three different characteristic maps. By Theorem 2.2, every characteristic map is a geodesic map (mapping the geodesics of the first space to the geodesics of the second). The Beltrami theorem now implies that U has constant curvature k. By the Minding-Riemann theorem we see that U isometric to a part of the elliptic space if $k > 0$, Euclidean if $k = 0$, and hyperbolic if $k < 0$. □

Remark 3.1. First of all, this result is absolutely different from that of Beltrami's. In the Beltrami theorem the geodesics are already given and we look for the corresponding rectification. But here we deal with a six-dimensional family of circles in $\mathbf{R}^3$ and we wish to choose among this family all four-dimensional families such that they could be families of geodesics. The problem of choosing these families relies heavily on the rectification problem.

Secondly, the Beltrami theorem holds in any dimension, but our result can not be extended to dimension 4. As we have already mentioned, a very natural example of Riemannian metrics in dimension 4 in which the geodesics are circles but the curvature is not constant, is the *Fubini-Study* metric.

To have a view of the nature of the metric g in the space of rectifiable families of circles we need to draw our attention to some models of geometries in which the geodesics are straight lines, cf. [**8**].

Remark 3.2. The following discussion is also true for $n = 2$. Clearly for $k = 0$, the model is the Euclidean space $\mathbf{R}^3$ with the corresponding Riemannian metric

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2.$$

Hence by the affine characteristic map $\Phi(z) = x$ we have

$$ds^2 = \sum_{i=1}^{3} dx_i^2.$$

For $k < 0$, we use the gnomonic projection of D^3 onto H^3, where

$$D^3 = \Big\{z \in \mathbf{R}^3 : \sum_{i=1}^{3} z_i^2 < 1\Big\},$$

and $H^3 = \{z \in \mathbf{R}^4 : z_1^2 + z_2^2 + z_3^2 - z_4^2 = -1, z_4 > 0\}$. Identify $\mathbf{R}^3$ with $\mathbf{R}^3 \times \{0\}$ in $\mathbf{R}^4$. The gnomonic projection μ of D^3 onto H^3 is defined to be the composition of vertical translation of D^3 by e_4 followed by the radial projection to H^3. An explicit formula for μ is given by

$$\mu(y) = \frac{y + e_4}{\|y + e_4\|}$$

where

$$\|y + e_4\|^2 = 1 - |y|^2 = 1 - \sum_{i=1}^{3} y_i^2.$$

First we note that the element of hyperbolic arc length of H^3 is

$$\|dz\|^2 = \sum_{i=1}^{3} dz_i^2 - dz_4^2.$$

If $y.dy$ denotes the standard inner product between $y = (y_1, y_2, y_3)$ and $dy = (dy_1, dy_2, dy_3)$, then

$$|dz|^2 = \sum_{i=1}^{3} dz_i^2 - dz_4^2 = \frac{(1 - |y|^2)|dy|^2 + (y.dy)^2}{(1 - |y|^2)^2}. \tag{9}$$

For $k > 0$, we similarly use the gnomonic projection of $\mathbf{R}^3$ onto S^3, the unit sphere. To do this, we identify $\mathbf{R}^3$ with $\mathbf{R}^3 \times \{0\}$ in $\mathbf{R}^4$. Then the gnomonic projection

$$\nu : \mathbf{R}^3 \longrightarrow S^3$$

is defined to be the composition of the vertical translation of $\mathbf{R}^3$ by e_4 followed by the radial projection to S^3. An explicit formula for ν is given by

$$\nu(y) = \frac{y + e_4}{|y + e_4|}$$

where $|y + e_4|$ is the Euclidean norm of $y + e_4$. Since the element of spherical arc length of S^3 is the element of Euclidean arc length of $\mathbf{R}^4$ restricted to S^3, the arc length ds of S^3 is given by

$$ds^2 = \sum_{i=1}^{4} dz_i^2.$$

Thus

$$ds^2 = \sum_{i=1}^{4} dz_i^2 = \frac{\sum_{i=1}^{3} dy_i^2}{1+|y|^2} - \frac{(y.dy)^2}{(1+|y|^2)^2}. \tag{10}$$

This gives rise to an elliptic model in which all geodesics are straight lines in $\mathbf{R}^3$.

Now we are ready to calculate the Riemannian metrics of our three geometries in which the geodesics at every point are circles.

Clearly for $k = 0$ this is the standard metric

$$ds^2 = \sum_{i=1}^{3} dx_i^2.$$

For the hyperbolic case, we set $k = -1$ and use the affine characteristic map $\Phi : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$\Phi(x) = \frac{x}{1+|x|^2} = y$$

where

$$|x|^2 = \sum_{i=1}^{3} x_i^2.$$

Let $w = (1+|x|^2)^{-1}$. Then $y_i = x_i w$, and $dy_i = dx_i w - 2x_i(x.dx)w^2$, $i = 1, 2, 3$.

Substituting these expressions into hyperbolic metric with geodesics as straight lines namely in (4.1) we get the following metric with geodesics as circles.

$$ds^2 = \frac{1}{1+|x|^2+|x|^4}\left(|dx|^2 - \frac{3(x.dx)^2}{1+|x|^2+|x|^4}\right).$$

Finally for the elliptic case we set $k = 1$, and we use the affine characteristic map $\Phi : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$\Phi(x) = \frac{x}{1-|x|^2} = y.$$

Let $t = (1-|x|^2)^{-1}$. Then $y_i = x_i t$, and $dy_i = dx_i t + 2x_i(x.dx)t^2$, $i = 1, 2, 3$. Substituting these expressions into elliptic metric with geodesics as straight lines namely in (4.2) we get the following metric with geodesics as circles.

$$ds^2 = \frac{1}{1-|x|^2+|x|^4}\left(|dx|^2 + \frac{3(x.dx)^2}{1-|x|^2+|x|^4}\right).$$

3.2. Open questions. In the end, I state some open problems posed by A.G. Khovanskii and Andrei Gabrielov. This work opens a large field for further investigations. Some most obvious questions to be asked are as follows: Firstly, what is going on in higher dimension (both of the above main results are wrong in $\mathbf{R}^4$, for details see the recent paper by Timorin [**11**], in fact there are at least five different geometries having circle geodesics in $\mathbf{R}^4$.) Secondly, are there some results in the same spirit for more general classes of curves (say, algebraic curves of given degree with fixed leading term - the existence of a unique asymptotic cone seems to be important)?

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ENDNOTES

1. A discipline which was discovered by the French Civil Engineer, Docagne (1880). It turned out to have practical applications in many branches of science and technology, cf. [**4, 5**].

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