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FINITENESS OF INFINITESIMAL DEFORMATIONS AND INFINITESIMAL RIGIDITY OF HYPERSURFACES IN REAL EUCLIDEAN SPACES

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1. Introduction. We are concerned in this paper with the finite dimensionality of the space of infinitesimal deformations of embeddings. We study the following two cases: isometric embeddings of Riemannian n-manifolds, $n \geq 3$, into $\mathbf{R}^{n+1}$ and conformal embeddings of Riemannian n-manifolds, $n \geq 5$, into $\mathbf{R}^{n+1}$. In each case, embeddings are defined by an overdetermined system of nonlinear partial differential equations of first order.

Generically, an overdetermined system admits prolongation to a complete system of finite order, that is, we can solve for all the partial derivatives of the unknown functions of a certain order as functions of lower order derivatives of the unknown functions after differentiating the original equations sufficiently many times. This occurs when the coefficients to the highest order partial derivatives satisfy the nondegeneracy condition of the implicit function theorem.

In this paper we shall show that the linearized system at an embedding admits prolongation to a complete system of finite order under certain generic conditions on the embedding, which implies the finite-dimensionality of the space of infinitesimal deformations. In particular, we prove that the dimension of the space of infinitesimal deformations is minimal, that is, equal to the dimension of the automorphism group of the target manifold, from which we conclude that the embedding is infinitesimally rigid.

The first result on the rigidity of local embeddings seems to be the classical Beez-Killing theorem, which states that if a germ of hypersurface in $\mathbf{R}^{n+1}$, $n \geq 3$, has three nonzero principal curvatures then it is rigid, see [**17**, p. 244]. The Beez-Killing theorem is the case where the embedding is completely determined by the first jet at a

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point. More generally, dependence on finite jet at a point of isometric embeddings for more general codimensions has been studied in [**1**]. The local rigidity problem has also been studied in [**7, 9, 11, 15, 18**] for the case of Riemannian isometric embeddings. For the conformal cases, Cartan proved in [**3**] that a germ of hypersurface in $\mathbf{R}^{n+1}$, $n \geq 5$, is conformally rigid if the multiplicity of each principal curvature does not exceed $n-3$. This result has been generalized to higher codimensions in [**10**].

This paper is organized as follows. In Section 1 we explain the notions of prolongation and complete system. In Section 2 and Section 3 we construct complete systems for the infinitesimal deformations of isometric embeddings and conformal embeddings, respectively.

1. Prolongation and the complete systems. Let $m, n \in \mathbf{N}$. Let Ω be an open subset of $\mathbf{R}^n$ and let $\mathbf{R}^{(q)}$ be a Euclidean space whose coordinates represent all the partial derivatives of $\mathbf{R}^m$-valued smooth maps defined on Ω of all orders from 0 to q. A multi-index of order r is an unordered r-tuple of integers $I = (i_1, \dots, i_r)$, with $1 \leq i_s \leq n$. The order of a multi-index I is denoted by $|I|$. By u_I^a we denote the $|I|$th order partial derivative of u^a with respect to $x^{i_1}, \dots, x^{i_{|I|}}$, and we often drop the parentheses and commas in writing multi-indices, thus $u_i^a = u_{(i)}^a = \partial u^a / \partial x^i$, $u_{jk}^a = u_{(j,k)}^a = (\partial^2 u^a)/(\partial x^j \partial x^k)$, and so forth. A point in $\mathbf{R}^{(q)}$ will be denoted by $u^{(q)}$, so that $u^{(q)} = (u_I^a)_{1 \leq a \leq m,\, 0 \leq |I| \leq q}$.

The product space $\mathcal{J}^q(\Omega, \mathbf{R}^m) = \Omega \times \mathbf{R}^{(q)}$ is called the qth order jet space of the space $\Omega \times \mathbf{R}^m$. If $f = (f^1, \dots, f^m) : \Omega \to \mathbf{R}^m$ is smooth, let $(\jmath^q f)(x) = (x, \partial_I f^a(x) : 1 \leq a \leq m, |I| \leq q)$, then $\jmath^q f$, called the q-graph of f, is a smooth section of $\mathcal{J}^q(\Omega, \mathbf{R}^m)$.

Consider a system of partial differential equations of order q, $q \geq 1$, for unknown functions $u = (u^1, \dots, u^m)$ of independent variables $x = (x^1, \dots, x^n)$:

$$\Delta_\nu(x, u^{(q)}) = 0, \quad \nu = 1, \dots, l, \tag{1.1}$$

where each $\Delta_\nu(x, u^{(q)})$ is a smooth function in its arguments. Then $\Delta = (\Delta_1, \dots, \Delta_l)$ is a smooth map from $\mathcal{J}^q(\Omega, \mathbf{R}^m)$ into $\mathbf{R}^l$. The subset $\mathcal{S}_\Delta$ of $\mathcal{J}^q(\Omega, \mathbf{R}^m)$ defined by $\Delta = 0$ is called the solution subvariety of (1.1). Then, a smooth solution of (1.1) is a smooth map $f : \Omega \to \mathbf{R}^m$ whose q-graph is contained in $\mathcal{S}_\Delta$.

A differential function $P(x, u^{(q)})$ of order q is a smooth function defined on an open subset of $\mathcal{J}^q(\Omega, \mathbf{R}^m)$. The total derivatives of $P(x, u^{(q)})$ with respect to x^i is the differential function of order $q+1$ defined by

$$D_i P(x, u^{(q+1)}) := \frac{\partial P}{\partial x^i}(x, u^{(q)}) + \sum_{a=1}^{m} \sum_{|J| \leq q} \frac{\partial P}{\partial u_J^a}(x, u^{(q)}) u_{J,i}^a,$$

where J, i denotes the multi-index $(j_1, \ldots, j_{|J|}, i)$, for $J = (j_1, \ldots, j_{|J|})$. For each nonnegative integer r, the rth-prolongation $\Delta^{(r)}$ of the system (1.1) is the system consisting of all the total derivatives of (1.1) of order up to r. Let $(\Delta^{(r)})$ be the ideal generated by $\Delta^{(r)}$ of the ring of differential functions on $\mathcal{J}^{q+r}(\Omega, \mathbf{R}^m)$. If $\widetilde{\Delta} \in (\Delta^{(r)})$ for some r, the equation

$$\widetilde{\Delta}(x, u^{(q+r)}) = 0 \tag{1.2}$$

is called a prolongation of (1.1). Note that any smooth solution of (1.1) must satisfy (1.2). If k is the order of the highest derivative involved in $\widetilde{\Delta}$, we call (1.2) a prolongation of order k.

We now define the complete system.

Definition 1.1. We say that (1.1) admits prolongation to a complete system of order k if there exist prolongations of (1.1) of order k

$$\widetilde{\Delta}_\nu(x, u^{(k)}) = 0, \quad \nu = 1, \ldots, N \tag{1.3}$$

which can be solved for all the kth order partial derivatives as smooth functions of lower order derivatives of u, namely, for each $a = 1, \ldots, m$ and for each multi-index J with $|J| = k$,

$$u_J^a = H_J^a(x, u^{(k-1)}) \tag{1.4}$$

for some function H_J^a which is smooth in its arguments. Equation (1.4) is called a complete system of order k.

The complete system (1.4) is obtained from (1.3) when the coefficients to u_J^a, $|J| = k$, $1 \leq a \leq m$, of $\tilde{\Delta}$ satisfies the nondegeneracy condition of the implicit function theorem. Therefore, generically an over-determined system admits prolongation to a complete system of some finite order.

Now we recall that solving the given system of partial differential equations (1.1) is equivalent to finding an integral manifold of the corresponding exterior differential system

$$du_I^a - \sum_{i=1}^{n} u_{I,i}^a dx^i = 0$$

for all multi-index I with $|I| < q$ and $a = 1, \dots, m$, with an independence condition $dx_1 \wedge \cdots \wedge dx_n \neq 0$ on $\mathcal{S}_\Delta$, see [**2**]. If (1.1) admits prolongation to a complete system of order k then we have the following Pfaffian system on $\mathcal{J}^{k-1}(\Omega, \mathbf{R}^m)$:

$$\begin{cases} du^a - \sum_{i=1}^{n} u_i^a dx^i = 0, & \\ \vdots & \\ du_I^a - \sum_{i=1}^{n} u_{I,i}^a dx^i = 0 & |I| = k-2, \\ du_I^a - \sum_{i=1}^{n} H_{I,i}^a(x, u^{(k-1)}) dx^i = 0 & |I| = k-1 \end{cases} \tag{1.5}$$

with an independence condition $dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $H_{I,i}^a$ are as in (1.4). Thus, a solution $u = f(x)$ of (1.1) of class C^k satisfies a complete system of order k if and only if

$$(x) \mapsto (x, \partial_J f(x) : |J| \leq k-1)$$

is an integral manifold of the Pfaffian system (1.5). In particular, we have

Proposition 1.2. *Suppose that* (1.1) *admits a complete system* (1.4). *Then a solution is uniquely determined by its* $(k-1)$ *jet at a point and is* C^∞ *provided that it is* C^k. *Furthermore, if* (1.1) *is real analytic in its arguments then each* H_J^a *is real analytic and every* C^k *solution of* (1.1) *is real analytic.*

2. Isometric embedding of Riemannian manifolds. Let Ω be an open subset equipped with a Riemannian metric g and let $g_{ij}(x) = g(\partial_{x^i}, \partial_{x^j})$, where $x = (x^1, \dots, x^n)$ is a Riemannian normal coordinate system. A mapping $u = (u^1, \dots, u^{n+1}) : \Omega \to \mathbf{R}^{n+1}$ is a local isometric embedding if u satisfies

$$\sum_{\alpha=1}^{n+1} u_i^\alpha u_j^\alpha = g_{ij}(x), \quad i, j = 1, \dots, n. \tag{2.1}$$

For mappings u and v of Ω into $\mathbf{R}^{n+1}$ we define a symmetric (0,2) tensor field on Ω by

$$\langle du, dv \rangle = \frac{1}{2} \sum_{\alpha=1}^{n+1} (du^\alpha \otimes dv^\alpha + dv^\alpha \otimes du^\alpha).$$

Then (2.1) can be written as

$$\langle du, du \rangle = g. \tag{2.1}$$

Let f be a solution of (2.1). A one-parameter family of solutions $\{u_\tau\}_{|\tau|<\varepsilon}$, of (2.1) is called a bending of f if $u_0 = f$. By substituting u_τ for u in (2.1) and differentiating with respect to τ at $\tau = 0$ we get

$$\left\langle df, d\left(\frac{\partial}{\partial \tau} \Big|_{\tau=0} u_\tau \right) \right\rangle = 0. \tag{2.2}$$

This motivates the following definition:

Definition 2.1. A $\mathbf{R}^{n+1}$-valued map v on Ω is an infinitesimal deformation of an isometric embedding f if and only if v satisfies

$$\langle df, dv \rangle = 0. \tag{2.3}$$

Let us denote by $\rho^i(f)$ the dimension of the solution space of (2.3). We observe that the composition of an infinitesimal rigid motion of $\mathbf{R}^{n+1}$ with f is trivially an infinitesimal deformation of f: It is the variation vector field of the composition of a Euclidean transformation of $\mathbf{R}^{n+1}$ with f, which is clearly a bending of f, see (2.2). Thus $\rho^i(f) \geq 1/2(n+1)(n+2)$. If the equality holds, we say that f is infinitesimally rigid.

We take up the question whether the space of the infinitesimal deformations of a given embedding f is finite dimensional. If the Riemannian manifold is flat, $\rho^i(f)$ is infinity as the following example shows:

Example 2.2. Suppose that $(\gamma_1(s), \gamma_2(s))$ is a curve in the coordinate plane parametrized by arclength s. Consider an isometric immersion f of $\mathbf{R}^2$ into $\mathbf{R}^3$ defined by $f(t,s) = (t, \gamma_1(s), \gamma_2(s))$.

A mapping $v(t,s) = (0, v_1(s), v_2(s))$ is an infinitesimal deformation of f if and only if v satisfies (2.3), that is

$$\langle df, dv \rangle = (\gamma_1'(s)v_1'(s) + \gamma_2'(s)v_2'(s))\, ds^2 = 0. \tag{2.4}$$

Since $\gamma'(s)$ is of unit length, we let $(\gamma_1'(s), \gamma_2'(s)) = (\cos\theta(s), \sin\theta(s))$, for a function θ. Then for any pair of functions $(v_1(s), v_2(s))$ such that $(v_1'(s), v_2'(s)) = (-\sin\theta(s), \cos\theta(s))\phi(s)$, where ϕ is arbitrary, the mapping $v(t,s) = (0, v_1(s), v_2(s))$ satisfies (2.4). Since ϕ is arbitrary, the solution space of (2.4) is infinite dimensional.

Let f be a solution of (2.1). Then in terms of the local coordinates (2.3) reads as

$$\sum_{\alpha=1}^{n+1} \left[\frac{\partial f^\alpha}{\partial x^i}(x) v_j^\alpha + \frac{\partial f^\alpha}{\partial x^j}(x) v_i^\alpha \right] := \Phi^{ij}(x, v^{(1)}) = 0, \tag{2.5}$$

for each $i, j = 1, \dots, n$.

Now we calculate the prolongations of (2.5). First, for each fixed quadruple (i,j,k,l) of integers $\{1, \dots, n\}$, we consider the second derivatives of (2.5):

$$\frac{1}{2}\left[D_{(j,k)}\Phi^{il} + D_{(i,l)}\Phi^{jk} - D_{(j,l)}\Phi^{ik} - D_{(i,k)}\Phi^{jl} \right] = 0. \tag{2.6}$$

In the lefthand side of (2.6) all the terms involving the third order derivatives of v cancel out and we get

$$\sum_{\alpha=1}^{n+1}\left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^k}(x)v^\alpha_{j\ell}+\frac{\partial^2 f^\alpha}{\partial x^j \partial x^\ell}(x)v^\alpha_{ik}-\frac{\partial^2 f^\alpha}{\partial x^i \partial x^\ell}(x)v^\alpha_{jk}-\frac{\partial^2 f^\alpha}{\partial x^j \partial x^k}(x)v^\alpha_{i\ell}\right]:=\mathcal{G}^{ijk\ell}(x,v^{(2)})=0. \tag{2.7}$$

On the other hand, for a fixed triple (i,j,k) of integers $\{1,\dots,n\}$, the first derivative of (2.5):

$$\frac{1}{2}\left[D_i\Phi^{jk}+D_j\Phi^{ki}-D_k\Phi^{ij}\right]=0$$

becomes

$$\sum_{\alpha=1}^{n+1}\left[\frac{\partial f^\alpha}{\partial x^k}(x)v^\alpha_{ij}+\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}(x)v^\alpha_{k}\right]:=\mathcal{E}^{ijk}(x,v^{(2)})=0. \tag{2.8}$$

The main result of this section is the following

Theorem 2.3. *Let (Ω, g) be a Riemannian manifold of dimension n, $n \geq 3$. Suppose that $f:\Omega\to\mathbf{R}^{n+1}$ is an isometric embedding such that $f(\Omega)$ has at least three nonzero principal curvatures at each point. Then the system* (2.5) *admits prolongation to a complete system of order* 2. *Furthermore, the space of the infinitesimal deformations is of dimension $(1/2)(n+1)(n+2)$, which is the dimension of the group of Euclidean motions of $\mathbf{R}^{n+1}$, therefore, f is infinitesimally rigid.*

Proof. We shall construct a complete system from (2.7) and (2.8).

Let $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ be the principal curvatures at the reference point 0. We may assume that $(x^1,\dots,x^n)$ is a Riemannian normal coordinate system at $0=(0,\dots,0)$ such that $df(0)(\partial_{x^i}|_0)$ is the principal direction corresponding to λ_i. By a suitable orthogonal change of coordinates in $\mathbf{R}^{n+1}$ we may assume that

$$\frac{\partial f^\alpha}{\partial x^i}(0)=\begin{cases}1 & \text{if } i=\alpha\\ 0 & \text{if } i\neq\alpha\end{cases} \tag{2.9}$$

$$\frac{\partial^2 f^{n+1}}{\partial x^i \partial x^j}(0)=\begin{cases}\lambda_i & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases} \tag{2.10}$$

Without loss of generality we assume that λ_1, λ_2 and λ_3 are nonzero. Note that the number of second derivatives of v is $(1/2)n(n+1)^2$ and the number of equations in (2.8) is $(1/2)n^2(n+1)$. Let us denote by Γ the subsystem of the second prolongation, which is composed of (2.8) and the following $(1/2)n(n+1)$ equations from (2.7):

$$(2.11) \qquad \begin{cases} \mathcal{G}^{2323}(x, v^{(2)}) = 0 \\ \mathcal{G}^{1i1i}(x, v^{(2)}) = 0 & i = 2, \dots, n, \\ \mathcal{G}^{3132}(x, v^{(2)}) = 0 \\ \mathcal{G}^{212i}(x, v^{(2)}) = 0 & i = 3, \dots, n, \\ \mathcal{G}^{1i1j}(x, v^{(2)}) = 0 & 2 \le i < j \le n. \end{cases}$$

Then, Γ is a determined linear system for the second order derivatives. Now we show that the coefficient matrix of Γ is nonsingular at the reference point 0.

We consider Γ with (2.8) arranged in the lexicographic order of the indices (i, j, k) and then followed by (2.11).

The second derivatives of unknown functions will be ordered as follows: $\{v^\alpha_{ij}\}_{1\le\alpha\le n,\, 1\le i\le j\le n}$ in the lexicographic order of the indices (i, j, α), and then $\{v^{n+1}_{ii}\}_{1\le i\le n}$ in the increasing order of the index i, and then $\{v^{n+1}_{ij}\}_{1\le i<j\le n}$ in the lexicographic order of the indices (i, j). For example, if $n = 3$, then the order is

$$v^1_{11}, v^2_{11}, v^3_{11}, v^1_{12}, v^2_{12}, v^3_{12}, v^1_{13}, v^2_{13}, v^3_{13}, v^1_{22}, v^2_{22}, v^3_{22}, v^1_{23}, v^2_{23}, v^3_{23},$$
$$v^1_{33}, v^2_{33}, v^3_{33}, v^4_{11}, v^4_{22}, v^4_{33}, v^4_{12}, v^4_{13}, v^4_{23}.$$

Then, by (2.9) and (2.10) the coefficient matrix of Γ with respect to the second order derivatives at 0 is of the block diagonal form

$$(2.12) \qquad \begin{bmatrix} \mathrm{A} & \mathrm{O} \\ \mathrm{O} & \mathrm{B} \end{bmatrix},$$

where the size of A is $(1/2)n^2(n+1) \times (1/2)n^2(n+1)$ and that of B is $(1/2)n(n+1) \times (1/2)n(n+1)$. The first $(1/2)n^2(n+1)$ rows correspond to (2.8) and the remaining rows correspond to (2.11). The first $(1/2)n^2(n+1)$ columns are the partial derivatives with respect to $\{u^\alpha_{ij}\}_{1\le\alpha\le n,\ 1\le i\le j\le n}$ and the remaining columns are partial derivatives with respect to $\{u_{ij}^{n+1}\}_{1\leq i\leq j\leq n}$. By (2.9) A is the identity matrix. The block B is again of the block diagonal form

$$\mathrm{B} = \begin{pmatrix} \mathrm{B}_1 & & & & \\ & \mathrm{B}_2 & & & \\ & & \mathrm{B}_3 & & \\ & & & \ddots & \\ & & & & \mathrm{B}_n \end{pmatrix}, \tag{2.13}$$

where all other elements are zeros. The block B_k, $k = 1, \dots, n$, of (2.13) is of size $[n-(k-1)] \times [n-(k-1)]$. The first n rows containing the block B_1 correspond to the first n equations of (2.11), the next $(n-1)$ rows containing the block B_2 correspond to the next $(n-1)$ equations, and so on. By (2.10) each block is as follows:

$$\mathrm{B}_1 = \begin{pmatrix} 0 & \lambda_3 & \lambda_2 & & 0 \\ \lambda_2 & \lambda_1 & 0 & \dots & 0 \\ \lambda_3 & 0 & \lambda_1 & & 0 \\ \vdots & \vdots & & \ddots & \\ \lambda_n & 0 & 0 & & \lambda_1 \end{pmatrix},$$

$$\mathrm{B}_2 = \begin{pmatrix} \lambda_3 & & & & \\ & \lambda_2 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_2 \end{pmatrix},$$

$$\mathrm{B}_k = \lambda_1 \, \mathrm{Id}_{n-(k-1)}, \quad k = 3, \dots, n,$$

where Id denotes the identity matrix. Then $(\det \mathrm{B}) = \prod_{j=1}^{n} (\det \mathrm{B}_j) = -2\lambda_1^{(1/2)(n-2)(n+1)} \lambda_2^{n-1} \lambda_3^2$, which is nonzero by the assumption, and hence the matrix (2.12) is nonsingular. Now we observe that Γ is smooth on $\mathcal{J}^2(\Omega, \mathbf{R}^{n+1})$. So, the coefficient matrix of Γ is nonsingular on a neighborhood of the reference point, and consequently, we get a complete system of order 2:

$$v_J^\alpha = H_J^\alpha(x, v^{(1)}), \quad |J| = 2, \quad \alpha = 1, \dots, n+1, \tag{2.14}$$

for some smooth maps $\{H_J^\alpha\}$ defined on an open subset of $\mathcal{J}^1(\Omega, \mathbf{R}^{n+1})$. This proves the first assertion.

The number of the partial derivatives of $v = (v^1, \dots, v^{n+1})$ of order up to 1 is $(n+1)^2$. But those derivatives are related by (2.5), where the number of independent equations is $(1/2)n(n+1)$. By (2.14) and Proposition 1.2 we know that the dimension of the space of infinitesimal deformations is the number of free partial derivatives of v of order up to 1 at a point, which is $(n+1)^2 - (1/2)n(n+1) = (1/2)(n+1)(n+2)$. The proof is completed. $\square$

3. Conformal embedding of Riemannian manifolds. Let Ω be an open subset equipped with a Riemannian metric g and let $g_{ij}(x) = g(\partial_{x^i}, \partial_{x^j})$, where $x = (x^1, \dots, x^n)$ is a Riemannian normal coordinate system. A mapping $u = (u^1, \dots, u^{n+1}) : \Omega \to \mathbf{R}^{n+1}$ is a local conformal embedding if u satisfies

$$\sum_{\alpha=1}^{n+1} u_i^\alpha u_j^\alpha = g_{ij}(x)\zeta, \quad i, j = 1, \dots, n, \tag{3.1}$$

for some positive function ζ, which is called the conformal factor. The linearization of (3.1) at a solution (f, ζ) is

$$\begin{aligned} 0 &= \sum_{\alpha=1}^{n+1} \left[\frac{\partial f^\alpha}{\partial x^i}(x) v_j^\alpha + \frac{\partial f^\alpha}{\partial x^j}(x) v_i^\alpha \right] - g_{ij}(x) w \\ &:= \Phi^{ij}(x, v^{(1)}, w^{(0)}), \end{aligned} \tag{3.2}$$

for each $i, j = 1, \dots, n$. That is, if $\{u_\tau\}_{|\tau|<\varepsilon}$ is a 1-parameter family of solutions of (3.1) with corresponding conformal factors $\{\zeta_\tau\}_{|\tau|<\varepsilon}$ satisfying $(u_0, \zeta_0) = (f, \zeta)$, then the variation vector field $v = (\partial/\partial\tau)|_{\tau=0} u_\tau$ satisfies (3.2) with the choice $w = (\partial/\partial\tau)|_{\tau=0}\zeta_\tau$. So, as in the isometric case, a $\mathbf{R}^{n+1}$-valued map v defined on Ω is called an infinitesimal conformal deformation of a conformal embedding f if and only if v satisfies (3.2) with some real-valued function w. Note that the unknown functions in (3.2) are $v = (v^1, \dots, v^{n+1})$ and w, whereas the number of equations are $(1/2)n(n+1)$.

Now we consider the second prolongations of (3.2): For each triple (i,j,k) of integers $\{1,\dots,n\}$, we get

(3.3)

$$\begin{aligned}
0 &= \frac{1}{2}\left[D_i\Phi^{jk} + D_j\Phi^{ki} - D_k\Phi^{ij}\right] \\
&= \sum_{\alpha=1}^{n+1}\left[\frac{\partial f^\alpha}{\partial x^k}(x)v^\alpha_{ij} + \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}(x)v^\alpha_k\right] + \frac{1}{2}[g_{ij}(x)w_k - g_{jk}(x)w_i - g_{ki}(x)w_j] \\
&\quad + \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial x^k}(x) - \frac{\partial g_{jk}}{\partial x^i}(x) - \frac{\partial g_{ki}}{\partial x^j}(x)\right]w \\
&:= \mathcal{E}^{ijk}(x, v^{(2)}, w^{(1)});
\end{aligned}$$

For each quadruple (i,j,k,l) of integers $\{1,\dots,n\}$, we get

(3.4)

$$\begin{aligned}
0 &= \frac{1}{2}\left[D_{(j,k)}\Phi^{il} + D_{(i,l)}\Phi^{jk} - D_{(j,l)}\Phi^{ik} - D_{(i,k)}\Phi^{jl}\right] \\
&= \sum_{\alpha=1}^{n+1}\left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^k}(x)v^\alpha_{jl} + \frac{\partial^2 f^\alpha}{\partial x^j \partial x^l}(x)v^\alpha_{ik} - \frac{\partial^2 f^\alpha}{\partial x^i \partial x^l}(x)v^\alpha_{jk} - \frac{\partial^2 f^\alpha}{\partial x^j \partial x^k}(x)v^\alpha_{il}\right] \\
&\quad + \frac{1}{2}\left[g_{il}(x)w_{jk} + g_{jk}(x)w_{il} - g_{ik}(x)w_{jl} - g_{jl}(x)w_{ik}\right] \\
&\quad + \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^l}(x) - \frac{\partial g_{jl}}{\partial x^k}(x)\right]w_i + \frac{1}{2}\left[\frac{\partial g_{il}}{\partial x^k}(x) - \frac{\partial g_{ik}}{\partial x^l}(x)\right]w_j \\
&\quad + \frac{1}{2}\left[\frac{\partial g_{il}}{\partial x^j}(x) - \frac{\partial g_{jl}}{\partial x^i}(x)\right]w_k + \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i}(x) - \frac{\partial g_{ik}}{\partial x^j}(x)\right]w_l \\
&\quad + \frac{1}{2}\left[\frac{\partial^2 g_{il}}{\partial x^j \partial x^k}(x) + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}(x) - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}(x) - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}(x)\right]w \\
&:= \mathcal{G}^{ijkl}(x, v^{(2)}, w^{(2)}).
\end{aligned}$$

Let us denote by $\rho^c(f)$ the dimension of the space of infinitesimal conformal deformations of f, which is equal to the dimension of the solution space of (3.2). Like the isometric case, we observe that the infinitesimal conformal motions of $\mathbf{R}^{n+1}$ composited with f are clearly infinitesimal conformal deformations of f. So, we have $\rho^c(f) \geq (1/2)(n+2)(n+3)$. If the equality holds, we say that f is infinitesimally conformally rigid.

As in the previous section, we shall construct under a certain nondegeneracy condition on f a complete system for (v, w) to conclude the finiteness of $\rho^c(f)$. First, we observe that the equation (3.2) can not be prolonged to a complete system of order 2 in any case: If such a prolongation is possible, then it follows from Proposition 1.2 that $\rho^c(f) \leq$ [the number of partial derivatives of (v, w) of order up to 1] $-$ [the number of independent equations in (3.2)] $= (1/2)(n^2+5n+4)$, which contradicts the fact $\rho^c(f) \geq (1/2)(n+2)(n+3)$. More specifically, we cannot construct a complete system of order 2 from (3.3) and (3.4).

Now we proceed one step further to the third prolongation. In the following theorem we shall construct a complete system of order 3 for the infinitesimal deformations of conformal embeddings under the same nondegeneracy assumptions as in [**3**] and [**4**].

Theorem 3.1. *Let* (Ω, g) *be a Riemannian manifold of dimension* n, $n \geq 5$. *Suppose that* $f : \Omega \to \mathbf{R}^{n+1}$ *is a conformal embedding with a corresponding conformal factor* ζ *such that the multiplicity of principal curvatures of* $f(\Omega)$ *does not exceed* $n-3$ *at each point. Then the system* (3.2) *for the infinitesimal deformations of* f *admits prolongation to a complete system of order* 3. *Furthermore, the space of the infinitesimal deformations is of dimension* $(1/2)(n+2)(n+3)$, *which is the dimension of the group of conformal motions of* $\mathbf{R}^{n+1}$, *therefore,* f *is infinitesimally conformally rigid.*

Proof. The system (3.2) and its prolongations are smooth in their arguments and linear in (v, w) and their partial derivatives. We shall show that at each fixed reference point the matrix of coefficients to the third order derivatives of v and w is nonsingular.

Let $(x^1, \dots, x^n)$ be a Riemannian normal coordinate system at the reference point $0 = (0, \dots, 0)$ with the coordinate vector fields $\partial_{x^1}, \dots, \partial_{x^n}$ and let $(y^1, \dots, y^{n+1})$ be a standard coordinate system of $\mathbf{R}^{n+1}$ at $f(0)$ with the coordinate vector fields $\partial_{y^1}, \dots, \partial_{y^{n+1}}$. Assume further that $T_{f(0)}\Omega$ is spanned by $\{\partial_{y^1}, \dots, \partial_{y^n}\}$, and that $\{\partial_{y^{n+1}}\}$ generate $[T_{f(0)}\Omega]^{\perp}$. Without loss of generality we may assume that

$df(0)(\partial_{x^i}|_0) = \sqrt{\zeta(0)}\partial_{y^i}|_{f(0)}$, $i = 1, \ldots, n$. Then we have

$$g(\partial_{x^i}|_0, \partial_{x^j}|_0) = g_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j; \end{cases} \tag{3.5}$$

$$\frac{\partial f^\alpha}{\partial x^i}(0) = \begin{cases} \sqrt{\zeta(0)} & \text{if } i = \alpha \\ 0 & \text{if } i \neq \alpha; \end{cases} \tag{3.6}$$

$$\frac{\partial^2 f^{n+1}}{\partial x^i \partial x^j}(0) = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.7}$$

For convenience we assume that the principal curvatures $\{\lambda_s\}_{1\leq s\leq n}$ are arranged in the descending order in their absolute values. (If two eigenvalues have the same absolute value, the positive one precedes the negative one). Then by the hypothesis on the maximal multiplicity of principal curvatures we have

$$\lambda_\nu \neq \lambda_\sigma, \quad \text{if} \quad |\nu - \sigma| \geq n - 3.$$

In particular we have $|\lambda_\nu| > |\lambda_n| \geq 0$, if $\nu \leq 3$, and hence

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 \neq 0. \tag{3.8}$$

First, we prove the following

Lemma 3.2. *The system* (3.3) *and* (3.4)*, the totality denoted by* Γ*, can be written as*

$$v^\alpha_J = H^\alpha_J(x, v^{(1)}, w^{(1)}) + \gamma^\alpha_J(x) w_{11}, \quad 1 \leq \alpha \leq n+1, \ |J| = 2, \tag{3.9}$$

$$w_J = \tilde{H}_J(x, v^{(1)}, w^{(1)}) + \tilde{\gamma}_J(x) w_{11}, \quad |J| = 2, \ J \neq (1,1), \tag{3.10}$$

for some smooth functions $\{H^\alpha_J, \tilde{H}_J\}$ *defined on an open subset of* $\mathcal{J}^1(\Omega, \mathbf{R}^{n+2})$ *and some smooth functions* $\{\gamma^\alpha_J, \tilde{\gamma}_J\}$ *defined on an open subset of the reference point* 0 *such that*

$$\gamma^\alpha_{(r,s)}(0) = \begin{cases} 0 & \textit{if } r \neq s \textit{ or } \alpha \neq n+1, \\ (2\lambda_1)^{-1} & \textit{if } r = s \textit{ and } \alpha = n+1, \end{cases} \tag{3.11}$$

$$\tilde{\gamma}_{(r,s)}(0) = \begin{cases} \lambda_s \lambda_1^{-1} & \textit{if } r = s, \\ 0 & \textit{if } r \neq s. \end{cases} \tag{3.12}$$

Proof of Lemma 3.2. Let $\Gamma(0)$ denote the system Γ with $x = 0$, the reference point. It suffices to show that $\Gamma(0)$ determines $\{v_J^\alpha\}_{1\le\alpha\le n+1,\,|J|=2}$ and $\{w_J\}_{|J|=2,\,J\neq(1,1)}$ uniquely when $\{v_I^\alpha\}_{\substack{1\le\alpha\le n+1,\\ |I|\le 1}}$, $\{w_I\}_{|I|\le 1}$ and w_{11} are specified. This is because Γ is smooth on $\mathcal{J}^2(\Omega, \mathbf{R}^{n+2})$ and linear with respect to the second order derivatives.

Rearrange $\Gamma(0)$ as follows: Put the equations $\mathcal{E}^{ijk}(0, v^{(2)}, w^{(1)}) = 0$, $1 \le i \le j \le n$, $1 \le k \le n$, in the lexicographic order of the indices (i, j, k), then $\mathcal{G}^{ijij}(0, v^{(2)}, w^{(2)}) = 0$, $1 \le i < j \le n$, in the lexicographic order of the indices (i, j), and then $\mathcal{G}^{ijik}(0, v^{(2)}, w^{(2)}) = 0$, $1 \le i$, $j, k \le n$, $i \neq j$, $j < k$, $k \neq i$, in the lexicographic order of the indices (j, k, i).

The second order derivatives of unknown functions will be ordered as follows: $\{v_{ij}^\alpha\}_{1\le\alpha\le n,\,1\le i\le j\le n}$ in the lexicographic order of the indices (i, j, α), then $\{v_{ii}^{n+1}\}_{1\le i\le n}$ in the increasing order of the index i, then $\{v_{ij}^{n+1}\}_{1\le i<j\le n}$ in the lexicographic order of the indices (i, j), then $\{w_{ii}\}_{1\le i\le n}$ in the increasing order of the index i, and then $\{w_{ij}\}_{1\le i<j\le n}$ in the lexicographic order of the indices (i, j).

Then by (3.5)–(3.7), the matrix of the coefficients to the second order terms in the linear system $\Gamma(0)$ becomes

$$\mathrm{M} = \begin{bmatrix} \mathrm{A} & \mathrm{O} \\ \mathrm{O} & \mathrm{B} \end{bmatrix}, \tag{3.13}$$

where the size of A is $(1/2)n^2(n+1) \times (1/2)n^2(n+1)$ and that of B is $(1/2)n(n-1)^2 \times n(n+1)$. The first $(1/2)n^2(n+1)$ rows correspond to (3.3) and the remaining rows correspond to (3.4). The first $(1/2)n^2(n+1)$ columns are the coefficients of $\{v_{ij}^\alpha\}_{1\le\alpha\le n,\ 1\le i\le j\le n}$, the next $(1/2)n(n+1)$ columns are the ones of $\{v_{ij}^{n+1}\}_{1\le i\le j\le n}$, and the remaining columns are the ones of $\{w_{ij}\}_{1\le i\le j\le n}$. By (3.6), $\mathrm{A} = \sqrt{\zeta(0)}\mathrm{Id}_{(1/2)n^2(n+1)}$. By (3.5) and (3.7), B is of the form

$$\mathrm{B} = \begin{bmatrix} \mathrm{B}^1 & \mathrm{O} & \mathrm{B}^2 & \mathrm{O} \\ \mathrm{O} & \mathrm{B}^3 & \mathrm{O} & \mathrm{B}^4 \end{bmatrix}, \tag{3.14}$$

where

$$\mathrm{B}^1 = \begin{pmatrix} \lambda_2 & \lambda_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \lambda_3 & 0 & \lambda_1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & & \vdots \\ \lambda_{n-1} & 0 & 0 & 0 & \dots & 0 & \lambda_1 & 0 \\ \lambda_n & 0 & 0 & 0 & \dots & 0 & 0 & \lambda_1 \\ 0 & \lambda_3 & \lambda_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda_4 & 0 & \lambda_2 & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & \lambda_{n-1} & 0 & 0 & \dots & 0 & \lambda_2 & 0 \\ 0 & \lambda_n & 0 & 0 & \dots & 0 & 0 & \lambda_2 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & & \lambda_{n-1} & \lambda_{n-2} & 0 \\ 0 & 0 & 0 & 0 & & \lambda_n & 0 & \lambda_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_n & \lambda_{n-1} \end{pmatrix};$$

B^2 is obtained from B^1 by replacing all λ_is by $-(1/2)$; B^3 is of the block diagonal form

$$\mathrm{B}^3 = \begin{pmatrix} \mathrm{B}^3_{1,2} & & & & & & & & \\ & \ddots & & & & & & & \\ & & \mathrm{B}^3_{1,n} & & & & & & \\ & & & \mathrm{B}^3_{2,3} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \mathrm{B}^3_{2,n} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \mathrm{B}^3_{n-2,n-1} & & \\ & & & & & & & & \mathrm{B}^3_{n-2,n} & \\ & & & & & & & & & \mathrm{B}^3_{n-1,n} \end{pmatrix},$$

where

$$\mathrm{B}^3_{r,s} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{r-1} \\ \lambda_{r+1} \\ \vdots \\ \lambda_{s-1} \\ \lambda_{s+1} \\ \vdots \\ \lambda_n \end{pmatrix}, \quad 1 \le r < s \le n;$$

and B^4 is obtained from B^3 by replacing all λ_is by $-(1/2)$. It can be shown that the matrix B is of rank $n(n+1)-1$. Indeed, from our hypothesis on the maximal multiplicity of principal curvatures, we see that the submatrix

$$\begin{bmatrix} \mathrm{B}^3 & \mathrm{B}^4 \end{bmatrix}$$

of B is of maximal rank $n(n-1)$, and the submatrix

$$\begin{bmatrix} \mathrm{B}^1 & \mathrm{B}^2 \end{bmatrix}$$

is of rank $2n-1$: We observe that

$$\frac{1}{2}\left[\mathrm{B}^1_1 + \mathrm{B}^1_2 + \cdots + \mathrm{B}^1_n\right] + \lambda_1 \mathrm{B}^2_1 + \lambda_2 \mathrm{B}^2_2 + \cdots + \lambda_n \mathrm{B}^2_n = 0 \tag{3.15}$$

holds, and hence rank $\begin{bmatrix} \mathrm{B}^1 & \mathrm{B}^2 \end{bmatrix} \le 2n-1$. We also observe that a sequence of suitable elementary operations on the matrix

$$[\mathrm{B}^1_1 \quad \dots \quad \mathrm{B}^1_n \quad \mathrm{B}^2_2 \quad \dots \quad \mathrm{B}^2_n]$$

yields, without changing the rank, an upper triangular matrix T whose (i,i)th element is

$$\begin{cases} 1 & \text{if } 1 \le i \le n-1, \\ \lambda_1 - \lambda_n & \text{if } n \le i \le 2n-3, \\ (\lambda_1 - \lambda_{n-1})(\lambda_1 - \lambda_{n-2}) & \text{if } i = 2n-2, \\ \lambda_1(\lambda_2 - \lambda_{n-1})(\lambda_3 - \lambda_n) & \text{if } i = 2n-1, \end{cases}$$

which is not zero by (3.8). Thus rank $[\mathrm{B}^1 \quad \mathrm{B}^2] = 2n-1$. In conclusion, we have shown that the column vectors (except the ones corresponding to the variable w_{11}) of the coefficient matrix M of the linear system $\Gamma(0)$ are linearly independent. This proves the equivalence of (3.3)–(3.4) and (3.9)–(3.10) with appropriate choices of $\{H^\alpha_J, \tilde{H}_J\}$ and $\{\gamma^\alpha_J, \tilde{\gamma}_J\}$. Substituting (3.9) and (3.10) into $\Gamma(0)$ we know that the vector

$$(\gamma^\alpha_I(0), \tilde{\gamma}_J(0))_{1\le\alpha\le n+1, |I|=2, |J|=2} \quad \text{with} \quad \tilde{\gamma}_{(1,1)} = 1$$

is a solution to $\mathrm{Mx} = 0$. On the other hand (3.15) states that the vector $(\varkappa^\alpha_I, \tilde{\varkappa}_J)_{1\le\alpha\le n+1, |I|=2, |J|=2}$ defined by

$$\varkappa^\alpha_{(r,s)} = \begin{cases} 0 & \text{if } r \ne s \text{ or } \alpha \ne n+1, \\ (1/2) & \text{if } r = s \text{ and } \alpha = n+1, \end{cases}$$

$$\tilde{\varkappa}_{(r,s)} = \begin{cases} \lambda_s & \text{if } r = s, \\ 0 & \text{if } r \ne s \end{cases}$$

is also a solution to $\mathrm{Mx} = 0$. Since $\dim(\operatorname{Ker} \mathrm{M}) = 1$, we should have (3.11)–(3.12). This completes the proof of Lemma 3.2. $\square$

Now we take total derivatives of (3.9) and (3.10) to get

$$(3.16) \qquad \begin{gathered} v^\alpha_{rsi} = D_i H^\alpha_{(r,s)}(x, v^{(2)}, w^{(2)}) + \frac{\partial \gamma^\alpha_{(r,s)}}{\partial x^i}(x) w_{11} + \gamma^\alpha_{(r,s)}(x) w_{i11}, \\ 1 \le \alpha \le n+1,\ 1 \le r, s, i \le n, \end{gathered}$$

$$(3.17) \qquad \begin{gathered} w_{rsi} = D_i \tilde{H}_{(r,s)}(x, v^{(2)}, w^{(2)}) + \frac{\partial \tilde{\gamma}_{(r,s)}}{\partial x^i}(x) w_{11} + \tilde{\gamma}_{(r,s)}(x) w_{i11}, \\ 1 \le r, s, i \le n,\ (r,s) \ne (1,1). \end{gathered}$$

We can solve (3.17) for all the third order derivatives of w as we show now:

As before we work at the reference point 0. We denote (3.17) by $\Psi^{rsi}(x, v^{(3)}, w^{(3)}) = 0$ and rearrange the equations as follows: Put $\Psi^{rsi} = 0$, $1 \le r < s < i \le n$, in the lexicographic order of the indices (r,s,i), then $\Psi^{rsr} = 0$, $2 \le r \le n$, $1 \le s \le n$, $r \ne s$, in the lexicographic order of the indices (r,s), then $\Psi^{1s1} = 0$, $2 \le s \le n$, in the increasing order of the index s, then $\Psi^{rr1} = 0$, $2 \leq r \leq n$, in the increasing order of the index r, and then finally $\Psi^{sss} = 0$, $1 < s \leq n$, in the increasing order of the index s.

The third order derivatives of w are ordered as follows:

$\{w_{rsi}\}_{1\leq r<s<i\leq n}$ in the lexicographic order of the indices (r, s, i), then $\{w_{rrs}\}_{2\leq r\leq n,\, 1\leq s\leq n,\, r\neq s}$ in the lexicographic order of the indices (r, s), then $\{w_{11s}\}_{2\leq s\leq n}$ in the increasing order of the index s, and then $\{w_{sss}\}_{1\leq s\leq n}$ in the increasing order of the index s. Then, the coefficient matrix becomes

$$\mathrm{C} = \begin{bmatrix} \mathrm{C}^1 & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{C}^2 & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{C}^3 & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{C}^6 & \mathrm{O} & \mathrm{C}^4 & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{C}^7 & \mathrm{O} & \mathrm{C}^5 \end{bmatrix},$$

where the sizes of C^1, C^2, C^3, C^4, and C^5 are $(n(n-1)(n-2))/6 \times (n(n-1)(n-2))/6$, $(n-1)^2 \times (n-1)^2$, $(n-1) \times (n-1)$, $(n-1) \times 1$, and $(n-1) \times (n-1)$, respectively:

$$\begin{aligned} \mathrm{C}^1 &= \mathrm{Id}_{(n(n-1)(n-2))/6}, \\ \mathrm{C}^2 &= \mathrm{Id}_{(n-1)^2}, \\ \mathrm{C}^3 &= \mathrm{Id}_{(n-1)}, \\ \mathrm{C}^4 &= -\frac{1}{\lambda_1}\begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{pmatrix}, \\ \mathrm{C}^5 &= \mathrm{Id}_{(n-1)}. \end{aligned}$$

It is easily verified from (3.8) that the matrix C is of maximal rank. Thus we can solve for all the third order derivatives of w:

$$w_K = \tilde{H}_K(x, v^{(2)}, w^{(2)}), \quad |K| = 3, \tag{3.18}$$

for some smooth maps $\{\tilde{H}_K\}$ defined on an open subset of $\mathcal{J}^2(\Omega, \mathbf{R}^{n+2})$.

Now, replace the term w_{i11} in (3.16) by the function $\tilde{H}_{(i,1,1)}(x, v^{(2)}, w^{(2)})$ in (3.18), to get

$$v_K^\alpha = H_K^\alpha(x, v^{(2)}, w^{(2)}), \quad |K| = 3, \ \alpha = 1, \dots, n+1, \tag{3.19}$$

for some smooth functions $\{H_K^\alpha\}$ defined on an open subset of $\mathcal{J}^2(\Omega, \mathbf{R}^{n+2})$.

Consequently, we get a complete system (3.18)–(3.19) of order 3 for conformal embeddings.

The number of the partial derivatives of $(v, w) = (v^1, \dots, v^{n+1}, w)$ of order up to 2 is $(1/2)(n+2)^2(n+1)$. But those derivatives are related by (3.2), (3.3) and (3.4), where the number of free partial derivatives of v and w of order up to 2 is $(1/2)(n^2+5n+6)$. By Proposition 1.2, the dimension of the space of infinitesimal deformations is $(1/2)(n^2+5n+6)$, and this proves the second assertion of the theorem. $\square$

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