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REMARKS ON TOPOLOGICAL PROPERTIES OF BOEHMIANS

JÓZEF BURZYK, PIOTR MIKUSIŃSKI AND DENNIS NEMZER

ABSTRACT. Spaces of Boehmians are equipped with a topology defined in a canonical way, but properties of that topology can differ significantly for different spaces of Boehmians. First we discuss an example of a space of Boehmians with trivial dual space. Then we show that the space of periodic Boehmians has a nontrivial dual space, but the topology is not locally convex. Finally, we give an example of a space of Boehmians that is a locally convex space.

1. Introduction. In [**3**], the second author investigates spaces of generalized functions called Boehmians. Unlike the space of Schwartz distributions, the construction of these spaces is algebraic. There are two types of convergence structures given to spaces of Boehmians. One type is called δ-convergence while the other is called Δ-convergence. Under some relatively mild conditions, a sequence of Boehmians (F_n) is Δ-convergent to F if and only if every subsequence of (F_n) contains a subsequence that is δ-convergent to F. If certain additional conditions are satisfied, the space of Boehmians with Δ-convergence is a complete topological vector space where the topology is given by an invariant metric, see [**3**].

Thus two questions arise. Can we give a concrete description of elements in the dual space? And, is the topology locally convex?

In this note we will discuss three different spaces of Boehmians. For the standard space of Boehmians defined on $\mathbf{R}^N$, we show that the dual space contains only the trivial functional, and therefore, the space of Boehmians is not locally convex. On the other hand, the space of periodic Boehmians has enough continuous linear functionals to separate points. However, we will show that it also is not locally convex. Elements in the third space are called rapidly decreasing Boehmians. We will show that this space has a locally convex topology.

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2. Preliminaries. In this section we give a brief introduction to Boehmians on $\mathbf{R}^N$.

The convolution of two functions f, g on $\mathbf{R}^N$, denoted by $f * g$, is defined by

$$(f * g)(x) = \int_{\mathbf{R}^N} f(x-y)g(y)\,dy,$$

whenever the integral is well defined.

Let $\mathcal{F}$ denote a space of real or complex-valued functions defined on $\mathbf{R}^N$, and let

$$\mathcal{F}^* = \{g \in \mathcal{F} : f * g \in \mathcal{F} \text{ for every } f \in \mathcal{F}\}.$$

Let Δ be a collection of sequences of functions $\varphi_1, \varphi_2, \ldots \in \mathcal{F}^*$ such that the following conditions are satisfied:

(a) If $f \in \mathcal{F}$, $(\varphi_n) \in \Delta$, and $f * \varphi_n = 0$ for every $n \in \mathbf{N}$, then $f = 0$.

(b) If $(\varphi_n), (\psi_n) \in \Delta$, then $(\varphi_n * \psi_n) \in \Delta$.

Sequences in Δ will be called *delta sequences*.

Let $\mathcal{F}^{\mathbf{N}}$ denote the collection of all sequences of elements of $\mathcal{F}$, and let $\mathcal{A} \subseteq \mathcal{F}^{\mathbf{N}} \times \Delta$ be defined as follows:

$$\mathcal{A} = \{((f_n), (\varphi_n)) : f_k * \varphi_n = f_n * \varphi_k \text{ for all } n, k \in \mathbf{N}\}.$$

Two elements $((f_n), (\varphi_n)), ((g_n), (\psi_n)) \in \mathcal{A}$ are said to be *equivalent* if $f_k * \psi_n = g_n * \varphi_k$ for all $n, k \in \mathbf{N}$. A straightforward calculation shows this is an equivalence relation on $\mathcal{A}$. The equivalence classes are called *Boehmians*. Define

$$\mathcal{B}(\mathcal{F}, \Delta) = \{[((f_n), (\varphi_n))] : ((f_n), (\varphi_n)) \in \mathcal{A}\}.$$

For convenience, a typical element of $\mathcal{B}(\mathcal{F}, \Delta)$ will be written as $F = f_n/\varphi_n$.

By defining a natural addition and scalar multiplication on $\mathcal{B}(\mathcal{F}, \Delta)$, i.e.,

$$\frac{f_n}{\varphi_n} + \frac{g_n}{\psi_n} = \frac{f_n * \psi_n + g_n * \varphi_n}{\varphi_n * \psi_n}$$

and

$$\alpha \frac{f_n}{\varphi_n} = \frac{\alpha f_n}{\varphi_n},$$

where α is a scalar, $\mathcal{B}(\mathcal{F},\Delta)$ becomes a vector space. Moreover, if $f_n/\varphi_n, g_n/\psi_n \in \mathcal{B}(\mathcal{F},\Delta)$ and $g_n \in \mathcal{F}^*$, then we can define

$$\frac{f_n}{\varphi_n} * \frac{g_n}{\psi_n} = \frac{f_n * g_n}{\varphi_n * \psi_n}.$$

Note that if $(\varphi_n) \in \Delta$, then $\delta = \varphi_n/\varphi_n \in \mathcal{B}(\mathcal{F},\Delta)$ and $F * \delta = F$ for all $F \in \mathcal{B}(\mathcal{F},\Delta)$.

Let (φ_n) be a fixed delta sequence. The space $\mathcal{F}$ can be identified with a subspace of $\mathcal{B}(\mathcal{F},\Delta)$ by identifying f with $(f * \varphi_n)/\varphi_n$. It is easy to show that this identification is independent of (φ_n). If for some $F \in \mathcal{B}(\mathcal{F},\Delta)$ there is an $f \in \mathcal{F}$ such that $F = (f * \varphi_n)/\varphi_n$, we will simply write $F = f$ and $F \in \mathcal{F}$. For example, if $F = f_n/\varphi_n$ we can write $F * \varphi_n \in \mathcal{F}$ and $F * \varphi_n = f_n$. This is a slight abuse of notation, but it will not lead to any misunderstanding.

Now suppose that $\mathcal{F}$ is equipped with a convergence. In this case it is usually assumed that $f * \varphi_n \to f$ for any delta sequence (φ_n). A sequence (F_n) of Boehmians is said to be Δ-*convergent* to F if there exists a delta sequence (φ_n) such that $(F_n - F) * \varphi_n \in \mathcal{F}$ for all $n \in \mathbf{N}$ and $(F_n - F) * \varphi_n \to 0$ in $\mathcal{F}$. If the convergence in $\mathcal{F}$ satisfies some additional conditions, then $\mathcal{B}(\mathcal{F},\Delta)$ with Δ-convergence is an F-space, see [**3**], that is, a complete topological vector space in which the topology is given by an invariant metric.

It is often more convenient to use another type of convergence in $\mathcal{B}(\mathcal{F},\Delta)$, called δ-convergence. A sequence (F_n) of Boehmians is said to be δ-*convergent* to a Boehmian F if there exists a delta sequence (φ_n) such that $(F_n - F) * \varphi_k \in \mathcal{F}$ for all $k, n \in \mathbf{N}$ and $(F_n - F) * \varphi_k \to 0$ as $n \to \infty$ in $\mathcal{F}$. This convergence is usually a nontopological convergence. Under natural conditions, one can prove that a sequence of Boehmians (F_n) is Δ-convergent to F if and only if every subsequence of (F_n) has a subsequence δ-convergent to F, see [**3**].

3. $\mathcal{B}(\mathcal{L},\Delta_{\mathcal{L}})$. The standard space of Boehmians is $\mathcal{B}(\mathcal{L},\Delta_{\mathcal{L}})$, where $\mathcal{L}$ is the space of complex-valued locally integrable functions on $\mathbf{R}^N$ and $\Delta_{\mathcal{L}}$ is defined as the family of all sequences $\varphi_1, \varphi_2, \ldots \in \mathcal{L}$ satisfying the following conditions:

(i) $\int_{\mathbf{R}^N} \varphi_n = 1$ for all $n \in \mathbf{N}$,

(ii) $\int_{\mathbf{R}^N} |\varphi_n| \leq M$ for some M and all $n \in \mathbf{N}$,

(iii) For every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $\varphi_n(x) = 0$ whenever $n > n_0$ and $\|x\| > \varepsilon$.

The space $\mathcal{D}'$ of Schwartz distributions can be identified with a subspace of $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$: a distribution $f \in \mathcal{D}'$ is identified with the Boehmian $(f * \varphi_n)/\varphi_n$, where $(\varphi_n) \in \Delta_{\mathcal{L}}$ is an arbitrary delta sequence such that $\varphi_n \in \mathcal{D}$ ($\mathcal{D}$ denotes the space of all $\mathcal{C}^\infty$-functions with compact support).

The convergence in $\mathcal{L}$ is defined by the semi-norms

$$\|f\|_r = \int_{B_r} |f(x)|\, dx,$$

where $B_r = \{x \in \mathbf{R}^N : \|x\| \leq r\}$. The corresponding Δ-convergence in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$ is metrizable and complete. Moreover, if $f_n \to f$ in $\mathcal{D}'$, then $f_n \to f$ in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$, see [**3**]. The Fourier transform of a function $f \in L^1(\mathbf{R}^N)$ is defined as follows

$$\hat{f}(z) = \int_{\mathbf{R^N}} g(x) e^{-iz\cdot x}\, dx.$$

The following theorem was proved by T.K. Boehme in the early eighties. The proof was never published.

Theorem 3.1. *There are no nontrivial bounded linear functionals on* $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$.

Proof. Suppose Λ is a continuous linear functional on $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. Then Λ is a bounded functional on $\mathcal{L}$. Consequently,

$$\Lambda(f) = \int_{\mathbf{R}^N} f(x) g(x)\, dx,$$

for some L^∞-function g with bounded support. The Fourier transform $\widehat{g}$ is an entire function on $\mathbf{C}^N$. From Lemma 7.21 in [**10**], it follows that there exist a $z = (z_1, \ldots, z_N) \in \mathbf{R}^N$ and a sequence of positive numbers (α_n) such that $z_1, \ldots, z_N$ are positive, $\alpha_n \to \infty$, and

$$\widehat{g}(\alpha_n z) \neq 0 \quad \text{for all } n \in \mathbf{N}.$$

Without loss of generality, we can assume that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n} < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} \frac{e^{i\alpha_n z\cdot x}}{\widehat{g}(\alpha_n z)} \tag{3.1}$$

converges in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. Indeed, let ω be the characteristic function of $[-\pi/z_1, \pi/z_1] \times \cdots \times [-\pi/z_N, \pi/z_N]$ and let $\omega_n(x) = \alpha_n^N \omega(\alpha_n x)$, $n \in \mathbf{N}$. Then the infinite convolution $\varphi_n = \omega_n * \omega_{n+1} * \cdots$ exists for every $n \in \mathbf{N}$ and $(\varphi_n) \in \Delta_{\mathcal{L}}$, see [**2**, Theorem 1.3.5] and [**3, 4**]. Moreover $e^{i\alpha_n z\cdot x} * \varphi_k = 0$ for all $n \geq k$.

Since (3.1) is convergent in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$,

$$\frac{e^{i\alpha_n z\cdot x}}{\widehat{g}(\alpha_n z)} \longrightarrow 0$$

in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. On the other hand,

$$\Lambda\left(\frac{e^{i\alpha_n z\cdot x}}{\widehat{g}(\alpha_n z)}\right) = \int_{\mathbf{R}^N} \frac{e^{i\alpha_n z\cdot x}}{\widehat{g}(\alpha_n z)}\, g(x)\, dx = 1. \qquad \square$$

The next theorem and its proof are similar to a result published by Józef Burzyk in [**1**] concerning type I convergence of Mikusiński's operators. The result for Boehmians was presented by Burzyk in 1981 at the seminar of Jan Mikusiński in Katowice, Poland, but was never published.

Theorem 3.2. *A set $A \subset \mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$ is precompact if and only if A is bounded.*

The proof of this theorem will be preceded by some definitions and lemmas.

By $\mathcal{K}$ we denote the space of integrable functions on $\mathbf{R}^N$ with bounded support. For a $\varphi \in \mathcal{K}$ let $||\varphi||$ denote the norm of φ in $L^1(\mathbf{R}^N)$, i.e., $||\varphi|| = \int_{\mathbf{R}^N} |\varphi|$. The subspace of $\mathcal{K}$ of all functions $\varphi \in \mathcal{K}$ such that $\operatorname{supp}(\varphi) \subset B_\varepsilon$ will be denoted by $\mathcal{K}_\varepsilon$. Note that

$$||f * \varphi||_r \leq ||f||_{r+\varepsilon} ||\varphi||$$

for all $f \in \mathcal{L},\ \varphi \in \mathcal{K}_\varepsilon$ and $r, \varepsilon > 0$.

For $\varepsilon > 0$ and $\eta \geq 1$, let $\Delta_{\varepsilon,\eta}$ denote the subset of $\mathcal{K}_\varepsilon$ which consists of all functions such that $\int_{\mathbf{R}^N} f = 1$ and $\int_{\mathbf{R}^N} |f| \leq \eta$. Finally, for $F \in \mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$ let

$$\mathcal{K}(F) = \{\varphi \in \mathcal{K} : \ F * \varphi \in \mathcal{L}\}$$

and

$$p_{r,\varepsilon,\eta}(F) = \inf\{||F * \varphi||_r : \varphi \in \mathcal{K}(F) \cap \Delta_{\varepsilon,\eta}\},$$

for each $r, \varepsilon > 0$ and $\eta > 1$.

It can be shown that a sequence (F_n) of Boehmians is Δ-convergent to F if and only if $p_{r,\varepsilon,\eta}(F_n - F) \to 0$ as $n \to \infty$ for every $r, \varepsilon > 0$ and $\eta > 1$.

Lemma 3.3. *Let the sequence $f_1, f_2, \ldots \in \mathcal{L}$ be such that for some $r, \varepsilon > 0$ the sequence $||f_n||_{r+\varepsilon}$ is bounded. Then there exists a subsequence (f_{k_n}) of (f_n) such that for each function $\varphi \in \mathcal{K}_\varepsilon$ there exists a function $f \in \mathcal{L}$ such that*

$$||f_{k_n} * \varphi - f||_r \to 0.$$

Proof. The operator L_n defined by

$$L_n(\varphi) = (f_n * \varphi)|_{B_r}$$

is a linear and continuous operator from $\mathcal{K}_\varepsilon$ to $L^1(B_r)$ such that $\|L_n\| \leq \|f_n\|_{r+\varepsilon}$. If $\varphi \in \mathcal{K}_\varepsilon$ is a continuous function, then $L_n(\varphi)$ is a continuous function on B_r, and it is easy to see that the set $\{L_n(\varphi)\}$ satisfies the assumptions of Arzelà's theorem, so this set is precompact in the space $C(B_r)$. Using the diagonal method for any countable subset $A \subset \mathcal{K}_\varepsilon$ we can find a subsequence (L_{k_n}) of (L_n) such that for each $\varphi \in A$ the sequence $(L_{k_n}(\varphi))$ is uniformly convergent on B_r. Since the space $\mathcal{K}_\varepsilon$ has a countable dense set whose elements are continuous functions, by the Banach-Steinhaus theorem, the sequence $(L_{k_n}(\varphi))$ is convergent for each function $\varphi \in \mathcal{K}_\varepsilon$. ◻

Lemma 3.4. *If (F_n) is a sequence in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$ such that for every $r > 0$, $\varepsilon > 0$ and $\eta > 1$ there exist $f, \varphi, \varphi_1, \varphi_2, \ldots \in \mathcal{K}$ such that*

$$\varphi_n \in \Delta_{\varepsilon,\eta} \cap \mathcal{K}(F_n) \quad \textit{for all } n \in \mathbf{N},$$
$$||F_n * \varphi_n - f||_r \to 0 \quad \textit{as } n \to \infty$$

and

$$||\varphi_n - \varphi|| \to 0 \quad \textit{as } n \to \infty,$$

then (F_n) is a convergent sequence in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$.

Proof. We assert that, under conditions of the lemma, (F_n) is a Cauchy sequence in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. In fact, assume that $r > 0$, $\varepsilon > 0$ and $\eta > 1$. Let (φ_n) and $f \in \mathcal{L}$, $\varphi \in \mathcal{K}$ be such that

$$\varphi_n \in \Delta_{\varepsilon/2,\sqrt{\eta}} \cap \mathcal{K}(F_n) \quad \text{for all } n \in \mathbf{N},$$

$$||F_n * \varphi_n - f||_{r+\varepsilon} \to 0 \quad \text{as } n \to \infty,$$

and

$$||\varphi_n - \varphi|| \to 0 \quad \text{as } n \to \infty.$$

Then for each for $n, m \in \mathbf{N}$ we have

$$\varphi_n * \varphi_m \in \Delta_{\varepsilon,\eta} \cap \mathcal{K}(F_n - F_m)$$

and

$$\begin{aligned} p_{r,\varepsilon,\eta}(F_n - F_m) &\leq ||(F_n - F_m) * \varphi_n * \varphi_m||_r \\ &\leq ||(F_n * \varphi_n - f) * \varphi_m||_r + ||(F_m * \varphi_m - f) * \varphi_n||_r \\ &\quad + ||f * (\varphi_n - \varphi_m)||_r \\ &\leq \sqrt{\eta}\, ||F_n * \varphi_n - f||_{r+\varepsilon} + \sqrt{\eta}\, ||F_m * \varphi_m - f||_{r+\varepsilon} \\ &\quad + ||f||_{r+\varepsilon} ||\varphi_n - \varphi_m||. \end{aligned}$$

Hence $p_{r,\varepsilon,\eta}(F_n - F_m)$ is small if n and m are sufficiently large. Consequently, (F_n) is a Cauchy sequence. Thus, by completeness of $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$, see [**3**], the sequence (F_n) is convergent in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. □

Proof of Theorem 3.2. Suppose that A is a bounded subset of $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. Let (F_n) be a sequence in A and let (α_n) be a sequence of scalars convergent to 0. Then for any $r > 0$, $\varepsilon > 0$ and $\eta > 1$, we have $p_{r,\varepsilon,\eta}(\alpha_n F_n) \to 0$. Since for each r, ε, η, the functional $p_{r,\varepsilon,\eta}$ is homogeneous, the sequence $(p_{r,\varepsilon,\eta}(F_n))$ is bounded. This implies that, for every $k \in \mathbf{N}$, there are functions

$$\varphi_{k,n} \in \Delta_{1/(2k),1+1/k} \cap \mathcal{K}(F_n)$$

such that

$$||F_n * \varphi_{k,n}||_{k+(1/k)} \leq M_k$$

for all $n, k \in \mathbf{N}$. Let γ_k be an arbitrary function such that $\gamma_k \in \Delta_{1/(2k),1}$. Using Lemma 3.3 and applying the diagonal method we can find a subsequence (F_{q_n}) of (F_n), functions $f_k \in \mathcal{L}$ and $\varphi_k \in \mathcal{K}$ such that

$$||F_{q_n} * (\varphi_{k,q_n} * \gamma_k) - f_k||_k \longrightarrow 0$$

and

$$||\varphi_{k,q_n} * \gamma_k - \varphi_k|| \longrightarrow 0,$$

as $n \to \infty$. Since the sequence (F_{q_n}) satisfies the assumptions of Lemma 3.4, it is convergent in $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$. □

4. Periodic Boehmians. In this section we will consider a space of Boehmians which is a subspace of $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$.

Let $\mathcal{C}_{2\pi}$ denote the space of all complex-valued continuous 2π-periodic functions on $\mathbf{R}$. Let $\Delta_{\mathcal{C}}^{+}$ denote the family of all sequences in $\Delta_{\mathcal{L}}$ whose members consist of nonnegative continuous functions.

Elements of the space $\mathcal{B}(\mathcal{C}_{2\pi}, \Delta_{\mathcal{C}}^{+})$ are called periodic Boehmians, see [**7, 8**]. The convergence in $\mathcal{C}_{2\pi}$ is uniform convergence on $\mathbf{R}$. The corresponding Δ-convergence in $\mathcal{B}(\mathcal{C}_{2\pi}, \Delta_{\mathcal{C}}^{+})$ is metrizable. We will show that $\mathcal{B}(\mathcal{C}_{2\pi}, \Delta_{\mathcal{C}}^{+})$ has a nontrivial dual, but its topology is not locally convex.

For the sake of our argument, it will be convenient to use an equivalent definition of periodic Boehmians. Let $\mathcal{C}(\Gamma)$ denote the collection of continuous complex-valued functions on the unit circle Γ. Whenever desirable, we will identify functions on Γ with 2π-periodic functions on the real line $\mathbf{R}$. The convolution of two functions $f, g \in \mathcal{C}(\Gamma)$, denoted by $f * g$, is given by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)\,dt.$$

A sequence of continuous nonnegative functions (φ_n) is called a delta sequence if

(i) $1/(2\pi)\int_{-\pi}^{\pi} \varphi_n(x)\,dx = 1$ for all $n \in \mathbf{N}$, and

(ii) $\operatorname{supp}\varphi_n \subseteq (-\varepsilon_n, \varepsilon_n)$, where $0 < \varepsilon_n$ and $\varepsilon_n \to 0$ as $n \to \infty$.

The collection of delta sequences will be denoted by $\Delta_{2\pi}^{+}$.

It is easy to see that the spaces $\mathcal{B}(\mathcal{C}_{2\pi}, \Delta_{\mathcal{C}}^{+})$ and $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ can be identified. Note that the space $\mathcal{D}'_{2\pi}$ of 2π-periodic Schwartz distributions can be viewed as a subspace of $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$.

The nth Fourier coefficient for a function $f \in \mathcal{C}(\Gamma)$ is defined in the usual way,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \quad n \in \mathbf{Z}.$$

The nth Fourier coefficient of $F = f_n/\varphi_n \in \mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$, denoted $\widehat{F}(n)$, is given by

$$\widehat{F}(n) = \lim_{k\to\infty} \widehat{f}_k(n).$$

It can be shown that this limit is independent of the representative.

A Boehmian $F \in \mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ is said to be zero on an open set Ω if there exists a delta sequence (φ_n) such that $F * \varphi_n \in \mathcal{C}(\Gamma)$ for all $n \in \mathbf{N}$ and $F * \varphi_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$. The support of $F \in \mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$, written $\operatorname{supp} F$, is the complement of the largest open set on which F is zero.

We see from the next theorem that $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$, unlike $\mathcal{B}(\mathcal{L}, \Delta_{\mathcal{L}})$, has a rich dual. The dual of $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$, with respect to Δ-convergence, will be denoted by $\mathcal{B}'(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$.

Theorem 4.1 [**9**]. *Let* $\Lambda \in \mathcal{B}'(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$. *There exists a unique trigonometric polynomial* $p(x) = \sum_{n=-m}^{m} \alpha_n e^{inx}$ *such that*

$$\Lambda(F) = \sum_{n=-m}^{m} \alpha_n \widehat{F}(n), \quad \text{for all } F \in \mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+}). \tag{4.1}$$

Conversely, any trigonometric polynomial $p(x) = \sum_{n=-m}^{m} \alpha_n e^{inx}$ *defines a bounded linear functional on* $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ *via* (4.1).

The next corollary provides a connection between Δ-convergence and weak convergence in $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$. Let ω be a real-valued even function defined on the integers $\mathbf{Z}$ such that $0 = \omega(0) \le \omega(n+m) \le \omega(n)+\omega(m)$ for all $n, m \in \mathbf{Z}$ and $\sum_{n=1}^{\infty} \omega(n)/n^2 < \infty$.

Corollary 4.2 [**9**]. *Suppose that the set of positive integers is partitioned into two disjoint sets* $\{t_n\}$ *and* $\{s_n\}$ *such that* $\sum_{n=1}^{\infty} 1/t_n < \infty$. *Let* (F_n) *be a sequence of Boehmians such that the sequence* $\big(e^{-\omega(s_k)}\widehat{F_n}(\pm s_k)\big)$ *is uniformly bounded for* $k, n \in \mathbf{N}$. *Then* (F_n) *is* Δ*-convergent to zero if and only if* (F_n) *converges weakly to zero.*

By the preceding corollary, it follows that the convergence structure in $\mathcal{D}'_{2\pi}$ is stronger than the convergence $\mathcal{D}'_{2\pi}$ inherits from $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$.

Now we prove the main result of this section.

Theorem 4.3. *The topology of* Δ*-convergence in* $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ *is not locally convex.*

Proof. It suffices to construct a proper closed subspace M of $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ such that if $\Lambda \in \mathcal{B}'(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ and $\Lambda(M) = \{0\}$, then Λ is identically zero, see [**10**, Theorem 3.5].

Let M denote the proper subspace of $\mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ consisting of all Boehmians F such that either $\operatorname{supp} F = \{0\}$ or $F = 0$. Clearly M is a proper subspace. It is not difficult, and hence it is left to the reader, to show that M is closed.

Now, suppose that $\Lambda \in \mathcal{B}'(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$ and $\Lambda(M) = \{0\}$. Then there exist an $m \in \mathbf{N}$ and constants $a_n \in \mathbf{C}$, $n = 0, \pm 1, \pm 2, \dots, \pm m$, such that $\Lambda(F) = \sum_{n=-m}^{m} a_n \widehat{F}(n)$, for all $F \in \mathcal{B}(\mathcal{C}(\Gamma), \Delta_{2\pi}^{+})$.

For $\nu = 0, 1, 2, \dots, 2m$, let $G_\nu = (-i)^\nu \delta^{(\nu)}$, where $\delta^{(\nu)} = \varphi_n^{(\nu)}/\varphi_n$ and (φ_n) is an infinitely differentiable delta sequence. Then $G_\nu \in M$ for $\nu = 0, 1, 2, \dots, 2m$. Thus

$$0 = \Lambda(G_\nu) = \sum_{n=-m}^{m} a_n \widehat{G}_\nu(n) = \sum_{n=-m}^{m} a_n n^\nu,$$

for $\nu = 0, 1, 2, \dots, 2m$. It follows that $a_n = 0$ for $n = 0, \pm 1, \pm 2, \dots, \pm m$. Hence Λ is identically zero. □

The condition that a delta sequence be nonnegative plays no role in the above proof. In the definition of a delta sequence, we may replace this condition by the condition that $1/2\pi \int_{-\pi}^{\pi} |\varphi_n(x)|\, \mathrm{d}x \le \gamma$ for some $\gamma > 0$ and all $n \in \mathbf{N}$. With this weaker condition, the previous theorem is still valid, although it is not known whether or not any new Boehmians are created.

For simplicity, the material in this section has been presented in the one-dimensional case. However, the ideas can easily be extended to $\mathbf{R}^N$.

5. Rapidly decreasing Boehmians. In this section we consider the space of Boehmians obtained when $\mathcal{F}$ is the space $\mathcal{S}$ of rapidly decreasing functions on $\mathbf{R}^N$. An infinitely differentiable complex-valued function f is called rapidly decreasing if for all nonnegative integers m and n we have

$$p_{m,n}(f) = \sup_{|\alpha| \le n} \sup_{x \in \mathbf{R}^N} \left(1 + x_1^2 + \cdots + x_N^2\right)^m |D^\alpha f(x)| < \infty,$$

where $x = (x_1, \dots, x_N)$, $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and D^α is the differential operator $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$. The topology of $\mathcal{S}$ is defined by the family of semi-norms $\{p_{m,n}\}_{m,n=0}^{\infty}$. It is known that the Fourier transform is a continuous map of $\mathcal{S}$ onto $\mathcal{S}$.

A sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{S}$ will be called a delta sequence if $\widehat{\varphi}_n \to 1$ in $\mathcal{C}^\infty$. The collection of these delta sequences will be denoted by $\Delta_\mathcal{S}$.

For $n = 1, 2, \ldots$, let $\omega_n \in \mathcal{S}$ be such that $\widehat{\omega}_n(x) = 1$ for $\|x\| \leq n$ and $\widehat{\omega}_n(x) = 0$ for $\|x\| \geq n+1$. Note that $(\omega_n) \in \Delta_\mathcal{S}$ and that $\omega_m * \omega_n = \omega_m$ if $m < n$.

Lemma 5.1. *For every $F \in \mathcal{B}(\mathcal{S}, \Delta_\mathcal{S})$, there exist $f_1, f_2, \ldots \in \mathcal{S}$ such that $F = f_n/\omega_n$.*

Proof. Let $F = g_n/\varphi_n \in \mathcal{B}(\mathcal{S}, \Delta_\mathcal{S})$. For every $n \in \mathbf{N}$ there exists an $m_n \in \mathbf{N}$ such that

$$\frac{n}{n+1} < |\widehat{\varphi}_{m_n}(x)| < \frac{n+1}{n} \quad \text{for } \|x\| \leq n+1.$$

Let $\psi_n \in \mathcal{S}$ be such that

$$\widehat{\psi}_n = \frac{\widehat{\omega}_n}{\widehat{\varphi}_{m_n}}. \tag{5.1}$$

Note that $(\psi_n) \in \Delta_\mathcal{S}$ and

$$F = \frac{g_n}{\varphi_n} = \frac{g_{m_n}}{\varphi_{m_n}} = \frac{g_{m_n} * \psi_n}{\varphi_{m_n} * \psi_n} = \frac{g_{m_n} * \psi_n}{\omega_n}. \qquad \square$$

Lemma 5.2. *Let $F_n \in \mathcal{B}(\mathcal{S}, \Delta_\mathcal{S})$. If $\delta\text{-}\lim F_n = 0$, then $F_n * \psi \to 0$ in $\mathcal{S}$ for every $\psi \in \mathcal{S}$ such that $\widehat{\psi} \in \mathcal{D}$.*

Proof. If $\delta\text{-}\lim F_n = 0$, then there exists a delta sequence $(\varphi_n) \in \Delta_\mathcal{S}$ such that, for every $k \in \mathbf{N}$, we have $F_n * \varphi_k \to 0$ in $\mathcal{S}$ as $n \to \infty$. Let $\psi \in \mathcal{S}$ be such that $\widehat{\psi} \in \mathcal{D}$. There exists a $k \in \mathbf{N}$ such that $\widehat{\varphi}_k \neq 0$ on $\operatorname{supp} \widehat{\psi}$. Then

$$F_n * \psi = F_n * \varphi_k * \gamma \to 0,$$

where $\gamma \in \mathcal{S}$ is defined by

$$\widehat{\gamma} = \frac{\widehat{\psi}}{\widehat{\varphi}_k}. \qquad \square$$

In the proof of the next lemma we use the Fourier transform of Boehmians. For the definition and basic properties, see [**6**].

Lemma 5.3. $\Delta\text{-}\lim F_n = 0$ *if and only if* $\delta\text{-}\lim F_n = 0$.

Proof. If $\Delta\text{-}\lim F_n = 0$, then $F_n * \varphi_n \to 0$ in $\mathcal{S}$ for some $(\varphi_n) \in \Delta_{\mathcal{S}}$. Then $\widehat{F}_n \widehat{\varphi}_n \to 0$ in $\mathcal{S}$. Let $k \in \mathbf{N}$. Since

$$\widehat{\omega}_k = \widehat{\varphi}_n \frac{\widehat{\omega}_k}{\widehat{\varphi}_n},$$

for all sufficiently large $n \in \mathbf{N}$, we have $\widehat{F}_n \widehat{\omega}_k \to 0$ in $\mathcal{S}$ as $n \to \infty$. Since k is arbitrary, $\delta\text{-}\lim F_n = 0$.

Now assume that $\delta\text{-}\lim F_n = 0$. Then there is $(\varphi_n) \in \Delta_{\mathcal{S}}$ such that, for every $k \in \mathbf{N}$, $F_n * \varphi_k \to 0$ in $\mathcal{S}$ as $n \to \infty$. Since $\mathcal{S}$ is a metric space, there exists a nondecreasing sequence of indices m_n such that $F_n * \varphi_{m_n} \to 0$ in $\mathcal{S}$. □

Theorem 5.4. *The topology of* Δ*-convergence in* $\mathcal{B}(\mathcal{S}, \Delta_{\mathcal{S}})$ *is locally convex.*

Proof. For $k, l, m = 0, 1, 2, \dots$ define

$$q_{k,l,m}(F) = \sup_{|\alpha| \le l} \sup_{x \in \mathbf{R}^N} (1 + x_1^2 + \cdots + x_N^2)^m |D^\alpha (F * \omega_k)(x)|.$$

Note that $q_{k,l,m}(F)$ is well-defined for all $F \in \mathcal{B}(\mathcal{S}, \Delta_{\mathcal{S}})$ and all $k, l, m = 0, 1, 2, \dots$, by Lemma 5.1. Moreover, $\delta\text{-}\lim F_n = 0$ if and only if $q_{k,l,m}(F_n) \to 0$ as $n \to \infty$ for all $k, l, m = 0, 1, 2, \dots$, by Lemma 5.2. Consequently, the topology of δ-convergence is locally convex. This proves that the topology of Δ-convergence is locally convex in view of Lemma 5.3. □

By a modification of the above argument, one can prove that there is a convergence preserving isomorphism between the space of tempered Boehmians $\mathcal{B}(\mathcal{T}, \Delta_{\mathcal{S}})$ and the space of Schwartz distributions $\mathcal{D}'$. Here $\mathcal{T}$ denotes the space of all complex-valued continuous slowly increasing functions. A function f is called slowly increasing if there exists a polynomial p such that $|f(x)| \leq p(x)$ for all $x \in \mathbf{R}^N$.

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Institute of Mathematics, Technical University of Silesia, Kaszubska 23, 44-100 Gliwice, Poland
E-mail address: Jozef.Burzyk@polsl.pl

Department of Mathematics, University of Central Florida, Orlando, FL 32816-1364
E-mail address: piotrm@mail.ucf.edu

Department of Mathematics, California State University, Stanislaus, Turlock, CA 95382
E-mail address: jclarke@csustan.edu