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SELF-TRANSVERSAL SPACES AND THEIR DISCRETE SUBSPACES

I. JUHÁSZ, M.G. TKACHENKO,
V.V. TKACHUK AND R.G. WILSON

ABSTRACT. A space X is called *self-transversal* if there is a bijection $\varphi : X \to X$ such that the family $\tau(X) \cup \varphi(\tau(X))$ forms a subbase of the discrete topology on X. We prove that, under CH, there exists a compact scattered space which is not self-transversal. It is shown that there exist compact self-transversal spaces of arbitrarily large cardinality with the Souslin property. We present examples of compact spaces which give a negative answer in ZFC to Problems 2 and 3 from [**8**] and a partial negative answer to Problem 1 of [**8**]. We also establish that it is independent of ZFC whether any metrizable space X is self-transversal if and only if $w(X) = |X|$. We show that any monotonically normal scattered space is self-transversal and that adding a single point to a self-transversal space can destroy self-transversality.

1. Introduction. Recall that two topologies τ and μ on the same set X are called *transversal* if $\tau \cup \mu$ is a subbase for the discrete topology on X. A natural way of exploring the properties of a given space (X, τ) is to study the interaction of its topology with its copies on the same set obtained by all possible bijections. If some of these copies are transversal to τ then the space (X, τ) is called self-transversal.

The study of transversal topologies was initiated in 1966 by Steiner who proved in [**9**] that no countable infinite set X admits a pair of Hausdorff transversal topologies whose intersection is the cofinite topology on X (such topologies are called T_2-complementary). Later, intensive study of T_1-complementary topologies was undertaken by

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E. Steiner and A. Steiner, see [**10, 11**], Watson, see [**14**–**16**], and three of the authors of this paper in various articles, see [**7, 8, 12**]. One of the results of this research was a complete answer to Watson's question whether every Hausdorff space has a T_1-complement. This question was published as Problem 162 in *Open problems in topology* [**14**] and repeated in many other papers.

Since transversality plays a crucial role in the study of complementary topologies, many results have been obtained on the existence of "nice" pairs of transversal topologies. Shakhmatov, Tkachenko and Wilson proved in [**8**], among other things, that every Hausdorff space admits a compact Hausdorff transversal topology and that every space embeds as a retract into a self-transversal space. It was also discovered in [**8**] that self-transversality of a space X has strong implications concerning the topology of X. One of them is the equality $nw(X) = |X|$ for any self-transversal space X. Consequently, weight and cardinality must coincide in any compact self-transversal space. This is a very strong restriction which shows that even the interval $[0, 1]$ with the natural topology is not self-transversal. On the other hand, Shakhmatov, Tkachenko and Wilson showed that many compact spaces are self-transversal and formulated in [**8**] some natural questions on self-transversal spaces.

We show that, under CH, there exists a scattered compact space which is not self-transversal, answering negatively Problem 1 from [**8**]. If we do not require compactness then the respective example exists in ZFC. We also give a general construction which furnishes ZFC examples of compact self-transversal spaces of arbitrarily large cardinality with the Souslin property. This gives a negative answer to Problem 2 of [**8**]. The same method makes it possible to obtain compact self-transversal spaces without points of countable π-character; this provides a strong negative answer to Problem 3 from [**8**].

Let (FH) denote the following statement: "For any infinite cardinal κ there are at most finitely many cardinals between κ and κ^ω"; since (FH) is weaker than GCH, it is consistent with ZFC. We establish that (FH) is equivalent to the following assertion: "A metrizable space X is self-transversal if and only if $w(X) = |X|$". We also give some examples of a "bad" behavior of self-transversality; in particular, we prove that a dense subspace of a self-transversal space can fail to be self-transversal and that adding one point can also destroy self-transversality.

2. Notation and terminology. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\varnothing\}$. If $A \subset X$, then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$; we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. The space $\mathbf{D}$ is the doubleton $\{0, 1\}$ with the discrete topology. A space X is *scattered* if every non-empty subspace of X has an isolated point. The space X is *monotonically normal* if, for every pair (x, U) with $x \in U \in \tau(X)$ there exists a set $H(x, U) \in \tau(x, U)$ such that, for any $U, V \in \tau(X)$ if $H(x, U) \cap H(y, V) \neq \varnothing$ then $x \in V$ or $y \in U$. The expression $X \simeq Y$ denotes that the spaces X and Y are homeomorphic; the symbol $\square$ stands for the end of a proof.

Given a point $x \in X$, a family $\mathcal{B} \subset \tau^*(X)$ is called *a π-base at x* if every $U \in \tau(x, X)$ contains an element of $\mathcal{B}$. The π-character $\pi\chi(x, X)$ of the point x in X is the minimal cardinality of a π-base at x. The *spread* $s(X)$ (*extent* $\operatorname{ext}(X)$) of a space X is the supremum of cardinalities of (closed) discrete subspaces of X. If $\mathcal{A}$ is a family of subsets of a set Y such that $\cup\mathcal{A} = Y$ then $\langle\mathcal{A}\rangle$ is the topology on Y generated by $\mathcal{A}$ as a subbase.

As usual, the symbol ω stands for the set of natural numbers and $\mathbf{N} = \omega\backslash\{0\}$. The space $\mathbf{R}$ is the real line with the usual topology. The hypothesis (FH) says that the set $\{\mu : \mu \text{ is a cardinal and } \kappa \leq \mu \leq \kappa^\omega\}$ is finite for any infinite cardinal κ. Given a family $\mathcal{T} = \{X_t : t \in T\}$ of any spaces, the expression $\oplus\{X_t : t \in T\}$ is used for the free union of the family $\mathcal{T}$. The rest of our terminology is standard and follows [**2**].

3. Self-transversal topologies. Shakhmatov, Tkachenko and Wilson described a number of properties every self-transversal space must have. We are also going to give some necessary and some sufficient conditions for self-transversality; they will be applied to answer three questions from [**8**] and give characterizations of self-transversality in some "nice" classes of spaces.

The following statement was proved and applied several times in [**8**]; since it has not been stated there explicitly, it is worth citing here.

3.1. Proposition. *Given any space X let O be the set of all isolated points of X. If $|X| = |O|$ then X is self-transversal.*

3.2. Proposition. *Given any infinite space X, if there is a closed discrete $D \subset X$ with $|D| = |X|$ then X is self-transversal.*

Proof. Making the set D smaller if necessary, we can assume, without loss of generality, that $|D| = |X \backslash D|$ and hence there exists a bijection $\varphi : X \to X$ such that $\varphi(D) = X \backslash D$. Let $\tau = \tau(X)$ and $\mu = \varphi(\tau(X))$; given any $x \in D$, we can find $U \in \tau(x, X)$ such that $U \cap D = \{x\}$. If $y = \varphi^{-1}(x)$ then $y \in X \backslash D \in \tau(y, X)$ and hence $D = \varphi(X \backslash D) \in \mu$. As a consequence, $U \cap D \in \langle \tau \cup \mu \rangle$ and $\{x\} = U \cap D$, so x is isolated in $\langle \tau \cup \mu \rangle$. Now, if $x \in X \backslash D$ then, for $y = \varphi^{-1}(x) \in D$, take any $W \in \tau(y, X)$ with $W \cap D = \{y\}$ and observe that $\varphi(W) \cap (X \backslash D) \in \langle \tau \cup \mu \rangle$ and $\varphi(W) \cap (X \backslash D) = \{x\}$, so the point x is again isolated in $\langle \tau \cup \mu \rangle$. □

3.3. Lemma. *Let X be a Hausdorff space*; *assume that we are given families $\{X_n : n \in \omega\}$ and $\{D_n : n \in \omega\}$ of infinite subsets of X with the following properties*:

(1) $\cup\{X_n : n \in \omega\}$ *is open in X and $X_i \cap X_j = \varnothing$ whenever $i \neq j$*;

(2) *if $P = X \backslash (\cup\{X_n : n \in \omega\})$ then $|P| \leq |D_0|$ and $\overline{X}_1 \cap P = \varnothing$*;

(3) *D_n is a discrete subspace of X and $D_n \subset \operatorname{Int}(X_n)$ for every $n \in \omega$*;

(4) *$|X_n| = |X_n \backslash D_n| = |D_{n+1}|$ for all $n \in \omega$*;

(5) *$X_n \cap \overline{X}_{n+2} = \varnothing$ for any $n \in \omega$.*

Then X is self-transversal.

Proof. We will construct a bijection $\varphi : X \to X$ which is actually an involution, i.e., $\varphi(\varphi(x)) = x$ for all $x \in X$. To this end apply (2) to find a bijection $a_0 : P \to Q$ of P onto some $Q \subset D_0$; let $E_0 = Q$ and $E_n = D_n$ for all $n \in \mathbf{N}$. Observe that the set P can be empty in which case $Q = \varnothing$ and there is no need to define a_0. Choose a bijection $b_n : X_n \backslash E_n \to E_{n+1}$ for every $n \in \omega$ (this choice is possible by (4)).

Now, if $x \in P$, let $\varphi(x) = a_0(x)$; to make the mapping φ an involution, let $\varphi(x) = a_0^{-1}(x)$ for each $x \in Q = E_0$. Now assume that $x \in X \backslash (P \cup E_0)$; there is a unique $n \in \omega$ such that $x \in X_n$. If $x \in X_n \backslash E_n$, then let $\varphi(x) = b_n(x)$; if $x \in E_n$, then $n > 0$ so we can define $\varphi(x) = b_{n-1}^{-1}(x)$. It is immediate that $\varphi : X \to X$ is an involution; let $\tau = \tau(X)$ and $\mu = \varphi(\tau)$. To prove that $\langle \tau \cup \mu \rangle$ is discrete take any $x \in X$ and let $y = \varphi(x)$; we must consider the following cases.

Case 1. $x \in P$. Then $y \in E_0$ so, by (3), there exists a set $V \in \tau(y, X)$ such that $V \subset X_0$ and $V \cap E_0 = \{y\}$. If $U = X \backslash \overline{X}_1$, then $U \in \tau(x, X)$ by (2). Consequently, $\varphi(V) = b_0(V \backslash \{y\}) \cup \{x\}$; it follows from $b_0(V \backslash \{y\}) \subset X_1$, that $U \cap \varphi(V) = \{x\}$ which shows that x is isolated in $\langle \tau \cup \mu \rangle$.

Case 2. $x \in E_0$. Then $y \in P$ so the result of Case 1 is applicable to the point y. Thus y is isolated in $\langle \tau \cup \mu \rangle$ and hence there exist $V \in \tau(y, X)$ and $U \in \tau(x, X)$ such that $V \cap \varphi(U) = \{y\}$. Since φ is an involution, we have $\varphi(V) \cap U = \{x\}$ so x is isolated in $\langle \tau \cup \mu \rangle$.

Case 3. $x \in X_n \backslash E_n$ for some $n \in \omega$. Then $y \in E_{n+1}$ and (3) shows that there exists $V \in \tau(y, X)$ such that $V \subset X_{n+1}$ and $V \cap E_{n+1} = \{y\}$. If $U = X \backslash \overline{X}_{n+2}$, then we have $U \in \tau(x, X)$ by (5); since $\varphi(V \backslash \{y\}) = b_{n+1}(V \backslash \{y\}) \subset X_{n+2}$, we obtain $\varphi(V) \cap U = \{x\}$ which shows that x is isolated in $\langle \tau \cup \mu \rangle$.

Case 4. $x \in E_{n+1}$ for some $n \in \omega$. Then $y \in X_n \backslash E_n$ so the result of Case 3 is applicable to the point y. Consequently, y is isolated in $\langle \tau \cup \mu \rangle$ and hence there exist $U \in \tau(x, X)$ and $V \in \tau(y, X)$ such that $V \cap \varphi(U) = \{y\}$. Since φ is an involution, we have $\varphi(V) \cap U = \{x\}$ so x is also isolated in $\langle \tau \cup \mu \rangle$. $\square$

3.4. Proposition. *Let X be a self-transversal space with a bijection $\varphi : X \to X$ which witnesses the self-transversality of X. Then, for any subspaces $Y, Z \subset X$, the set $D = \{(y, \varphi(y)) : y \in Y$ and $\varphi(y) \in Z\}$ is a discrete subspace of the product $Y \times Z$. Therefore $|Z \cap \varphi(Y)| \le s(Y \times Z) \le \min\{s(Y) \cdot nw(Z),\ nw(Y) \cdot s(Z)\}$.*

Proof. Given any $t = (y, z) \in D$, we have $z = \varphi(y) \in Z$; since $\varphi(\tau(X))$ is transversal to $\tau(X)$, there is $U \in \tau(y, X)$ and $V \in \tau(z, X)$ such that $\varphi(U) \cap V = \{z\}$. It is straightforward that $W = (U \cap Y) \times (V \cap Z) \in \tau(t, Y \times Z)$ and $W \cap D = \{t\}$. $\square$

3.5. Corollary. *If X is a self-transversal space, then $|Y| \le nw(Y) \cdot s(X)$ for any $Y \subset X$; besides, there is a discrete $D \subset X \times X$ such that $|D| = |X|$ and, in particular, $s(X \times X) = |X|$.*

Proof. Take any bijection $\varphi : X \to X$ which witnesses the self-transversality of X; to prove the first statement apply Proposition 3.4 to the sets Y and $Z = X$; if we let $Y = Z = X$ in Proposition 3.4 then we obtain the second statement. □

3.6. *Remark.* It was proved in [**8**] that, for any self-transversal space X, we have $nw(X) = |X|$. It is worth mentioning that this equality is an easy consequence of Corollary 3.5 because $nw(X) \le |X| = s(X \times X) \le nw(X \times X) = nw(X)$ for any infinite self-transversal space X.

3.7. Theorem. *Every countable Hausdorff space is self-transversal.*

Proof. Take any countable Hausdorff space X. If X is compact then it is self-transversal by Corollary 2.3 of [**8**]. If X is not compact then, being countable, it is not countably compact and hence there is an infinite closed discrete $D \subset X$ so we can apply Proposition 3.2 to conclude our proof. □

3.8. Proposition. *If X is a metrizable self-transversal space, then there is a discrete $D \subset X$ such that $|D| = |X|$. In particular, $w(X) = s(X) = |X|$. However, there exists a metrizable self-transversal space Y such that $|E| < |Y|$ for any closed discrete $E \subset Y$.*

Proof. We have $w(X) = nw(X) = |X|$ by [**8**, Theorem 2.11]; moreover, it is well known that any metrizable space X has a discrete subspace of size $w(X)$.

Now, let A be the set of ordinals less than ω_ω; choose a point $w \notin A$ and consider the space $Y = A \cup \{w\}$ in which all points of A are isolated and $U \in \tau(w, Y)$ if and only if $w \in U$ and $Y \backslash U \subset \omega_n$ for some $n \in \omega$. We omit the trivial verification that Y is metrizable. The space Y has no closed discrete subspace of cardinality ω_ω because every subspace of this cardinality accumulates to w. Finally, X is self-transversal by Proposition 3.1. □

3.9. Theorem. *The following statements are equivalent*:

(FH) *for any cardinal* $\kappa \geq \omega$, *the set* $\{\mu : \mu$ *is a cardinal and* $\kappa \leq \mu \leq \kappa^\omega\}$ *is finite*;

(MH) *a metrizable space* X *is self-transversal if and only if* $w(X) = |X|$.

Proof. Assume that (FH) holds, and take any metrizable space X. If X is self-transversal, then $nw(X) = |X|$ by Theorem 2.11 from [**8**]. Since X is metrizable, we have $w(X) = nw(X)$ which proves the necessity in (MH).

Now assume that X is a metrizable space with $w(X) = |X|$; it is evident that $\mathrm{ext}\,(X) = w(X) = |X|$. If $|X| = \omega$ the result follows from Theorem 3.7. Thus, we can assume that X is uncountable.

If there exists a closed discrete $D \subset X$ with $|D| = |X|$ then Proposition 3.2 is applicable to conclude that X is self-transversal. If such a set D does not exist then $\kappa = |X|$ has countable cofinality and there exists a compact subspace $K \subset X$ such that $w(X \backslash O) < \kappa$ for each $O \in \tau(K, X)$ [**3**]. It is easy to construct a sequence $\{U_n : n \in \omega\}$ of open subsets of X such that $\cap\{U_n : n \in \omega\} = K$ and $\overline{U}_{n+1} \subset U_n$ for all $n \in \omega$. Observe that $\lambda_n = w(X \backslash U_n) < \kappa$ for each $n \in \omega$ and $\sup\{\lambda_n : n \in \omega\} = \kappa$; an immediate consequence of (FH) is that $\lambda_n^\omega < \kappa$ for every $n \in \omega$. Since $|Z| \leq (w(Z))^\omega$ for every metrizable space Z, we have $|X \backslash U_n| \leq (w(X \backslash U_n))^\omega = \lambda_n^\omega < \kappa$. This makes it possible to choose a sequence $\{O_n : n \in \omega\} \subset \tau(K, X)$ and a sequence $\{\kappa_n : n \in \omega\}$ of uncountable cardinals with the following properties:

(i) $\kappa_n < \kappa_{n+1}$ for each $n \in \omega$ and $\sup\{\kappa_n : n \in \omega\} = \kappa$;

(ii) $O_0 = X$, $\cap\{O_n : n \in \omega\} = K$ and $\overline{O}_{n+1} \subset O_n$ for all $n \in \omega$;

(iii) there is a discrete $D_n \subset O_n \backslash \overline{O}_{n+1}$ with $|D_n| = \kappa_n$ for each $n \in \omega$;

(iv) $|K| < \kappa_0$ and $|O_n \backslash O_{n+1}| = |(O_n \backslash O_{n+1}) \backslash D_n| = \kappa_{n+1}$ for all $n \in \omega$.

Observe that (FH) implies $\kappa > \mathfrak{c}$ so, to obtain the property $|K| < \kappa_0$ in (iv) we can assume that $\kappa_0 > \mathfrak{c} \geq |K|$ (recall that K is a metrizable compact space and hence $|K| \leq \mathfrak{c}$).

To finish our proof, observe that letting $X_n = O_n \backslash O_{n+1}$ we obtain the sequences $\{X_n : n \in \omega\}$ and $\{D_n : n \in \omega\}$ which satisfy the conditions (1)–(5) of Lemma 3.3. Therefore, X is self-transversal and we have proved that (FH) implies (MH).

Now assume that (MH) holds and there is an infinite cardinal κ such that the set $P(\kappa) = \{\mu : \mu \text{ is a cardinal and } \kappa \leq \mu \leq \kappa^\omega\}$ is infinite. Let $\kappa_0 = \kappa^+$ and $\kappa_{n+1} = \kappa_n^+$ for all $n \in \omega$. Then $\kappa^\omega > \kappa_n$ for every $n \in \omega$ and hence $\kappa^\omega > \lambda = \lim\{\kappa_n : n \in \omega\}$.

Note first that, for each $n \in \omega$, there exists a metrizable space X_n such that $w(X_n) = \kappa_n$ and $|U| = \lambda$ for every $U \in \tau^*(X_n)$. To construct such a space, let A_n be the discrete space of cardinality κ_n for each $n \in \omega$. Observe that $w((A_n)^\omega) = \kappa_n$ and $|W| = \kappa^\omega$ for any $W \in \tau^*((A_n)^\omega)$. Now take a base $\{B_\alpha : \alpha < \kappa_n\}$ of the space $(A_n)^\omega$ and choose any set $C_\alpha \subset B_\alpha$ with $|C_\alpha| = \lambda$ for each $\alpha < \kappa_n$. Then $X_n = \cup\{C_\alpha : \alpha < \kappa_n\}$ is as promised.

Take any point $w \notin \cup\{X_n : n \in \omega\}$, and let $X = \{w\} \cup (\cup\{X_n : n \in \omega\})$; every set X_n is clopen in X and carries the topology of X_n. The local base at w is given by the family $\{\{w\} \cup (\cup\{X_k : k \geq n\}) : n \in \omega\}$. It is easy to see that X is a metrizable space and $w(X) = |X| = \lambda$.

To obtain a contradiction, assume that X is self-transversal and take a bijection $\varphi : X \to X$ which witnesses this. We claim that

($*$) for any $U \in \tau^*(X)$ the set $N_U = \{n \in \omega : \varphi(U) \cap X_n \neq \varnothing\}$ is infinite.

Making U smaller, if necessary, we can assume that $U \subset X_n$ for some $n \in \omega$. If ($*$) is not true, then there is $k > n$ such that $\varphi(U) \subset Z = \{w\} \cup X_0 \cup \ldots \cup X_k$. Observe that $nw(Z) \leq \kappa_k$ and $nw(U) \leq nw(X_n) \leq \kappa_n < \kappa_k$ so we can apply Proposition 3.4 to conclude that $\lambda = |\varphi(U)| = |\varphi(U) \cap Z| \leq nw(U) + nw(Z) \leq \kappa_k$; this contradiction proves ($*$).

Finally, let $u = \varphi^{-1}(w)$; since $\varphi(\tau(X))$ is transversal to $\tau(X)$, we can find $G \in \tau(u, X)$ and $H \in \tau(w, X)$ such that $\varphi(G) \cap H = \{w\}$. However, H contains a set $P_n = \cup\{X_k : k > n\}$ for some $n \in \omega$ while $\varphi(G) \cap P_n \neq \varnothing$ for all $n \in \omega$ by ($*$). This shows that $\varphi(G) \cap H \neq \{w\}$ which is again a contradiction. $\square$

3.10. Corollary. *The statement "a metrizable space X is self-transversal if and only if $w(X) = |X|$" is independent of* ZFC.

3.11. Corollary. *The following statements are both equivalent to* (FH):

(FU) *if a metrizable space X is a union of $\leq w(X)$-many of its self-transversal subspaces, then X is self-transversal*;

(OP) *if X is a metrizable space and $X\backslash\{x\}$ is self-transversal for some $x \in X$, then X is self-transversal.*

Proof. Assume that (FH) is false. In the (MH) $\Longrightarrow$ (FH) part of the proof of Theorem 3.9, we established that if (FH) is false then there exists a metrizable space X with the following properties:

(i) $X = \{w\} \cup (\cup\{X_n : n \in \omega\})$ where X_n is a clopen subset of X for each $n \in \omega$;

(ii) $X_i \cap X_j = \varnothing$ if $i \neq j$ and $w \notin \cup\{X_n : n \in \omega\}$;

(iii) $\kappa_n = w(X_n)$ is a regular (in fact, a successor) cardinal for all $n \in \omega$;

(iv) X is not self-transversal and $\lambda = |X| = \sup\{\kappa_n : n \in \omega\}$.

It follows from (iii) that there is a closed discrete set $D_n \subset X_n$ with $|D_n| = \kappa_n$ for every $n \in \omega$. It is evident that $Y = X\backslash\{w\}$ is homeomorphic to $\oplus\{X_n : n \in \omega\}$ so the set $D = \cup\{D_n : n \in \omega\}$ is closed and discrete in the space Y. Since $|D| = \sup\{\kappa_n : n \in \omega\} = \lambda = |Y|$, the space Y is self-transversal by Proposition 3.2. Therefore the space X and the point $w \in X$ show that (OP) fails, proving that (OP) $\Longrightarrow$ (FH).

Now assume that (FH) is true and we have an (infinite) metrizable space X with $w(X) = \kappa$ such that $X = \cup\{Y_\alpha : \alpha < \kappa\}$ where Y_α is self-transversal and hence $|Y_\alpha| = w(Y_\alpha) \leq \kappa$ for each $\alpha < \kappa$. But then

$$|X| \leq \sum\{|Y_\alpha| : \alpha < \kappa\} = \sum\{w(Y_\alpha) : \alpha < \kappa\} \leq \kappa \cdot \kappa = \kappa = w(X),$$

whence $w(X) = |X|$ which, together with (FH) implies that X is self-transversal by Theorem 3.9. This proves that (FH) $\Longrightarrow$ (FU); since the implication (FU) $\Longrightarrow$ (OP) is clear, we have (FU) $\Longleftrightarrow$ (OP) $\Longleftrightarrow$ (FH). $\square$

Proposition 3.2 makes it natural to ask whether a space X is self-transversal if it has a discrete (not necessarily closed) subspace $D \subset X$ with $|D| = |X|$. The following ZFC example shows that this is not true.

3.12. Example. There exists a space X with the following properties:

(a) $|X| = \mathfrak{c}$ and X is a dense subspace of $\mathbf{R}^{\mathfrak{c}}$;

(b) X is not self-transversal;

(c) X contains a subspace homeomorphic to the one-point compactification of a discrete space of cardinality $\mathfrak{c}$.

In particular, X has a discrete subspace of cardinality $\mathfrak{c} = |X|$. Thus X is a ZFC example showing that it is essential that the discrete set in Proposition 3.2 be also closed. In addition, if a is the unique non-isolated point of this one-point compactification, then the subspace $T = X\backslash\{a\}$ is self-transversal by Proposition 3.2. Thus, adding one point to the space T destroys its self-transversality.

Proof. The space $Y = C_p(\mathbf{R})$ of all continuous real-valued functions on $\mathbf{R}$ is dense in $\mathbf{R}^{\mathbf{R}}$ [**1**, Proposition 0.3.6] and the latter is homeomorphic to $\mathbf{R}^{\mathfrak{c}}$. Take any point $a \in \mathbf{R}^{\mathbf{R}}\backslash Y$; it is easy to construct a subspace $A \subset \mathbf{R}^{\mathbf{R}}$ homeomorphic to the one-point compactification of a discrete space of cardinality $\mathfrak{c}$ for which a is the unique non-isolated point of A. Since $s(Y) \le nw(Y) = \omega$ [**1**, Theorem 1.1.3], the set $A \cap Y$ is at most countable so $B = A\backslash Y$ is the one-point compactification of a discrete space of cardinality $\mathfrak{c}$ with $B \subset \mathbf{R}^{\mathbf{R}}\backslash Y$. We claim that the space $X = Y \cup B$ has all required properties.

Since (a) and (c) are clear, we only have to check that X is not self-transversal. To obtain a contradiction, assume that $\varphi : X \to X$ is a bijection witnessing the self-transversality of X. Then

$(**)$ $\varphi(U) \cap Y$ is countable for any $U \in \tau^*(Y)$,

because $|\varphi(U) \cap Y| \le nw(U) + nw(Y) = \omega$ by Proposition 3.4. Now, if $b = \varphi^{-1}(a)$, then there exists $V \in \tau(b, X)$ and $W \in \tau(a, X)$ such that $\varphi(V) \cap W = \{a\}$. Observe that $U = V \cap Y$ is a non-empty open subset of Y and hence U is uncountable. By $(**)$, the set $\varphi(U) \cap A$ is also uncountable; since W contains all points of A except for a finite set, we have $\varphi(V) \cap W \supset \varphi(U) \cap W \supset \varphi(U) \cap W \cap A$; the last set being uncountable, the set $\varphi(V) \cap W$ is also uncountable which is a contradiction. Therefore, X is not self-transversal. $\square$

The following theorem gives a negative answer in a strong form to Problems 2 and 3 from [**8**].

3.13. Theorem. *Let λ be a strong limit cardinal of countable cofinality. Then, for any infinite cardinal $\kappa < \lambda$, there exists a self-transversal compact space X with the Souslin property for which $|X| = \lambda$ and $\pi\chi(x, X) \geq \kappa$ for each $x \in X$.*

Proof. Let us first choose a sequence $\{\lambda_n : n \in \omega\}$ of infinite cardinals such that $\sup\{\lambda_n : n \in \omega\} = \lambda$ while $2^\kappa < \lambda_0$ and $2^{\lambda_n} < \lambda_{n+1}$ for each $n \in \omega$. Let $Z_n = \mathbf{D}^{\lambda_n}$ for each $n \in \omega$ and consider the space $Z = \oplus_{n\in\omega} Z_n$. Clearly, Z is a locally compact, σ-compact space.

Denote by $A(Z)$ the one-point compactification of the space Z. The space $X = A(Z) \times \mathbf{D}^\kappa$ is a countable union of Cantor cubes and hence $c(X) = \omega$. It is also obvious that $|X| = \lambda$. Furthermore, X is a compact space with $\pi\chi(x, X) \geq \kappa$ for every $x \in X$ because the projection onto the second factor maps X openly onto $\mathbf{D}^\kappa$. Now if $X_n = Z_n \times \mathbf{D}^\kappa$ then X_n is a clopen subspace of X and we can choose a discrete subspace $D_n \subset X_n \simeq \mathbf{D}^{\lambda_n}$ in such a way that $2^\kappa \leq |D_0|$ and $|D_{n+1}| = 2^{\lambda_n}$ for each $n \in \omega$. It is immediate that the families $\{X_n : n \in \omega\}$ and $\{D_n : n \in \omega\}$ satisfy all the conditions of Lemma 3.3 and hence X is self-transversal. $\square$

3.14. *Remark.* If we don't want to guarantee a large π-character at all points, then we can take instead of X the one-point compactification $A(Z)$ of the space Z from the proof of Theorem 3.13. The resulting space also settles Problems 2 and 3 from [**8**]; its only "defect" is to have one point of countable character.

3.15. *Remark.* The compact space X (or $A(Z)$) from Theorem 3.13 is self-transversal and hence $|X| = w(X)$ while $c(X) = \omega$. On the other hand, if we take any cardinal μ with $\mu^\omega = \mu$ then there is no compact space K satisfying both $|K| = w(K) = \mu^+$ and $c(K) = \omega$. Indeed, if

K has a dense set of points of π-character $\leq \mu$, then by a well-known result of Shapirovskii we have $w(K) \leq \mu^{c(K)} = \mu$. If not, then there is a closed $F \subset K$ such that $\chi(z, F) \geq \mu^+$ for all $z \in F$ and therefore, by the Čech-Pospišil theorem, we have $|K| \geq |F| \geq 2^{\mu^+} > \mu^+$. It would be interesting to clarify what happens for the cardinals not covered by Theorem 3.13 and this observation. For example, it is not clear whether there exists, under GCH, a compact space K with $c(K) = \omega$ and $|K| = w(K) = \omega_{\omega+1}$.

The following result gives a consistent negative answer to Problem 1 of [**8**].

3.16. Theorem. *Under* CH*, there exists a scattered compact space which is not self-transversal.*

Proof. Under CH, Kunen constructed a compact scattered space X such that $|X| = \omega_1$, and the space X^n is hereditarily separable for all $n \in \mathbf{N}$ (this example seems to have never been published but another one with stronger properties was given in [**13**, Theorem 2.5]). An immediate consequence is $s(X \times X) = \omega$ so X cannot be self-transversal by Corollary 3.5. $\square$

We do not know if the conclusion of this result could be obtained in ZFC. However, this can be done if we do not require that the space be compact.

3.17. Theorem. *There is a zero-dimensional scattered space X which is not self-transversal.*

Proof. Juhász proved that there exists in ZFC a scattered subspace X of $\mathbf{D}^{\omega_2}$ such that $|X| = \omega_2$ while $s(X \times X) = \omega_1$ [**5**]. But then Corollary 3.5 can be applied again to conclude that X is not self-transversal. $\square$

However, there is a large class of scattered spaces whose elements are all self-transversal.

3.18. Proposition. *Any scattered monotonically normal space is self-transversal.*

Proof. Take any scattered monotonically normal space X. The set D of isolated points of X is dense in X. If $|D| = \kappa$ then $d(X) = \kappa$ and hence $hl(X) = c(X) \leq d(X) \leq \kappa$ [**6**]. The space X being scattered, we have $hl(X) = |X|$, and therefore $|X| \leq \kappa$. Now apply Proposition 3.1 to conclude that X is self-transversal. $\square$

3.19. Corollary. *Any scattered metrizable space as well as any scattered generalized ordered space is self-transversal.*

In the sequel we prove some simple facts about general categorical properties of self-transversal spaces.

3.20. Proposition. *If (X, τ) is a self-transversal space and μ is a topology on X with $\tau \subset \mu$, then (X, μ) is also self-transversal.*

Proof. Let $\varphi : X \to X$ be a bijection which witnesses the self-transversality of (X, τ). The topology $\langle \varphi(\mu) \cup \mu \rangle$ is discrete because it is stronger than the discrete topology $\langle \varphi(\tau) \cup \tau \rangle$. $\square$

3.21. Example. A dense subspace of a self-transversal space is not necessarily self-transversal.

Proof. If X is the Niemytzky plane (also known as the bubble space), then X has a closed discrete subspace $D \subset X$ such that $|D| = \mathfrak{c} = |X|$; observe that the space $Y = X \backslash D$ is homeomorphic to the upper half-plane of $\mathbf{R} \times \mathbf{R}$. By Proposition 3.2 the space X is self-transversal while Y is dense in X and $nw(Y) = w(Y) = \omega < |Y|$, so Y is not self-transversal by Theorem 2.11 of [**8**]. $\square$

3.22. Theorem. *Any finite product of self-transversal spaces is self-transversal.*

Proof. It suffices to show that our statement holds for the product of two self-transversal spaces, say, X and Y. Let $\varphi_X : X \to X$ and $\varphi_Y : Y \to Y$ be the respective bijections; denote by τ the topology of $X \times Y$. Define a map $\varphi : X \times Y \to X \times Y$ by $\varphi(x, y) = (\varphi_X(x), \varphi_Y(y))$ for any $(x, y) \in X \times Y$. It is immediate that φ is a bijection. Given any $z = (x, y) \in X \times Y$, let $x' = \varphi_X^{-1}(x)$ and $y' = \varphi_X^{-1}(y)$. Then $z = \varphi(z')$ where $z' = (x', y')$. The pairs of topologies $(\varphi_X(\tau(X)), \tau(X))$ and $(\varphi_Y(\tau(Y)), \tau(Y))$ being transversal, we can choose $U' \in \tau(x', X)$, $U \in \tau(x, X)$ and $V' \in \tau(y', Y)$, $V \in \tau(y, Y)$ such that $\varphi_X(U') \cap U = \{x\}$ and $\varphi_Y(V') \cap V = \{y\}$. If $W' = U' \times V'$ and $W = U \times V$ then $W \in \tau(z, X \times Y)$, $W' \in \tau(z', X \times Y)$ and $\varphi(W') \cap W = \{z\}$ so z is isolated in $\langle \tau \cup \varphi(\tau) \rangle$. $\square$

3.23. Examples. The space $\mathbf{D}^\omega$ shows that a countable product of discrete, and hence self-transversal, spaces can fail to be self-transversal. It is less trivial to see that there exists a non-self-transversal space X such that $X \times X$ is self-transversal.

Proof. Let $X = \mathbf{R} \oplus S$ where S is the Sorgenfrey line. The space $X \times X$ has a closed discrete subset of cardinality $\mathfrak{c}$ because such a subset exists in $S \times S$. Therefore $X \times X$ is self-transversal by Proposition 3.2.

To see that X is not self-transversal, assume the contrary and fix a bijection $\varphi : X \to X$ witnessing this. Let $\tau = \tau(X)$ and $\mu = \varphi(\tau)$; furthermore, take any countable base $\mathcal{B} = \{B_n : n \in \omega\}$ of the space $\mathbf{R}$. For each $n \in \omega$ consider the set $D_n = \{x \in B_n : W \cap B_n = \{x\}$ for some $W \in \mu\}$. It is obvious that the D_n is discrete in (X, μ) for each $n \in \omega$. It follows from the transversality of τ and μ that $\mathbf{R} = \cup\{D_n : n \in \omega\}$ and therefore $|D_n| > \omega$ for some $n \in \omega$. However the spread of (X, μ) is countable because it is homeomorphic to (X, τ); this contradiction shows that X is not self-transversal. $\square$

4. Open problems. The following list features the most interesting questions we could not answer while working on this paper.

4.1. Problem. *Is there in* ZFC *an example of a compact scattered space which is not self-transversal?*

4.2. Problem. *Take a dense left-separated subspace X of a Souslin line. Must X be self-transversal?*

4.3. Problem. *Assume that X is a compact space for which there exists a discrete $D \subset X$ such that $|D| = |X|$. Must X be self-transversal?*

4.4. Problem. *Suppose that X is a space which is a union of two of its closed self-transversal subspaces. Must X be self-transversal? What happens if X is compact?*

4.5. Problem. *Let X be a Hausdorff space with a σ-locally finite base. Is it true* (*under* GCH *or in* ZFC) *that $w(X) = |X|$ implies that X is self-transversal?*

4.6. Problem. *How can hereditarily self-transversal spaces be characterized? For example, must a compact Hausdorff hereditarily self-transversal space be scattered?*

4.7. Problem. *Given a space Z, let $N(Z)$ be the set of its non-isolated points. Assume that X is a compact scattered space such that $N(N(N(X)))$ is a finite set* (*such scattered spaces are said to have height* 4). *Must X be self-transversal?*

4.8. Problem. *Must every compact σ-discrete space be self-transversal? How about an arbitrary strongly σ-discrete space* (*i.e., a space which is a countable union of closed discrete subspaces*)*?*

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, P.O. Box 127, H-1364, Budapest, Hungary
E-mail address: juhasz@renyi.hu

Dpto. de Matemáticas, Universidad Autónoma Metropolitana, San Rafael Atlixco, 186, Col. Vicentina, Iztapalapa, C.P. 09340, México D.F.
E-mail address: mich@xanum.uam.mx

Dpto. de Matemáticas, Universidad Autónoma Metropolitana, San Rafael Atlixco, 186, Col. Vicentina, Iztapalapa, C.P. 09340, México D.F.
E-mail address: vova@xanum.uam.mx

Dpto. de Matemáticas, Universidad Autónoma Metropolitana, San Rafael Atlixco, 186, Col. Vicentina, Iztapalapa, C.P. 09340, México D.F.
E-mail address: rgw@xanum.uam.mx