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GENERALIZED CONDITIONAL YEH-WIENER INTEGRAL

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ABSTRACT. In this paper, we introduce the generalized conditional Yeh-Wiener integral which includes the conditional Yeh-Wiener integral and the modified conditional Yeh-Wiener integral. We also show that some of the results in the conditional Yeh-Wiener integral and the modified conditional Yeh-Wiener integral can be obtained as corollaries of our result. We also treat the generalized conditional Yeh-Wiener integral for the functional containing a generalized quasi-polyhedric function.

1. Introduction. Kitagawa [**5**] introduced the Wiener space of functions of two variables which is the collection of the continuous functions $x(s,t)$ on the unit square $[0,1] \times [0,1]$ satisfying $x(s,t) = 0$ for $st = 0$, and he treated the integration on this space. Yeh [**7**] treated the integration of this space for more general functions and made a firm logical foundation of this space. We call this space a Yeh-Wiener space and the integral a Yeh-Wiener integral.

In [**8, 9**], Yeh introduced the conditional expectation and the conditional Wiener integral. He also evaluated conditional Wiener integrals for a real-valued conditioning function using the inversion formulae. Chang and the first author [**4**] treated the conditional Wiener integral for vector-valued conditioning function. Park and Skoug [**6**] introduced a simple formula for the conditional Yeh-Wiener integral which is very useful in evaluating the conditional Yeh-Wiener integrals.

Recently the first author [**1**] introduced the modified conditional Yeh-Wiener integral and evaluated it for various functionals. In [**6**], Park and Skoug treated the conditional Yeh-Wiener integral for the functional on a set of continuous functions which are defined only on a rectangular region Ω. But in [**1**], the first author considered the set

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of continuous functions on various regions Ω, for example, triangular, parabolic and circular regions. In this paper we consider even more general region Ω than were considered in [**1**].

The purpose of this paper is to introduce the generalized conditional Yeh-Wiener integral which includes the conditional Yeh-Wiener integral in [**6**] and the modified conditional Yeh-Wiener integral in [**1**]. To do so, we consider the space of continuous functions on the region $\Omega = \{(s,t) \mid a \leq s \leq b,\ 0 \leq t \leq g(s)\}$ where g is a monotone decreasing and continuous function which is sectionally decreasing or constant on $[a,\ b]$ with $g(b) \geq 0$. To make a partition of the region Ω, we use a similar, but slightly different notation than the one used in [**4**] and we divide the partitions of the region Ω into the two different types depending on whether $g(s)$ is constant immediately to the right of $s = a$ or $g(s)$ is strictly decreasing just to the right of $s = a$. We call the new resulting space and the resulting new integral the generalized Yeh-Wiener space and the generalized Yeh-Wiener integral, respectively.

We also obtain a simple formula for the generalized conditional Yeh-Wiener integral using the generalized quasi-polyhedric function. Using this formula we show that some of the results in [**1, 6, 9**] can be obtained as corollaries of our result. Finally we treat the generalized conditional Yeh-Wiener integral for the functional F given by $F(x) = \int_\Omega ([x](s,t))^k \, ds \, dt$ where k is a nonnegative integer and $[x]$ is the generalized quasi-polyhedric function on Ω.

2. Generalized conditional Yeh-Wiener integral. Let g be a monotone decreasing and continuous function which is sectionally constant or decreasing on $[a,b]$ with $g(b) \geq 0$. Let $a = \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k+1} = b$ be chosen in such a way that on each interval $[\tau_{i-1}, \tau_i]$, g is either constant or (strictly) decreasing and g is not constant or decreasing on two consecutive intervals. Thus, if $k = 0$, then g is either a constant function or a decreasing continuous function on $[a,b]$.

To make a partition $\{s_0, s_1, \dots, s_d\}$ of $[a,b]$, we use the notation

$$
\begin{aligned}
a = s_0 < s_1 < \cdots < s_{l_1} = \tau_1 < \cdots < s_{l_1+l_2} = \tau_2 \\
< \cdots < s_{l_1+\cdots+l_k} = \tau_k < \cdots < s_d = b
\end{aligned}
\tag{2.1}
$$

where $d = l_1 + \cdots + l_{k+1}$ and $l_i \geq 1$ for $i = 1, \dots, k+1$. The notation (2.1) is similar but slightly different than the notation used in [**4**]. For notational convenience, we let $l_i = 0$ for $i \leq 0$, and we let $A_i = (\tau_{i-1}, \tau_i)$ for $i = 1, \dots, k+1$.

We first consider the case g is constant on A_1. Let $g(a) = T$, and let

$$\hat{k} = \begin{cases} k+1 & k \ : \text{odd} \\ k & k \ : \text{even}. \end{cases} \tag{2.2}$$

For $n > l_2 + l_4 + \cdots + l_{\hat{k}}$ when $g(b) > 0$ and $n = l_2 + l_4 + \cdots + l_{\hat{k}}$ when $g(b) = 0$, construct a partition $\{t_0, t_1, \dots, t_n\}$ of $[0, T]$ satisfying $0 = t_0 < t_1 < \cdots < t_n = T$ and the following properties:

(2.3)

i. $g(s) = t_{n-l_2-l_4-\cdots-l_{2i-2}}$ on A_{2i-1}, $i = 1, 2, \dots, \left\langle \frac{k+2}{2} \right\rangle$;

ii. for $0 \leq p \leq l_{2i}$, $g(s_{l_1+l_2+\cdots+l_{2i-1}+p}) = t_{n-l_2-\cdots-l_{2i-2}-p}$ on A_{2i}, $i = 1, 2, \dots, \hat{k}/2$,

where, for real number y, $\langle y \rangle$ denotes the greatest integer less than or equal to y.

Let $\Omega = \{(s,t) \mid a \leq s \leq b,\ 0 \leq t \leq g(s)\}$, and let $C(\Omega)$ be the space of continuous functions x on Ω satisfying $x(s,0) = x(a,t) = 0$ for all (s,t) in Ω. In [**2, 5**–**7**], the various authors worked with the rectangle $\Omega = [a,b] \times [0,T]$, i.e., $g(s) = T$ on $A_1 = [a,b]$, which is a special case of (2.3) for $k = 0$.

Let $L_p = l_1 + \cdots + l_p$, and let Λ be the partition of Ω given by

$$\Lambda = \{(s_i, t_j) \mid t_1 \leq t_j \leq g(s_i),\ 1 \leq i \leq L_{k+1}\} \tag{2.4}$$

where $g(s_i)$ is given by (2.3). Let N be the number of elements in Λ. If we let $M_p = n - l_2 - \cdots - l_{2p}$, then we have

$$N = dr + \sum_{i=1}^{\hat{k}/2} \left[d - l_{k+1} - l_k - \cdots - l_{2i} + \frac{1}{2}(l_{2i} - 1) \right] l_{2i}, \tag{2.5}$$

where $d = L_{k+1}$ and $r = M_{\hat{k}/2}$.

Let X_Λ be a random vector from $C(\Omega)$ to R^N, and let $I = X_\Lambda^{-1}(B)$, $B \in \mathcal{B}^N$, the Borel σ-algebra of N-dimensional Euclidean space. Define the set function $\widetilde{m}$ of a set I by

$$\widetilde{m}(I) = \int_B W(\Lambda, \vec{u}) d\vec{u} \tag{2.6}$$

where

$$(2.7)\quad W(\Lambda,\vec{u}) = \Big\{(2\pi)^N\Big[\prod_{j=1}^{r}(\Delta_j t)^d\Big]\Big[\prod_{i=0}^{\langle k/2\rangle}\big[(s_{L_{2i}+1}-\tau_{2i})\cdots(\tau_{2i+1}-s_{L_{2i+1}-1})\big]^{M_i}\Big] \Big[\prod_{i=0}^{\hat{k}/2-1}\prod_{j=1}^{l_{2i+2}}(\Delta_{L_{2i+1}+j}s)^{M_i-j}(\Delta_{M_i-j+1}t)^{L_{2i+1}+j-1}\Big]\Big\}^{-1/2}$$

$$\cdot \exp\Big\{-\sum_{i=1}^{d}\sum_{j=1}^{r}\frac{(\Delta_{i,j}\vec{u})^2}{2\Delta_i s\Delta_j t}-\sum_{p=1}^{\hat{k}/2}\sum_{i=1}^{L_{2p-1}}\sum_{j=M_p+1}^{M_{p-1}}\frac{(\Delta_{i,j}\vec{u})^2}{2\Delta_i s\Delta_j t} -\sum_{i=1}^{\hat{k}/2}\sum_{p=1}^{l_{2i}-1}\sum_{j=M_i+1}^{M_{i-1}-p}\frac{(\Delta_{L_{2i-1}+p,j}\vec{u})^2}{2\Delta_{L_{2i-1}+p}s\Delta_j t}\Big\}$$

with $\Delta_i s = s_i - s_{i-1}$, $\Delta_j t = t_j - t_{j-1}$, $\Delta_{i,j}\vec{u} = u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}$ and $u_{0,j} = u_{i,0} = 0$ for all i and j.

Let $\mathcal{I}$ be the collection of subsets of type I. Then it can be shown that $\mathcal{I}$ is a semi-algebra of subsets of $C(\Omega)$ and the set function $\widetilde{m}$ is a measure defined on $\mathcal{I}$ and the factor $W(\Lambda,\vec{u})$ is chosen to make $\widetilde{m}(C(\Omega)) = 1$. The measure $\widetilde{m}$ can be extended to a measure on the Caratheodory extension of interval class $\mathcal{I}$ in the usual way. With this Caratheodory extension, measurable functionals on $C(\Omega)$ may be defined and their integration on $C(\Omega)$ can be considered.

The case when g is decreasing on A_1 can be dealt with in a similar manner (with obvious adjustments in subscripts) as the case where g is constant on A_1 handled above. Thus we may conclude the following.

Let Ω be a region given by $\Omega = \{\ (s,t) \mid 0 \le t \le g(s),\ a \le s \le b\}$ for a monotone decreasing and continuous function g which is sectionally constant or decreasing on $[a,b]$ with $g(b) \ge 0$. Let N be the number of elements of $\Lambda = \{(s_i,t_j) \mid 0 < t_j \le g(s_i),\ 0 \le i \le d\}$ and $\tilde{m}$ the measure satisfying $\widetilde{m}(C(\Omega)) = 1$. Here we call the space $C(\Omega)$ with the measure $\widetilde{m}$ a generalized Yeh-Wiener space which can be obtained by the similar method as in [**5**]. And we call $E(F) = \int_{C(\Omega)} F(x)\, d\widetilde{m}(x)$ a generalized Yeh-Wiener integral of F on $C(\Omega)$ if it exists and the process $\{x(s,t),\ (s,t) \in \Omega\}$ a generalized Yeh-Wiener process. We can easily obtain mean $E(x(s,t)) = 0$ and covariance $E(x(s,t)x(u,v)) = \min\{s,u\}\min\{t,v\}$ for x in $C(\Omega)$, and we can also state the existence of a generalized Yeh-Wiener process.

Let P_{X_Λ} be the probability distribution induced by the random vector X_Λ, that is, $P_{X_\Lambda}(B) = \widetilde{m}(X_\Lambda^{-1}(B))$ for B in $\mathcal{B}^N$. Then, by the definition of conditional expectation [**8**], for each function F in $L_1(C(\Omega))$,

$$\int_{X_\Lambda^{-1}(B)} F(x)\, d\widetilde{m}(x) = \int_B E(F(x) \mid X_\Lambda(x) = \vec{u})\, dP_{X_\Lambda}(\vec{u}) \tag{2.8}$$

for B in $\mathcal{B}^N$ and $E(F(x) \mid X_\Lambda(x) = \vec{u})$ is a Borel measurable function of $\vec{u}$ which is unique up to Borel null sets in R^N. Here we call $E(F \mid X_\Lambda)(\vec{u}) \equiv E(F(x) \mid X_\Lambda(x) = \vec{u})$ a generalized conditional Yeh-Wiener integral of F given X_Λ.

For each partition Λ of Ω and x in $C(\Omega)$, we define the generalized quasi-polyhedric function $[x]$ of x on Ω by

$$\begin{aligned} [x](s,t) = x(s_{i-1}, t_{j-1}) & \\ &+ \frac{s - s_{i-1}}{\Delta_i s}\,(x(s_i, t_{j-1}) - x(s_{i-1}, t_{j-1})) \\ &+ \frac{t - t_{j-1}}{\Delta_j t}\,(x(s_{i-1}, t_j) - x(s_{i-1}, t_{j-1})) \\ &+ \frac{(s - s_{i-1})(t - t_{j-1})}{\Delta_i s \Delta_j t}\,\Delta_{ij} x(s,t) \end{aligned} \tag{2.9}$$

on each $\Omega_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j]$, $t_1 \le t_j \le g(s_i)$, $1 \le i \le d$, and

$$\begin{aligned} [x](s,t) = x(s_{i-1}, g(s_i)) & \\ &+ \frac{s - s_{i-1}}{\Delta_i s}\,\big(x(s_i, g(s_i)) - x(s_{i-1}, g(s_i))\big) \\ &+ \frac{t - g(s_i)}{\Delta_i g}\,\big(x(s_{i-1}, g(s_{i-1})) - x(s_{i-1}, g(s_i))\big) \end{aligned} \tag{2.10}$$

on $\Omega_i = \{(s,t) \mid s_{i-1} < s \le s_i,\ g(s_i) < t \le g(s)\}$, where $\Delta_i s = s_i - s_{i-1}$, $\Delta_j t = t_j - t_{j-1}$, $\Delta_i g = g(s_{i-1}) - g(s_i)$, and $\Delta_{ij} x(s,t) = x(s_i, t_j) - x(s_{i-1}, t_j) - x(s_i, t_{j-1}) + x(s_{i-1}, t_{j-1})$, and $[x](s,t) = 0$ if $(s-a)t = 0$. Here the function $[x]$ in (2.10) is defined on the set Ω_i

with $\Delta_i g \neq 0$ and the generalized quasi-polyhedric function $[x]$ defined by the function g is different from the quasi-polyhedric function in [**6**] and modified quasi-polyhedric function in [**1**].

Similarly, for $\vec{u}$ in R^N, we define the generalized quasi-polyhedric function $[\vec{u}]$ of $\vec{u}$ on Ω by

$$
\begin{aligned}
[\vec{u}](s,t) = u_{i-1,j-1} &+ \frac{s-s_{i-1}}{\Delta_i s}\,(u_{i,j-1} - u_{i-1,j-1}) \\
&+ \frac{t-t_{j-1}}{\Delta_j t}\,(u_{i-1,j} - u_{i-1,j-1}) \\
&+ \frac{(s-s_{i-1})(t-t_{j-1})}{\Delta_i s \Delta_j t}\,\Delta_{ij}\vec{u},
\end{aligned} \tag{2.11}
$$

on each Ω_{ij}, and

$$
\begin{aligned}
[\vec{u}](s,t) = u_{i-1,\bar{i}} &+ \frac{s-s_{i-1}}{\Delta_i s}\,(u_{i,\bar{i}} - u_{i-1,\bar{i}}) \\
&+ \frac{t-g(s_i)}{\Delta_i g}\,(u_{i-1,\overline{i-1}} - u_{i-1,\bar{i}})
\end{aligned} \tag{2.12}
$$

on each Ω_i, where $t_{\bar{i}} = g(s_i)$, $u_{0,j} = u_{i,0} = 0$ for all i, j, and $[\vec{u}](s,t) = 0$ for $(s-a)t = 0$. Here the function $[\vec{u}]$ in (2.12) is defined on the set The following theorem plays a key role in this paper.

Theorem 2.1. *If* $\{x(s,t) \mid (s,t) \in \Omega\}$ *is the generalized Yeh-Wiener process, then the two processes* $\{x(s,t) - [x](s,t) \mid (s,t) \in \Omega\}$ *and* $X_\Lambda(x)$ *are stochastically independent.*

Proof. Let (s_p, t_q) be in Λ. By (2.10), we have

$$
\begin{aligned}
x(s,t) - [x](s,t) = x(s,t) &- x(s_{i-1}, g(s_i)) \\
&- \frac{s-s_{i-1}}{\Delta_i s}\big(x(s_i, g(s_i)) - x(s_{i-1}, g(s_i))\big) \\
&- \frac{t-g(s_i)}{\Delta_i g}\big(x(s_{i-1}, g(s_{i-1})) - x(s_{i-1}, g(s_i))\big)
\end{aligned} \tag{2.13}
$$

for (s,t) in $\Omega_i = \{(s,t) \mid s_{i-1} < s \le s_i,\ g(s_i) < t < g(s)\}$. For each Ω_i and (s_p, t_q) in Λ, we have three cases:

$$\begin{aligned} &\text{(i)} && s_p \le s_{i-1},\ t_q \le g(s_i) \\ &\text{(ii)} && s_p \ge s_i,\quad\ \ t_q \le g(s_i) \\ &\text{(iii)} && s_p \le s_{i-1},\ t_q \ge g(s_{i-1}). \end{aligned} \tag{2.14}$$

For each case in (2.14), we can easily obtain $E[x(s_p,t_q)(x(s,t) - [x](s,t))] = 0$ using (2.13) and $E(x(s,t)x(u,v)) = (s \wedge u)(t \wedge v)$. For (s,t) in Ω_{ij}, we already know that $E[x(s_p,t_q)(x(s,t) - [x](s,t))] = 0$ [**8**]. Since both $x(s_p,t_q)$ and $\{x(s,t) - [x](s,t) \mid (s,t) \in \Omega\}$ are Gaussian and uncorrelated, we may conclude that they are stochastically independent. □

Using Theorem 2.1 and the similar technique in the proof of Theorem 2 in [**6**], we have the following theorem.

Theorem 2.2. *Let F be in $L_1(C(\Omega), \widetilde{m})$. Then we have*

$$\int_{X_\Lambda^{-1}(B)} F(x)\, d\widetilde{m}(x) = \int_B E(F(x - [x] + [\vec{u}]))\, dP_{X_\tau}(\vec{u}) \tag{2.15}$$

for $\mathcal{B}$ in $\mathcal{B}^N$, and

$$E(F \mid X_\Lambda)(\vec{u}) = \hat{E}\big[F(x - [x] + [\vec{u}])\big], \tag{2.16}$$

where the righthand side of (2.16) *is any Borel measurable function of $\vec{u}$ which is equal to $E(F(x - [x] + [\vec{u}]))$ for almost every $\vec{u}$ in R^N. In particular, if F is Borel measurable, then*

$$E(F \mid X_\Lambda)(\vec{u}) = E\big[F(x - [x] + [\vec{u}])\big]. \tag{2.17}$$

The equalities in (2.16) *and* (2.17) *mean that both sides are Borel measurable functions of $\vec{u}$ and they are equal except for Borel null sets.*

Equation (2.17) in Theorem 2.2 is a simple formula for the generalized conditional Yeh-Wiener integral which is very convenient to apply in application.

3. Evaluation of the generalized conditional Yeh-Wiener integral for various regions. For c in $[a,b]$ and $0 \le S \le T$, let $t = g(s)$ be a function on $[a,b]$ defined by $g(s) = T$ on $[a,c]$ and $g(s) = \eta s + \delta$ on $[c,b]$ where $\eta = (S-T)/(b-c)$ and $\delta = (Tb - Sc)/(b-c)$. Let

$$\Omega = \{(s,t) \mid a \le s \le b,\ 0 \le t \le g(s)\} \tag{3.1}$$

and Λ be a partition of Ω given by

$$\Lambda = \{(s_i, t_j) \mid t_1 \le t_j \le g(s_i),\ 1 \le i \le d\} \tag{3.2}$$

which satisfies the properties:

(3.3)

i. $\{s_0, s_1, \dots, s_d\}$ is a partition of $[a,b]$ satisfying $a = s_0 < s_1 < \cdots < s_{l_1} = c < s_{l_1+1} < \cdots < s_d = b$, and $d = l_1 + l_2$;

ii. $\{t_0, t_1, \dots, t_n\}$ is a partition of $[0,T]$ satisfying $0 = t_0 < t_1 < \cdots < t_n = T,\ g(s) = T$ on A_1, and $g(s_{l_1+p}) = t_{n-p}$ for $0 \le p \le l_2$.

Let N be the number of elements of Λ. Then we have $N = dn - (l_2(l_2+1))/2$. Let X_Λ be a random vector on $C(\Omega)$ given by $X_\Lambda(x) = (x(s_1, t_1), \dots, x(s_1, t_n), x(s_2, t_1), \dots, x(s_d, t_l), \dots, x(s_d, t_{n-l_2}))$ in R^N.

Theorem 3.1. *Let F be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x(s,t)\, ds\, dt$. Then the generalized conditional Yeh-Wiener integral $E(F \mid X_\Lambda)(\vec{u})$ given conditioning function X_Λ at $\vec{u}$ in R^N is*

$$\begin{aligned} E(F \mid X_\Lambda)(\vec{u}) &= \frac{1}{4} \sum_{i=1}^{d} \sum_{j=1}^{n-l_2} (u_{i-1,j-1} + u_{i-1,j} + u_{i,j-1} + u_{i,j}) \Delta_i s \Delta_j t \\ &\quad + \frac{1}{4} \sum_{j=n-l_2+1}^{n} \sum_{i=1}^{n+l_1-j} (u_{i-1,j-1} + u_{i-1,j} + u_{i,j-1} + u_{i,j}) \Delta_i s \Delta_j t \\ &\quad + \frac{1}{6} \sum_{i=l_1+1}^{d} (\alpha_i + \beta_i + \gamma_i) \Delta_i s \Delta_{n+l_1-i+1} t \end{aligned} \tag{3.4}$$

at $\vec{u}$ in R^N, where $\alpha_i = u_{i-1,n+l_1-i}$, $\beta_i = u_{i,n+l_1-i}$ and $\gamma_i = u_{i-1,n+l_1-i+1}$.

Proof. Using Theorem 2.2 and the Fubini theorem, we have

$$
\begin{aligned}
E(F \mid X_\Lambda)(\vec{u}) &= \int_\Omega E(x(s,t) - [x](s,t) + [\vec{u}](s,t))\, ds\, dt \\
&= \int_\Omega [\vec{u}](s,t)\, ds\, dt \\
&= \sum_{i=1}^{d} \sum_{j=1}^{n-l_2} \int_{\Omega_{ij}} [\vec{u}](s,t)\, ds\, dt \\
&\quad + \sum_{j=n-l_2+1}^{n} \sum_{i=1}^{n+l_1-j} \int_{\Omega_{ij}} [\vec{u}](s,t)\, ds\, dt \\
&\quad + \sum_{i=l_1+1}^{d} \int_{\Omega_i} [\vec{u}](s,t)\, ds\, dt
\end{aligned}
\tag{3.5}
$$

where $\Omega_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j]$ and $\Omega_i = \{(s,t) \mid s_{i-1} < s \le s_i, g(s_i) < t \le g(s)\}$. The second equality in (3.5) follows from the fact $E(x(s,t)) = E([x](s,t)) = 0$ and $\widetilde{m}(C(\Omega)) = 1$.

On Ω_i, $g(s_i) = t_{n+l_1-i}$ for $i = l_1+1, \ldots, d$. If we let $\alpha_i = u_{i-1,n+l_1-i}$, $\beta_i = u_{i,n+l_1-i}$ and $\gamma_i = u_{i-1,n+l_1-i+1}$, then we have, by (2.12),

$$
\begin{aligned}
[\vec{u}](s,t) &= \alpha_i + \frac{\beta_i - \alpha_i}{\Delta_i s}(s - s_{i-1}) \\
&\quad + \frac{\gamma_i - \alpha_i}{\Delta_{n+l_1-i+1} t}(t - t_{n+l_1-i}).
\end{aligned}
\tag{3.6}
$$

In (2.12), we know that $\Delta_i g = g(s_{i-1}) - g(s_i) = \Delta_{n+l_1-i+1} t$. Thus we obtain

$$
\begin{aligned}
\int_{\Omega_i} [\vec{u}](s,t)\, ds\, dt &= \alpha_i A(\Omega_i) + \frac{\beta_i - \alpha_i}{\Delta_i s} \int_{\Omega_i} (s - s_{i-1})\, ds\, dt \\
&\quad + \frac{\gamma_i - \alpha_i}{\Delta_{n+l_1-i+1} t} \int_{\Omega_i} (t - t_{n+l_1-i})\, ds\, dt
\end{aligned}
\tag{3.7}
$$

where the area of Ω_i is $A(\Omega_i) = (1/2)\Delta_i s\,\Delta_{n+l_1-i+1}t$. Using $g(s_i) = \eta s_i + \delta = t_{n+l_1-i}$, we have $\Delta_{n+l_1-i+1}t = -\eta\Delta_i s$ on Ω_i. Thus we obtain

$$(3.8)\qquad \int_{\Omega_i}(s - s_{i-1})\,dt\,ds = \int_{s_{i-1}}^{s_i}(s - s_{i-1})\eta(s - s_i)\,ds = -\frac{1}{6}\,\eta(\Delta_i s)^3$$

and

$$(3.9)\qquad \int_{\Omega_i}(t - t_{n+l_1-i})\,dt\,ds = \frac{1}{2}\int_{s_{i-1}}^{s_i}(\eta s + \delta - t_{n+l_1-i})^2\,ds = \frac{1}{6}\,\eta^2(\Delta_i s)^3.$$

From (3.7), (3.8), (3.9) and the fact $\Delta_{n+l_1-i+1}t = -\eta\Delta_i s$, we have

$$(3.10)\qquad \int_{\Omega_i}[\vec{u}](s,t)\,ds\,dt = \frac{1}{6}(\alpha_i + \beta_i + \gamma_i)\Delta_i s\Delta_{n+l_1-i+1}t.$$

It is a well-known fact [**1**] that

$$(3.11)\qquad \int_{\Omega_{ij}}[\vec{u}](s,t)\,ds\,dt = \frac{1}{4}(u_{i-1,j-1} + u_{i-1,j} + u_{i,j-1} + u_{i,j})\Delta_i s\Delta_j t.$$

From (3.5), (3.10), and (3.11), our theorem is proved. □

Corollary 3.2. *Let F be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x(s,t)\,ds\,dt$ where Ω is the region* (3.1) *with $g(s) = T$ on $[a,b]$. Then the conditional Yeh-Wiener integral $E(F \mid X_\Lambda)$ of a functional F given X_Λ is*

$$(3.12)\qquad E(F \mid X_\Lambda)(\vec{u}) = \frac{1}{4}\sum_{i=1}^{d}\sum_{j=1}^{n}(u_{i-1,j-1} + u_{i-1,j} + u_{i,j-1} + u_{i,j})\Delta_i s\Delta_j t$$

for $\vec{u}$ in R^N.

Corollary 3.3. *Let F be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x(s,t)\,ds\,dt$ where Ω is the region* (3.1) *with $g(s) = (S-T)/(b-a)s+$

$(Tb-Sa)/(b-a)$ on $[a,b]$ and $0\le S<T$. Then the modified conditional Yeh-Wiener integral $E(F|X_\Lambda)$ of a functional F given X_Λ is

$$\begin{aligned}E(F\mid X_\Lambda)(\vec{u}) &\\ &= \frac{1}{4}\sum_{i=1}^{d}\sum_{j=1}^{n-i}(u_{i-1,j-1}+u_{i-1,j}+u_{i,j-1}+u_{i,j})\Delta_i s\Delta_j t \\ &\quad+\frac{1}{6}\sum_{i=1}^{d}(\alpha_i+\beta_i+\gamma_i)\Delta_i s\Delta_{n-i+1}t.\end{aligned} \tag{3.13}$$

for $\vec{u}$ in R^N, where $\alpha_i=u_{i-1,n-i}$, $\beta_i=u_{i,n-i}$ and $\gamma_i=u_{i-1,n-i+1}$.

Corollary 3.2 and Corollary 3.3 are special cases of Theorem 3.1 for $l_2=0$ and $l_1=0$, respectively. The results [**6**, Example 1] and [**1**, Example 3.1] are the same as (3.12) and (3.13) with $d=m$, respectively.

Let τ_1 and τ_2 be the points in $[a,b]$ with $a\le\tau_1\le\tau_2\le b$, and let $0\le Q\le S\le T$. Define the function g on $[a,b]$ by $g(s)=\nu\sqrt{(\tau_1-a)^2-(s-a)^2}+S$ on $[a,\tau_1]$, $g(s)=S$ on $[\tau_1,\tau_2]$, and $g(s)=\omega\sqrt{s-\tau_2}+S$ on $[\tau_2,b]$ where $\nu=(T-S)/(\tau_1-a)$ and $\omega=(Q-S)/(\sqrt{b-\tau_2})$. Let

$$\Omega=\{(s,t)\mid a\le s\le b,\ 0\le t\le g(s)\}. \tag{3.14}$$

Let Λ be a partition of Ω given by

$$\Lambda=\{(s_i,t_j)\mid 1\le i\le d,\ t_1\le t_j\le g(s_i)\}. \tag{3.15}$$

which satisfies the properties:

(3.16)

i. $\{s_0,s_1,\dots,s_d\}$ is a partition of $[a,b]$ satisfying $a=s_0<s_1<\cdots<s_{l_1}=\tau_1<s_{l_1+1}<\cdots< s_{l_1+l_2}=\tau_2<s_{l_1+l_2+1}<\cdots<s_d=b$ and $d=l_1+l_2+l_3$;

ii. $\{t_0,t_1,\dots,t_n\}$ is a partition of $[0,T]$ satisfying $0=t_0<t_1<\cdots<t_n=T,\ g(s_p)=t_{n-p}$ on A_1 for $0\le p\le l_1,\ g(s)=t_{n-l_1}$ on A_2, and $g(s_{l_1+l_2+p}) =t_{n-l_1-p}$ on A_3 for $0\le p\le l_3$.

Let N be the number of elements of Λ. Then we have $N = dn - ((l_1(l_1+1) + l_3(l_3+1))/2) - l_1(l_2+l_3)$, and let X_Λ be a random vector on $C(\Omega)$ given by $X_\Lambda(x) = (x(s_1,t_1),\dots,x(s_d,t_{n-l_1-l_3}))$ in R^N.

Theorem 3.4. *Let F be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x(s,t)\,ds\,dt$ where the region Ω is given by* (3.14)*. Then the generalized conditional Yeh-Wiener integral $E(F \mid X_\Lambda)(\vec{u})$ given X_Λ at $\vec{u}$ in R^N is*

$$\begin{aligned} &E(F \mid X_\Lambda)(\vec{u}) \\ &\quad= \sum_{i=1}^{d}\sum_{j=1}^{n-l_1-l_3} A_{ij}(\vec{u}) + \sum_{j=n-l_1-l_3+1}^{n-l_1}\ \sum_{i=1}^{n+l_2-j} A_{ij}(\vec{u}) \\ &\quad\quad+ \sum_{j=n-l_1+1}^{n-1}\sum_{i=1}^{n-j} A_{ij}(\vec{u}) + \sum_{i=1}^{l_1} B_i(\vec{u}) + \sum_{i=l_1+l_2+1}^{d} C_i(\vec{u}) \end{aligned} \tag{3.17}$$

where $A_{ij}(\vec{u}) = \int_{\Omega_{ij}}[\vec{u}](s,t)\,ds\,dt$ is given by (3.11)*, and $B_i(\vec{u})$ and $C_i(\vec{u})$ are given by* (3.20) *and* (3.22)*, respectively.*

Proof. By Theorem 2.2, the Fubini theorem, $E(x) = E([x]) = 0$, and $\widetilde{m}(C(\Omega)) = 1$, we have

$$\begin{aligned} E(F|X_\Lambda)(\vec{u}) &= \int_\Omega E(x(s,t) - [x](s,t) + [\vec{u}](s,t))\,ds\,dt \\ &= \int_\Omega [\vec{u}](s,t)\,ds\,dt \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n-l_1-l_3} A_{ij}(\vec{u}) + \sum_{j=n-l_1-l_3+1}^{n-l_1}\ \sum_{i=1}^{n+l_2-j} A_{ij}(\vec{u}) \\ &\quad+ \sum_{j=n-l_1+1}^{n-1}\sum_{i=1}^{n-j} A_{ij}(\vec{u}) + \sum_{i=1}^{l_1}\int_{\Omega_i}[u](s,t)\,ds\,dt \\ &\quad+ \sum_{i=l_1+l_2+1}^{d}\int_{\Omega_i}[\vec{u}](s,t)\,ds\,dt \end{aligned} \tag{3.18}$$

where $A_{ij}(\vec{u}) = \int_{\Omega_{ij}}[\vec{u}](s,t)\,ds\,dt$. For $i = 1,\dots,l_1$, $g(s_i) = t_{n-i}$ on Ω_i and so, by (2.12), the generalized quasi-polyhedric function $[\vec{u}](s,t)$ is obtained by

$$
\begin{aligned}
[\vec{u}](s,t) = u_{i-1,n-i} &+ \frac{s-s_{i-1}}{\Delta_i s}\,(u_{i,n-i} - u_{i-1,n-i}) \\
&+ \frac{t-t_{n-i}}{\Delta_{n-i+1} t}\,(u_{i-1,n-i+1} - u_{i-1,n-i})
\end{aligned} \tag{3.19}
$$

on $\Omega_i = \{(s,t) \mid s_{i-1} < s \le s_i,\ g(s_i) < t < g(s)\}$ with $g(s) = \nu\sqrt{(\tau_1 - a)^2 - (s-a)^2} + S$. Then, using (3.19), we can evaluate

$$
B_i(\vec{u}) = \int_{\Omega_i} [\vec{u}](s,t)\,ds\,dt \tag{3.20}
$$

for $i = 1, 2, \ldots, l_1$. For $l_1 + l_2 + 1 \le i \le d$, $g(s_i) = t_{n+l_2-i}$ on Ω_i and so, by (2.12), the generalized quasi-polyhedric function $[\vec{u}](s,t)$ is obtained by

$$
\begin{aligned}
[\vec{u}](s,t) = u_{i-1,n+l_2-i} &+ \frac{s-s_{i-1}}{\Delta_i s}\,(u_{i,n+l_2-i} - u_{i-1,n+l_2-i}) \\
&+ \frac{t-t_{n+l_2-i}}{\Delta_{n+l_2-i+1} t}\,(u_{i-1,n+l_2-i+1} - u_{i-1,n+l_2-i})
\end{aligned} \tag{3.21}
$$

on $\Omega_i = \{(s,t) \mid s_{i-1} < s \le s_i,\ g(s_i) < t < g(s)\}$ with $g(s) = \omega\sqrt{s-\tau_2} + S$. Hence, using (3.21), we can evaluate

$$
C_i(\vec{u}) = \int_{\Omega_i} [\vec{u}](s,t)\,ds\,dt \tag{3.22}
$$

for $i = l_1 + l_2 + 1,\ l_1 + l_2 + 2, \ldots, d$. From (3.11), (3.18), (3.20), and (3.22), we can obtain the result (3.17).

4. Evaluation of the generalized conditional Yeh-Wiener integral for $F(x) = \int_\Omega ([x](s,t))^k\,ds\,dt$. In this section we will consider the generalized conditional Yeh-Wiener integral for the functional containing a generalized quasi-polyhedric function. Let $g(s)$ be a strictly decreasing and continuous function on $[0,S]$ such that $g(S) = 0$ and let $\Omega = \{(s,t) \mid 0 \le s \le S,\ 0 \le t \le g(s)\}$. And let $C(\Omega)$ denote the space of all real-valued continuous functions $x(s,t)$ on Ω such that $x(s,0) = x(0,t) = 0$ for every (s,t) in Ω, and let $g(0) = T$.

For each partition $\tau = \{(s_i, t_j) \mid 1 \le j \le n-i \text{ for } 1 \le i \le n-1\}$ of Ω with $0 = s_0 < s_1 < \cdots < s_n = S$ and $t_{n-i} = g(s_i)$, $i = 0, 1, 2, \ldots, n$, define $X_\tau : C(\Omega) \to R^N$ by $X_\tau(x) = (x(s_1, t_1), \ldots, x(s_1, t_{n-1}), x(s_2, t_1), \ldots, x(s_2, t_{n-2}), x(s_3, t_1), \ldots, x(s_{n-1}, t_1))$ for $N = (n(n-1))/2$.

For a nonnegative integer k, let F be a functional on Ω given by

$$F(x) = \int_\Omega ([x](s,t))^k \, ds\, dt \tag{4.1}$$

where $[x]$ is the generalized quasi-polyhedric function on Ω given by (2.9) and (2.10). We note that $g(s_i) = t_{n-i}$ and $\Delta_i g = g(s_{i-1}) - g(s_i) = \Delta_{n-i+1} t$ since g is strictly decreasing and continuous on $[0, S]$.

By (2.17) in Theorem 2.2 and the Fubini theorem, we have

$$\begin{aligned} E(F \mid X_\tau)(\vec{u}) &= \int_\Omega E\big([x - [x] + [\vec{u}]]^k(s,t)\big)\, ds\, dt \\ &= \int_\Omega ([\vec{u}](s,t))^k \, ds\, dt \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \int_{\Omega_{ij}} ([\vec{u}](s,t))^k \, ds\, dt \\ &\quad + \sum_{i=1}^{n} \int_{\Omega_i} ([\vec{u}](s,t))^k \, ds\, dt \end{aligned} \tag{4.2}$$

where the second equality in (4.2) comes from the fact that the quasi-polyhedric function satisfies the linearity, $[[x]](s,t) = [x](s,t)$ for (s,t) in Ω and $\widetilde{m}(C(\Omega)) = 1$.

Now, using (2.11) and the simple change of variable, we have

$$\begin{aligned} &\int_{\Omega_{ij}} ([\vec{u}](s,t))^k \, ds\, dt \\ &\quad = \int_{t_{j-1}}^{t_j} \left\{ \int_{s_{i-1}}^{s_i} \left[a(t) + \frac{s - s_{i-1}}{\Delta_i s} \,(b(t) - a(t)) \right]^k ds \right\} dt \\ &\quad = \int_{t_{j-1}}^{t_j} \left\{ \frac{\Delta_i s}{b(t) - a(t)} \int_{a(t)}^{b(t)} u^k \, du \right\} dt \\ &\quad = \frac{\Delta_i s}{k+1} \int_{t_{j-1}}^{t_j} \sum_{p=0}^{k} a(t)^p b(t)^{k-p} \, dt \end{aligned} \tag{4.3}$$

where

$$(4.4)\qquad \begin{aligned} a(t) &= u_{i-1,j-1} + \frac{t-t_{j-1}}{\Delta_j t}\,(u_{i-1,j}-u_{i-1,j-1}) \\ b(t) &= u_{i,j-1} + \frac{t-t_{j-1}}{\Delta_j t}\,(u_{i,j}-u_{i,j-1}). \end{aligned}$$

Doing the change of variable one more time, that is $y = b(t)$, the righthand side of the last equality in (4.3) becomes

$$(4.5)\quad \frac{\Delta_i s}{k+1}\,\frac{\Delta_j t}{u_{i,j}-u_{i,j-1}} \left\{\sum_{p=0}^{k}\Big[\int_{u_{i,j-1}}^{u_{i,j}}\Big(u_{i-1,j-1}+\frac{(y-u_{i,j-1})(u_{i-1,j}-u_{i-1,j-1})}{u_{i,j}-u_{i,j-1}}\Big)^{p} y^{k-p}\,dy\Big]\right\}$$
$$= \frac{\Delta_i s\Delta_j t}{(k+1)(u_{i,j}-u_{i,j-1})}\left\{\sum_{p=0}^{k}\Big[\sum_{q=0}^{p}\binom{p}{q}\Big(\frac{u_{i-1,j}-u_{i-1,j-1}}{u_{i,j}-u_{i,j-1}}\Big)^{p-q} \Big(u_{i-1,j-1}-\frac{u_{i,j-1}(u_{i-1,j}-u_{i-1,j-1})}{u_{i,j}-u_{i,j-1}}\Big)^{q}\int_{u_{i,j-1}}^{u_{i,j}} y^{k-q}\,dy\Big]\right\}.$$

Combining (4.2), (4.3) and (4.5), we have the following theorem.

Theorem 4.1. *Let F be a functional on $C(\Omega)$ given by (4.1). Then the generalized conditional Yeh-Wiener integral $E(F \mid X_\tau)$ of F given X_τ is*

$$(4.6)\quad E(F\mid X_\tau)(\vec{u}) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\frac{1}{k+1}\left\{\sum_{p=0}^{k}\left[\sum_{q=0}^{p}\frac{\binom{p}{q}}{k-p+1}\left(\sum_{r=0}^{k-q}u_{i,j}^{r}\,u_{i,j-1}^{k-q-r}\right) \frac{(u_{i-1,j}-u_{i-1,j-1})^{p-q}(u_{i,j}u_{i-1,j-1}-u_{i-1,j}u_{i,j-1})^{q}}{(u_{i,j}-u_{i,j-1})^{p}}\right]\right\}\Delta_i s\Delta_j t + \sum_{i=1}^{n}\int_{\Omega_i}([\vec{u}](s,t))^k\,ds\,dt,$$

for $\vec{u}$ in R^N and $\binom{p}{q} = (p(p-1)\cdots(p-q+1))/q!$.

The result of Theorem 4.1 can be used to evaluate the generalized conditional Yeh-Wiener integral for the functional F on $C(\Omega)$ given by $F(x) = \int_\Omega (x(s,t))^k \, ds \, dt$ where k is a nonnegative integer.

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