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ON GENERALIZATION OF BULLEN-SIMPSON'S INEQUALITY

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ABSTRACT. Generalization of Bullen-Simpson's inequality for $(2r)$-convex functions is given, by using some Euler type identities. A number of inequalities, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions in L_p-spaces, are proved.

1. Introduction. For any convex function $f : [0,1] \to \mathbf{R}$, the following pair of inequalities, usually referred in the literature as Hadamard's inequalities, hold

$$f\left(\frac{1}{2}\right) \le \int_0^1 f(t)\,dt \le \frac{f(0)+f(1)}{2}. \tag{1.1}$$

If f is concave, the inequalities are reversed. In [**9**] Hammer showed, by a simple geometric argument that for convex functions the absolute value of error in the mid-point quadrature rule is always smaller than absolute value of the error in the trapezoidal rule, i.e., the following inequalities are valid for a convex function f

$$0 \le \int_0^1 f(t)\,dt - f\left(\frac{1}{2}\right) \le \frac{1}{2}\,[f(0)+f(1)] - \int_0^1 f(t)\,dt. \tag{1.2}$$

An elementary analytic proof of (1.1) and (1.2), but stated on the interval $[-1,1]$, was given in [**3**].

The trapezoid rule is the simplest example of a closed quadrature rule, while the mid-point rule is the simplest open quadrature rule, [**4**]. The next simplest such pair is based on the Simpson's formula

$$\int_0^1 f(t)\,dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \frac{1}{2880}\,f^{(4)}(\eta) \tag{1.3}$$

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and the three-point formula (we call it the dual Simpson's formula)

$$\int_0^1 f(t)\,dt = \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + \frac{7}{23040} f^{(4)}(\xi), \tag{1.4}$$

which hold in this form for some η and ξ from $[0,1]$, for any function f with continuous fourth derivative $f^{(4)}$ on $[0,1]$. If additionally $f^{(4)}$ is nonnegative, that is, if $f^{(4)}(t) \geq 0$, for all $t \in [0,1]$, then from the above formulas we immediately have

$$\begin{aligned} \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] &\leq \int_0^1 f(t)\,dt \\ &\leq \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]. \end{aligned} \tag{1.5}$$

In the case when $f^{(4)}$ exists, the condition $f^{(4)}(t) \geq 0$, for all $t \in [0,1]$ is equivalent to the requirement that f is a four-convex function on $[0,1]$. However, a function f may be four-convex although $f^{(4)}$ does not exist.

Bullen in [**3**] proved that, if f is four-convex, then (1.5) is valid. Moreover, he proved that the dual Simpson's quadrature rule is more accurate than the Simpson's quadrature rule, that is, we have

$$\begin{aligned} 0 &\leq \int_0^1 f(t)\,dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] \\ &\leq \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)\,dt, \end{aligned} \tag{1.6}$$

provided f is four-convex. We shall call this inequality Bullen-Simpson's inequality.

We recall that a function $f : [0,1] \to \mathbf{R}$ is said to be n-convex on $[0,1]$, for some $n \geq 0$, if for any choice of $n+1$ distinct points $x_0, x_1, \dots, x_n$ from $[0,1]$ we have

$$f[x_0, x_1, \dots, x_n] \geq 0,$$

where $f[x_0, x_1, \dots, x_n]$ is the nth order divided difference of f. In the case when the above inequality is reversed, f is said to be n-concave on $[0,1]$. If f is n-convex (n-concave), then $f^{(n-2)}$ exists and is convex (concave) function in the ordinary sense. Especially, two-convex (two-concave) function f is convex (concave) in the ordinary sense. Also, if $f^{(n)}$ exists, then f is an n-convex (n-concave) if and only if $f^{(n)} \geq 0$, $f^{(n)} \leq 0$. For some further details on n-convexity see for example [**12**].

In the recent papers [**5, 6**], Dedić et al. considered a generalization of (1.3) and (1.4), based on the well-known Euler formula for expanding n times differentiable function f in terms of Bernoulli polynomials, see for example [**10**, p. 17]. Before stating the basic results from [**5, 6**] we recall that a sequence of Bernoulli polynomials $(B_k(t))_{k\geq 0}$ is uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1; \qquad B_0(t) = 1 \tag{1.7}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0. \tag{1.8}$$

The values $B_k = B_k(0)$ are usually called the Bernoulli numbers. For some further details on the Bernoulli polynomials and the Bernoulli numbers, see for example [**1, 2**]. Also, we need a sequence $(B_k^*(t))_{k\geq 0}$ of periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \quad \text{and} \quad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbf{R}$$

It is easy to see that $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. Also, since $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, the functions $B_k^*(t)$ are continuous for $k \geq 2$. Moreover, from (1.7) we get

$$B_k^{*\prime}(t) = kB_{k-1}^*(t), \quad k \geq 1$$

for every $t \in \mathbf{R}$ when $k \geq 3$, and for every $t \in \mathbf{R} \setminus Z$ when $k = 1, 2$.

Now we consider a function $f : [0,1] \to \mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Denote $r = [n/2]$, $s = [(n-1)/2]$, where $[a]$ is the greatest integer less than or equal to a.

For such a function f, the following identities were obtained in [**5**]:

$$\int_0^1 f(t)\,dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] + T_r^S(f) + \sigma_n^1(f) \tag{1.9}$$

and

$$\int_0^1 f(t)\,dt = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]+T_s^S(f)+\sigma_n^2(f). \tag{1.10}$$

The perturbations $T_r^S(f)$ and $T_s^S(f)$ are defined as $T_0^S(f)=T_1^S(f)=0$ and, for $m\geq 2$,

$$T_m^S(f)=\frac{1}{3}\sum_{k=2}^{m}\frac{1}{(2k)!}(1-2^{2-2k})B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right], \tag{1.11}$$

while the remainders $\sigma_n^1(f)$ and $\sigma_n^2(f)$ are given by

$$\sigma_n^1(f)=\frac{1}{3\,(n!)}\int_0^1 G_n^S(t)\;df^{(n-1)}(t)$$

and

$$\sigma_n^2(f)=\frac{1}{3\,(n!)}\int_0^1 F_n^S(t)\;df^{(n-1)}(t),$$

where

$$F_1^S(t)=G_1^S(t)=B_1(1-t)+2B_1^*\left(\frac{1}{2}-t\right)$$

and for $n\geq 2$

$$G_n^S(t)=B_n(1-t)+2B_n^*\left(\frac{1}{2}-t\right),\quad F_n^S(t)=G_n^S(t)-G_n^S(0).$$

Further, under the same assumptions on f, the following two identities were obtained in [**6**]:

$$\int_0^1 f(t)\,dt=\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]+T_r^D(f)+\rho_n^1(f) \tag{1.12}$$

and

$$\int_0^1 f(t)\,dt=\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]+T_s^D(f)+\rho_n^2(f). \tag{1.13}$$

Here, the perturbations $T_r^D(f)$ and $T_s^D(f)$ are defined as $T_0^D(f) = T_1^D(f) = 0$ and, for $m \geq 2$,

$$(1.14) \quad T_m^D(f) = -\frac{1}{3}\sum_{k=2}^{m} \frac{1}{(2k)!}\,(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1) \times B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right],$$

while the remainders $\rho_n^1(f)$ and $\rho_n^2(f)$ are given by

$$\rho_n^1(f) = \frac{1}{3\,(n!)}\int_0^1 G_n^D(t)\,df^{(n-1)}(t)$$

and

$$\rho_n^2(f) = \frac{1}{3\,(n!)}\int_0^1 F_n^D(t)\,df^{(n-1)}(t),$$

where, for all $n \geq 1$,

$$G_n^D(t) = 2B_n^*\left(\frac{1}{4} - t\right) - B_n^*\left(\frac{1}{2} - t\right) + 2B_n^*\left(\frac{3}{4} - t\right),$$
$$F_n^D(t) = G_n^D(t) - G_n^D(0).$$

The aim of this paper is to establish a generalization of the inequalities (1.5) and (1.6) for a class of $(2r)$-convex functions and also to obtain some estimates for the absolute value of difference between the absolute value of error in the dual Simpson's quadrature rule and the absolute value of error in the Simpson's quadrature rule. We shall make use of the following five-point quadrature formula

$$\int_0^1 f(t)\,dt \approx \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right],$$

obtained by adding the Simpson's and the dual Simpson's quadrature formulae. It is suitable for our purposes to rewrite the inequality (1.5) in the form

$$(1.15) \quad \int_0^1 f(t)\,dt \leq \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right].$$

As we mentioned earlier, this inequality is valid for any 4-convex function f and we call it the Bullen-Simpson's inequality.

It should be noted that each continuous n-convex function on $[0,1]$ is the uniform limit of the sequence of corresponding Bernstein's polynomials, see for example [**12**, p. 293]. Also, Bernstein's polynomials of continuous n-convex function are also n-convex functions. Therefore, when stating our results for a continuous $(2r)$-convex function f, without any loss of generality we assume that $f^{(2r)}$ exists and is continuous. Actually those results are valid for any continuous $(2r)$-convex function f.

2. Bullen-Simpson's formulae of Euler type. We consider the sequences of functions $(G_k(t))_{k\geq 1}$ and $(F_k(t))_{k\geq 1}$ defined for $t \in \mathbf{R}$ by

$$G_k(t) := G_k^S(t) + G_k^D(t), \quad F_k(t) := F_k^S(t) + F_k^D(t),$$

where $G_k^S(t)$, $G_k^D(t)$, $F_k^S(t)$ and $F_k^D(t)$ are defined as in the introduction. So we have

$$G_1(t) = F_1(t) = B_1(1-t) + 2B_1^*\left(\frac{1}{4} - t\right) + B_1^*\left(\frac{1}{2} - t\right) + 2B_1^*\left(\frac{3}{4} - t\right)$$

and, for $k \geq 2$,

$$G_k(t) = B_k(1-t) + 2B_k^*\left(\frac{1}{4} - t\right) + B_k^*\left(\frac{1}{2} - t\right) + 2B_k^*\left(\frac{3}{4} - t\right),$$
$$F_k(t) := G_k(t) - \widetilde{B}_k,$$

where

$$\widetilde{B}_k := G_k(0) = B_k + 2B_k\left(\frac{1}{4}\right) + B_k\left(\frac{1}{2}\right) + 2B_k\left(\frac{3}{4}\right).$$

Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$. We introduce the following notation

$$D(0,1) := \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right]$$

Further, we define $T_0(f) = T_1(f) := 0$ and, for $2 \le m \le [n/2]$,

$$T_m(f) := \frac{1}{2}\left[T_m^S(f) + T_m^D(f)\right],$$

where $T_m^S(f)$ and $T_m^D(f)$ are given by (1.11) and (1.14), respectively. It is easy to see that

(2.1)

$$T_m(f) = \frac{1}{3}\sum_{k=2}^{m}\frac{1}{(2k)!}\,2^{-2k}(1-4\cdot 2^{-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

In the next lemma we establish two formulae which play the key role in this paper. We call them Bullen-Simpson formulae of Euler type.

Lemma 1. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \ge 1$. Then we have*

$$\int_0^1 f(t)\,dt = D(0,1) + T_r(f) + \tau_n^1(f), \tag{2.2}$$

where $r = [n/2]$ and

$$\tau_n^1(f) = \frac{1}{6\,(n!)}\int_0^1 G_n(t)\;df^{(n-1)}(t).$$

Also,

$$\int_0^1 f(t)\,dt = D(0,1) + T_s(f) + \tau_n^2(f), \tag{2.3}$$

where $s = [(n-1)/2]$ and

$$\tau_n^2(f) = \frac{1}{6\,(n!)}\int_0^1 F_n(t)\;df^{(n-1)}(t).$$

Proof. First we multiply formulas (1.9) and (1.12) by the factor 1/2 and then add them up to obtain the identity (2.2). The identity (2.3) follows analogously from (1.10) and (1.13). □

Remark 1. The interval $[0,1]$ is used for simplicity and involves no loss in generality. The results which follow will apply,without comment, to any interval that is convenient.

Namely, it is easy to transform the identities (2.2) and (2.3) to the identities which hold for any function $f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n\geq 1$. We get

(2.4)
$$\int_a^b f(t)\,dt = D(a,b)+\widetilde{T}_r(f)+\frac{(b-a)^n}{6\,(n!)}\int_a^b G_n\left(\frac{t-a}{b-a}\right)\,df^{(n-1)}(t)$$

and

(2.5)
$$\int_a^b f(t)\,dt = D(a,b)+\widetilde{T}_s(f)+\frac{(b-a)^n}{6\,(n!)}\int_a^b F_n\left(\frac{t-a}{b-a}\right)\,df^{(n-1)}(t),$$

where

$$D(a,b):=\frac{b-a}{12}\left[f(a)+4\left(\frac{3a+b}{4}\right)+2\left(\frac{a+b}{2}\right)+4\left(\frac{a+3b}{4}\right)+f(b)\right],$$

while $\widetilde{T}_0(f)=\widetilde{T}_1(f)=0$ and

$$\widetilde{T}_m(f)=\frac{1}{3}\sum_{k=2}^{m}\frac{(b-a)^{2k}}{(2k)!}2^{-2k}(1-4\cdot 2^{-2k})B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right],$$

for $2\leq m\leq[n/2]$.

3. Bullen-Simpson's inequality for $(2r)$-convex functions. In this section we use Bullen-Simpson formulae of Euler type established in Lemma 1 to obtain a generalization of Bullen-Simpson's inequality for $(2r)$-convex functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$ defined in the previous section.

Since $B_1(t)=t-(1/2)$, we have

$$G_1(t)=F_1(t)=\begin{cases}-6t+1/2 & t\in[0,1/4]\\ -6t+5/2 & t\in(1/4,1/2]\\ -6t+7/2 & t\in(1/2,3/4]\\ -6t+11/2 & t\in(3/4,1].\end{cases}\tag{3.1}$$

Further, for $k \geq 2$, the functions $B_k^*(t)$ are periodic with period 1 and continuous. We have

$$G_k(0) = G_k(1/2) = G_k(1) = \widetilde{B}_k \quad \text{and} \quad F_k(0) = F_k(1/2) = F_k(1) = 0.$$

Moreover, it is enough to know the values of the functions $G_k(t)$ and $F_k(t)$, $k \geq 2$, only on the interval $[0, 1/2]$ since for $0 \leq t \leq 1/2$ we have

$$\begin{aligned} G_k\left(t + \frac{1}{2}\right) &= B_k\left(\frac{1}{2} - t\right) + 2B_k^*\left(-\frac{1}{4} - t\right) + B_k^*(-t) + 2B_k^*\left(\frac{1}{4} - t\right) \\ &= B_k^*\left(\frac{1}{2} - t\right) + 2B_k^*\left(\frac{3}{4} - t\right) + B_k(1-t) + 2B_k^*\left(\frac{1}{4} - t\right) \\ &= G_k(t). \end{aligned}$$

For $k = 2$ and $k = 3$, we have $B_2(t) = t^2 - t + (1/6)$ and $B_3(t) = t^3 - (3/2)t^2 + (1/2)t$, so that by direct calculation we get $\widetilde{B}_2 = \widetilde{B}_3 = 0$ and

$$G_2(t) = F_2(t) = \begin{cases} 6t^2 - t & t \in [0, 1/4] \\ 6t^2 - 5t + 1 & t \in (1/4, 1/2] \end{cases}, \tag{3.2}$$

$$G_3(t) = F_3(t) = \begin{cases} -6t^3 + (3/2)t^2 & t \in [0, 1/4] \\ -6t^3 + (15/2)t^2 - 3t + (3/8) & t \in (1/4, 1/2]. \end{cases} \tag{3.3}$$

The Bernoulli polynomials have a property of symmetry with respect to 1/2, that is, [**1**]

$$B_k(1-t) = (-1)^k B_k(t), \quad t \in \mathbf{R}, \quad k \geq 1. \tag{3.4}$$

Also, we have

$$B_k(1) = B_k(0) = B_k, \quad k \geq 2, \quad B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2r-1} = 0, \quad r \geq 2.$$

This implies

$$\widetilde{B}_{2r-1} = 0, \quad r \geq 2 \tag{3.5}$$

and

$$\widetilde{B}_{2r} = B_{2r} + 4B_{2r}\left(\frac{1}{4}\right) + B_{2r}\left(\frac{1}{2}\right), \quad r \geq 1.$$

Also, we have [**1**, 23.1.21, 23.1.22]

$$B_{2r}\left(\frac{1}{2}\right) = -\left(1 - 2^{1-2r}\right) B_{2r},$$
$$B_{2r}\left(\frac{1}{4}\right) = -2^{-2r}\left(1 - 2^{1-2r}\right) B_{2r}, \quad r \geq 1,$$

which gives the formula

$$\widetilde{B}_{2r} = 2 \cdot 2^{-2r}(4 \cdot 2^{-2r} - 1)B_{2r}, \quad r \geq 1. \tag{3.6}$$

Now, by (3.5) we have

$$F_{2r-1}(t) = G_{2r-1}(t), \quad r \geq 1. \tag{3.7}$$

Also,

$$F_{2r}(t) = G_{2r}(t) - 2 \cdot 2^{-2r}(4 \cdot 2^{-2r} - 1)B_{2r}, \quad r \geq 1. \tag{3.8}$$

Further, as we pointed out earlier, the points 0 and 1/2 are the zeros of $F_k(t)$, $k \geq 2$. As we shall see below, 0 and 1/2 are the only zeros of $F_k(t)$ in $[0, 1/2]$ for $k = 2r$, $r \geq 1$, while for $k = 2r - 1$, $r \geq 2$, we have $F_{2r-1}(1/4) = G_{2r-1}(1/4) = 0$. We shall see that 0, 1/4 and 1/2 are the only zeros of $F_{2r-1}(t) = G_{2r-1}(t)$ in $[0, 1/2]$ for $r \geq 2$. Also, note that for $r \geq 1$ we have

$$G_{2r}(0) = G_{2r}\left(\frac{1}{2}\right) = \widetilde{B}_{2r} = 2 \cdot 2^{-2r}(4 \cdot 2^{-2r} - 1)B_{2r}$$

and

$$G_{2r}\left(\frac{1}{4}\right) = 2B_{2r} + 2B_{2r}\left(\frac{1}{4}\right) + 2B_{2r}\left(\frac{1}{2}\right) = 2 \cdot 2^{-2r}(2 \cdot 2^{-2r} + 1)B_{2r},$$

while

$$F_{2r}\left(\frac{1}{4}\right) = G_{2r}\left(\frac{1}{4}\right) - \widetilde{B}_{2r} = 4 \cdot 2^{-2r}(1 - 2^{-2r})B_{2r}. \tag{3.9}$$

Lemma 2. *For $k \geq 2$, we have*

$$G_k\left(\frac{1}{2}-t\right) = (-1)^k G_k(t), \quad 0 \leq t \leq \frac{1}{2},$$

and

$$F_k\left(\frac{1}{2}-t\right) = (-1)^k F_k(t), \quad 0 \leq t \leq \frac{1}{2}.$$

Proof. As the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$, we obtain the above two identities by the simple argument similar to the one used in [**5, 6, 8**]. □

Note that the identities established in Lemma 2 are valid for $k = 1$, too, except at the points 0, 1/4 and 1/2.

Lemma 3. *For $r \geq 2$, the function $G_{2r-1}(t)$ has no zeros in the interval $(0, 1/4)$. The sign of this function is determined by*

$$(-1)^r G_{2r-1}(t) > 0, \quad 0 < t < \frac{1}{4}.$$

Proof. For $r = 2$, $G_3(t)$ is given by (3.3) and it is easy to see that

$$G_3(t) > 0, \quad 0 < t < \frac{1}{4}. \tag{3.10}$$

Thus, our assertion is true for $r = 2$. Now, using a simple induction, similarly as it was done in [**5, 6, 8**], we prove that $G_{2r-1}(t)$ cannot have a zero inside the interval $(0, 1/4)$. Further, if $G_{2r-3}(t) > 0$, $0 < t < (1/4)$, then from $G''_{2r-1}(t) = (2r-1)(2r-2)G_{2r-3}(t)$ it follows that $G_{2r-1}(t)$ is convex on $(0, 1/4)$ and hence $G_{2r-1}(t) < 0$, $0 < t < 1/4$, while in the case when $G_{2r-3}(t) < 0$, $0 < t < 1/4$ we have that $G_{2r-1}(t)$ is concave and hence $G_{2r-1}(t) > 0$, $0 < t < 1/4$. Since (3.10) is valid we conclude that

$$(-1)^r G_{2r-1}(t) > 0, \quad 0 < t < \frac{1}{4}. \qquad \square$$

Corollary 1. *For $r \geq 2$ the functions $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly increasing on the interval $(0,1/4)$ and strictly decreasing on the interval $(1/4,1/2)$. Consequently, 0 and $1/2$ are the only zeros of $F_{2r}(t)$ in the interval $[0,1/2]$ and*

$$\max_{t\in[0,1]} |F_{2r}(t)| = 4\cdot 2^{-2r}(1-2^{-2r})\,|B_{2r}|\,, \quad r\geq 1.$$

Also, we have

$$\max_{t\in[0,1]} |G_{2r}(t)| = 2\cdot 2^{-2r}\left(2\cdot 2^{-2r}+1\right)|B_{2r}|\,, \quad r\geq 1.$$

Proof. Using (1.7), we get

$$\left[(-1)^{r-1}F_{2r}(t)\right]' = \left[(-1)^{r-1}G_{2r}(t)\right]' = 2r(-1)^r G_{2r-1}(t)$$

and $(-1)^r G_{2r-1}(t) > 0$ for $0 < t < 1/4$ by Lemma 3. Thus, $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly increasing on the interval $(0,1/4)$. Also, by Lemma 2, we have $F_{2r}\left(1/2-t\right) = F_{2r}(t)$, $0\leq t\leq 1/2$ and $G_{2r}\left(1/2-t\right) = G_{2r}(t)$, $0\leq t\leq 1/2$. which implies that $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly decreasing on the interval $(1/4,1/2)$. Further, $F_{2r}(0) = F_{2r}\left(1/2\right) = 0$, which implies that $|F_{2r}(t)|$ achieves its maximum at $t = 1/4$, that is

$$\max_{t\in[0,1]} |F_{2r}(t)| = \left|F_{2r}\left(\frac{1}{4}\right)\right| = 4\cdot 2^{-2r}(1-2^{-2r})\,|B_{2r}|\,.$$

Also,

$$\begin{aligned}\max_{t\in[0,1]} |G_{2r}(t)| &= \max\left\{|G_{2r}\left(0\right)|\,, \left|G_{2r}\left(\frac{1}{4}\right)\right|\right\}\\ &= 2\cdot 2^{-2r}\left(1+2\cdot 2^{-2r}\right)|B_{2r}|\,,\end{aligned}$$

which completes the proof. □

Corollary 2. *Assume $r\geq 2$. Then we have*

$$\int_0^1 |G_{2r-1}(t)|\,dt = \frac{8\cdot 2^{-2r}\left(1-2^{-2r}\right)}{r}\,|B_{2r}|\,.$$

Also, we have

$$\int_0^1 |F_{2r}(t)|\, dt = \left|\widetilde{B}_{2r}\right| = 2 \cdot 2^{-2r}\left(1 - 4 \cdot 2^{-2r}\right)|B_{2r}|$$

and

$$\int_0^1 |G_{2r}(t)|\, dt \le 2\left|\widetilde{B}_{2r}\right| = 4 \cdot 2^{-2r}\left(1 - 4 \cdot 2^{-2r}\right)|B_{2r}|\,.$$

Proof. Using Lemma 2 and Lemma 3, we get

$$\begin{aligned}\int_0^1 |G_{2r-1}(t)|\, dt &= 4\left|\int_0^{1/4} G_{2r-1}(t)\, dt\right| = 4\left| -\frac{1}{2r}\, G_{2r}(t)|_0^{1/4}\,\right| \\ &= \frac{2}{r}\left|G_{2r}\left(\frac{1}{4}\right) - G_{2r}\,(0)\right| = \frac{8 \cdot 2^{-2r}\left(1-2^{-2r}\right)}{r}\ |B_{2r}|\,,\end{aligned}$$

which proves the first assertion. By Corollary 1, $F_{2r}(t)$ does not change the sign on the interval $(0, 1/2)$. Therefore, using (3.8) we get

$$\begin{aligned}\int_0^1 \left|F_{2r}(t)\right| dt &= 2\left|\int_0^{1/2} F_{2r}(t)\, dt\right| = 2\left|\int_0^{1/2}\ \left[G_{2r}(t) - \widetilde{B}_{2r}\right]\, dt\right| \\ &= 2\left|-\frac{1}{2r+1}\ G_{2r+1}(t)|_0^{1/2} - \frac{1}{2}\widetilde{B}_{2r}\right| = \left|\widetilde{B}_{2r}\right| \\ &= 2 \cdot 2^{-2r}\left(1 - 4 \cdot 2^{-2r}\right)|B_{2r}|\,.\end{aligned}$$

This proves the second assertion. Finally, we use (3.8) again and the triangle inequality to obtain the third formula. □

In the following discussion we assume that $f : [0, 1] \to \mathbf{R}$ has a continuous derivative of order n, for some $n \ge 1$. In this case the remainders $\tau_n^1(f)$ and $\tau_n^2(f)$ are given by

$$\tau_n^1(f) = \frac{1}{6(n!)}\int_0^1 G_n(s) f^{(n)}(s)\, ds \tag{3.11}$$

and

$$\tau_n^2(f) = \frac{1}{6(n!)} \int_0^1 F_n(s) f^{(n)}(s)\, ds. \tag{3.12}$$

Lemma 4. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r \geq 2$, then there exists a point $\eta \in [0,1]$ such that*

$$\tau_{2r}^2(f) = \frac{1}{3(2r)!}\, 2^{-2r}(1 - 4 \cdot 2^{-2r}) B_{2r} f^{(2r)}(\eta). \tag{3.13}$$

Proof. Using (3.12) with $n = 2r$, we can rewrite $\tau_{2r}^2(f)$ as

$$\tau_{2r}^2(f) = (-1)^{r-1} \frac{1}{6(2r)!}\, J_r, \tag{3.14}$$

where

$$J_r = \int_0^1 (-1)^{r-1} F_{2r}(s) f^{(2r)}(s)\, ds. \tag{3.15}$$

From Corollary 1 it follows that $(-1)^{r-1} F_{2r}(s) \geq 0$, $0 \leq s \leq 1$, so that (3.13) follows from the mean value theorem for integrals and Corollary 4.12. $\square$

Now, we prove the main result:

Theorem 1. *Assume $f : [0,1] \to \mathbf{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$ for some $r \geq 2$. If f is $(2r)$-convex function, then for even r we have*

$$\begin{aligned} 0 &\leq \int_0^1 f(t)\, dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - T_{r-1}^D(f) \\ &\leq \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] + T_{r-1}^S - \int_0^1 f(t)\, dt, \end{aligned} \tag{3.16}$$

while for odd r we have reversed inequalities in (3.16).

Proof. Let us denote by LHS and RHS, respectively, the lefthand side and the righthand side in the second inequality in (3.16). Then we have

$$LHS = \rho_{2r}^2(f)$$

and

$$RHS - LHS = -2\tau_{2r}^2(f),$$

where $\rho_{2r}^2(f)$ and $\tau_{2r}^2(f)$ are determined respectively by the identities (1.13) and (2.3). Dedić et al. [**6**] proved that, under the given assumption on f, there exists a point $\xi \in [0,1]$ such that

$$\rho_{2r}^2(f) = -\frac{1}{3(2r)!}\left(1 - 2\cdot 2^{-2r}\right)(1 - 4\cdot 2^{-2r})B_{2r}f^{(2r)}(\xi). \tag{3.17}$$

Also, by Lemma 4, we know that

$$-2\tau_{2r}^2(f) = -\frac{2}{3(2r)!}\, 2^{-2r}(1 - 4\cdot 2^{-2r})B_{2r}f^{(2r)}(\eta), \tag{3.18}$$

for some point $\eta \in [0,1]$. Finally, we know that [**1**]

$$(-1)^{r-1}B_{2r} > 0, \quad r = 1, 2, \dots . \tag{3.19}$$

Now, if f is a $(2r)$-convex function, then $f^{(2r)}(\xi) \geq 0$ and $f^{(2r)}(\eta) \geq 0$ so that using (3.17), (3.18) and (3.19), we get the inequalities

$$\begin{aligned} &LHS \geq 0, \quad RHS - LHS \geq 0, \quad \text{when } r \text{ is even;}\\ &LHS \leq 0, \quad RHS - LHS \leq 0, \quad \text{when } r \text{ is odd.} \end{aligned}$$

This proves our assertions. □

Remark 2. For $r = 2$, formula (3.13) reduces to

$$\tau_4^2(f) = -\frac{1}{46080}\, f^{(4)}(\eta).$$

Note that in this case the result stated in Theorem 1 reduces to Bullen's result that we mentioned in the introduction.

Theorem 2. *Assume $f:[0,1]\to\mathbf{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$ for some $r\geq 2$. If f is either a $(2r)$-convex or $(2r)$-concave function, then there exists a point $\vartheta\in[0,1]$ such that*

$$\tau_{2r}^2(f)=\vartheta\,\frac{2}{3(2r)!}\,2^{-2r}(1-2^{-2r})B_{2r}\left[f^{(2r-1)}(1)-f^{(2r-1)}(0)\right]. \tag{3.20}$$

Proof. We obtain (3.20) from (3.14) using (3.9) and arguing similarly as in [**5, 6, 8**]. □

Remark 3. If we approximate $\int_0^1 f(t)\,dt$ by

$$I_{2r}(f):=D(0,1)+T_{r-1}(f),$$

then the next approximation will be $I_{2r+2}(f)$. The difference

$$\Delta_{2r}(f)=I_{2r+2}(f)-I_{2r}(f)$$

is equal to the last term in $I_{2r+2}(f)$, that is,

$$\Delta_{2r}(f)=\frac{1}{3(2r)!}\,2^{-2r}(1-4\cdot 2^{-2r})B_{2r}\left[f^{(2r-1)}(1)-f^{(2r-1)}(0)\right].$$

We see that, under the assumptions of Theorem 2,

$$\tau_{2r}^2(f)=\frac{2\vartheta\left(1-2^{-2r}\right)}{1-4\cdot 2^{-2r}}\;\Delta_{2r}(f).$$

Theorem 3. *Assume $f:[0,1]\to\mathbf{R}$ is such that $f^{(2r+2)}$ is continuous on $[0,1]$ for some $r\geq 2$. If f is either a $(2r)$-convex and $(2r+2)$-convex or $(2r)$-concave and $(2r+2)$-concave function, then the remainder $\tau_{2r}^2(f)$ has the same sign as the first neglected term $\Delta_{2r}(f)$ and*

$$\left|\tau_{2r}^2(f)\right|\leq\left|\Delta_{2r}(f)\right|.$$

Proof. We have

$$\Delta_{2r}(f)+\tau_{2r+2}^2(f)=\tau_{2r}^2(f),$$

that is,

$$\Delta_{2r}(f) = \tau_{2r}^2(f) - \tau_{2r+2}^2(f). \tag{3.21}$$

By (3.12) we have

$$\tau_{2r}^2(f) = \frac{1}{6(2r)!} \int_0^1 F_{2r}(s) f^{(2r)}(s)\, ds$$

and

$$-\tau_{2r+2}^2(f) = \frac{1}{6(2r+2)!} \int_0^1 \left[-F_{2r+2}(s)\right] f^{(2r+2)}(s)\, ds.$$

Similarly as in [**5, 6, 8**] it follows that

$$\left|\tau_{2r}^2(f)\right| \le |\Delta_{2r}(f)| \quad \text{and} \quad \left|-\tau_{2r+2}^2(f)\right| \le |\Delta_{2r}(f)|. \qquad \square$$

4. Some inequalities related to Bullen-Simpson formulae of Euler type. In this section we use Bullen-Simpson formulae of Euler type established in Lemma 1 to estimate the absolute value of difference between the absolute value of error in the dual Simpson's quadrature rule and the absolute value of error in the Simpson's quadrature rule. We do this by proving a number of inequalities for various classes of functions.

First, let us denote

$$R_S := \int_0^1 f(t)\, dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

and

$$R_D := \int_0^1 f(t)\, dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right].$$

By the triangle inequality we have

$$||R_D| - |R_S|| \le |R_D + R_S|.$$

Now, if we define $R := R_D + R_S$, then

(4.1)
$$\begin{aligned}\frac{R}{2} &= \int_0^1 f(t)\,dt - \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] \\ &= \int_0^1 f(t)\,dt - D(0,1).\end{aligned}$$

Theorem 4. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geq 1$.*

If $n = 2r - 1$, $r \geq 2$, then

$$\begin{aligned}&\left|\int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f)\right| \\ &\qquad \leq \frac{1}{6(2r-1)!}\int_0^1 |G_{2r-1}(t)|\,dt \cdot L \\ &\qquad = \frac{8 \cdot 2^{-2r}(1 - 2^{-2r})}{3(2r)!}\,|B_{2r}| \cdot L.\end{aligned} \tag{4.2}$$

If $n = 2r$, $r \geq 2$, then

$$\begin{aligned}&\left|\int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f)\right| \\ &\qquad \leq \frac{1}{6(2r)!}\int_0^1 |F_{2r}(t)|,dt \cdot L \\ &\qquad = \frac{2^{-2r}(1 - 4 \cdot 2^{-2r})}{3(2r)!}\,|B_{2r}| \cdot L\end{aligned} \tag{4.3}$$

and also

$$\begin{aligned}&\left|\int_0^1 f(t)\,dt - D(0,1) - T_r(f)\right| \\ &\qquad \leq \frac{1}{6(2r)!}\int_0^1 |G_{2r}(t)|\,dt \cdot L \\ &\qquad \leq \frac{2 \cdot 2^{-2r}(1 - 4 \cdot 2^{-2r})}{3(2r)!}\,|B_{2r}| \cdot L.\end{aligned} \tag{4.4}$$

Proof. For any integrable function $\Phi : [0,1] \to \mathbf{R}$, we have

$$\left| \int_0^1 \Phi(t)\, df^{(n-1)}(t) \right| \le \int_0^1 |\Phi(t)|\; dt \cdot L, \tag{4.5}$$

since $f^{(n-1)}$ is L-Lipschitzian function. Applying (4.5) with $\Phi(t) = G_{2r-1}(t)$, we get

$$\left| \frac{1}{6(2r-1)!} \int_0^1 G_{2r-1}(t)\, df^{(2r-2)}(t) \right| \le \frac{1}{6(2r-1)!} \int_0^1 |G_{2r-1}(t)|\; dt \cdot L.$$

Applying the above inequality and the identity (2.3), we get the inequality in (4.2). Similarly, we can apply the inequality (4.5) with $\Phi(t) = F_{2r}(t)$ and again the identity (2.3) to get the inequality in (4.3). Finally, applying (4.5) with $\Phi(t) = G_{2r}(t)$ and the identity (2.2), we get the first inequality in (4.4). The equalities in (4.2) and (4.3) and the second inequality in (4.4) follow from Corollary 2. □

Corollary 3. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \ge 1$.*

If $n = 2r-1$, $r \ge 2$, then

$$\begin{aligned} |R - 2T_{r-1}(f)| &\le \frac{1}{3(2r-1)!} \int_0^1 |G_{2r-1}(t)|\, dt \cdot L \\ &= \frac{16 \cdot 2^{-2r}(1-2^{-2r})}{3(2r)!} |B_{2r}| \cdot L. \end{aligned} \tag{4.6}$$

If $n = 2r$, $r \ge 2$, then

$$\begin{aligned} |R - 2T_{r-1}(f)| &\le \frac{1}{3(2r)!} \int_0^1 |F_{2r}(t)|\, dt \cdot L \\ &= \frac{2 \cdot 2^{-2r}(1-4 \cdot 2^{-2r})}{3(2r)!} |B_{2r}| \cdot L \end{aligned} \tag{4.7}$$

and also

$$\begin{aligned} |R - 2T_{r}(f)| &\le \frac{1}{3(2r)!} \int_0^1 |G_{2r}(t)|\, dt \cdot L \\ &\le \frac{4 \cdot 2^{-2r}(1-4 \cdot 2^{-2r})}{3(2r)!} |B_{2r}| \cdot L. \end{aligned} \tag{4.8}$$

Proof. Follows from Theorem 4 and (4.1). □

Corollary 4. *Let $f : [0,1] \to \mathbf{R}$ be a given function.*
If f'' is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{4608}\,L, \quad |R| \le \frac{1}{2304}\,L.$$

If f''' is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{46080}\,L, \quad |R| \le \frac{1}{23040}\,L.$$

Proof. The first pair of inequalities follow from (4.2) and (4.6) with $r = 2$, while the second pair follow from (4.3) and (4.7) with $r = 2$. □

Remark 4. If f is L-Lipschitzian on $[0,1]$, then, as above,

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{6} \int_0^1 |G_1(t)|\,dt \cdot L.$$

Since $\int_0^1 |G_1(t)|\,dt = 5/12$, we get

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{5}{72} \cdot L \quad \text{and} \quad |R| \le \frac{5}{36} \cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{12} \int_0^1 |F_2(t)|\,dt \cdot L.$$

Since $\int_0^1 |F_2(t)|\,dt = 1/27$, we get

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{324} \cdot L \quad \text{and} \quad |R| \le \frac{1}{162} \cdot L.$$

Remark 5. Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is bounded on $[0,1]$ for some $n \geq 1$. In this case we have for all $t, s \in [0,1]$,

$$\left| f^{(n-1)}(t) - f^{(n-1)}(s) \right| \leq \|f^{(n)}\|_\infty \cdot |t-s| \,,$$

which means that $f^{(n-1)}$ is an $\|f^{(n)}\|_\infty$-Lipschitzian function on $[0,1]$. Therefore, the inequalities established in Theorem 4 hold with $L = \|f^{(n)}\|_\infty$. Consequently, under appropriate assumptions on f, the inequalities from Corollary 4 and Remark 4 hold with $L = \|f'\|_\infty$, $\|f''\|_\infty$, $\|f'''\|_\infty$, $\|f^{(4)}\|_\infty$.

Theorem 5. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \geq 1$.*

If $n = 2r-1$, $r \geq 2$, then

$$\begin{aligned} &\left| \int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f) \right| \\ &\qquad \leq \frac{1}{6(2r-1)!} \max_{t\in[0,1]} |G_{2r-1}(t)| \cdot V_0^1(f^{(2r-2)}). \end{aligned} \tag{4.9}$$

If $n = 2r$, $r \geq 2$, then

$$\begin{aligned} &\left| \int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f) \right| \\ &\qquad \leq \frac{1}{6(2r)!} \max_{t\in[0,1]} |F_{2r}(t)| \cdot V_0^1(f^{(2r-1)}) \\ &\qquad = \frac{2 \cdot 2^{-2r}(1-2^{-2r})}{3(2r)!} |B_{2r}| \cdot V_0^1(f^{(2r-1)}). \end{aligned} \tag{4.10}$$

Also, we have

$$\begin{aligned} \left| \int_0^1 f(t)\,dt - D(0,1) - T_r(f) \right| &\leq \frac{1}{6(2r)!} \max_{t\in[0,1]} |G_{2r}(t)| \cdot V_0^1(f^{(2r-1)}) \\ &= \frac{2^{-2r}(2 \cdot 2^{-2r}+1)}{3(2r)!} |B_{2r}| \cdot V_0^1(f^{(2r-1)}). \end{aligned} \tag{4.11}$$

Here $V_0^1(f^{(n-1)})$ denotes the total variation of $f^{(n-1)}$ on $[0,1]$.

Proof. If $\Phi : [0,1] \to \mathbf{R}$ is bounded on $[0,1]$ and the Riemann-Stieltjes integral $\int_0^1 \Phi(t)\, df^{(n-1)}(t)$ exists, then

$$\left| \int_0^1 \Phi(t)\, df^{(n-1)}(t) \right| \le \max_{t\in[0,1]} |\Phi(t)| \cdot V_0^1(f^{(n-1)}). \tag{4.12}$$

We apply this estimate to $\Phi(t) = G_{2r-1}(t)$ to obtain

$$\left| \frac{1}{6(2r-1)!} \int_0^1 G_{2r-1}(t)\; df^{(2r-2)}(t) \right| \le \frac{1}{6(2r-1)!} \max_{t\in[0,1]} |G_{2r-1}(t)| \cdot V_0^1(f^{(2r-2)}),$$

which is just the inequality (4.9) because of the identity (2.3). Similarly, we can apply the estimate (4.12) with $\Phi(t) = F_{2r}(t)$ and use the identity (2.3) and Corollary 1 to obtain (4.10). Finally, (4.11) follows from (4.12) with $\Phi(t) = G_{2r}(t)$, the identity (2.2) and Corollary 1. □

Corollary 5. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \ge 1$.*

If $n = 2r-1$, $r \ge 2$, then

$$|R - 2T_{r-1}(f)| \le \frac{1}{3(2r-1)!} \max_{t\in[0,1]} |G_{2r-1}(t)| \cdot V_0^1(f^{(2r-2)}). \tag{4.13}$$

If $n = 2r$, $r \ge 2$, then

$$\begin{aligned} |R - 2T_{r-1}(f)| &\le \frac{1}{3(2r)!} \max_{t\in[0,1]} |F_{2r}(t)| \cdot V_0^1(f^{(2r-1)}) \\ &= \frac{4 \cdot 2^{-2r}(1-2^{-2r})}{3(2r)!} |B_{2r}| \cdot V_0^1(f^{(2r-1)}). \end{aligned} \tag{4.14}$$

Also, we have

$$\begin{aligned} |R - 2T_r(f)| &\le \frac{1}{3(2r)!} \max_{t\in[0,1]} |G_{2r}(t)| \cdot V_0^1(f^{(2r-1)}) \\ &= \frac{2 \cdot 2^{-2r}(2 \cdot 2^{-2r}+1)}{3(2r)!} |B_{2r}| \cdot V_0^1(f^{(2r-1)}). \end{aligned} \tag{4.15}$$

Proof. Follows from Theorem 5 and (4.1). □

Corollary 6. *Let $f:[0,1]\to\mathbf{R}$ be a given function.*

If f'' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{1}{2592}\,V_0^1(f''), \quad |R| \le \frac{1}{1296}\,V_0^1(f'').$$

If f''' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{1}{18432}\,V_0^1(f'''), \quad |R| \le \frac{1}{9216}\,V_0^1(f''').$$

Proof. From explicit expressions (3.3), we get

$$\max_{t\in[0,1]} |G_3(t)| = \frac{1}{72},$$

so that the first pair of inequalities follow from (4.9) and (4.13) with $r=2$. The second pair of inequalities follow from (4.10) and (4.14) with $r=2$. □

Remark 6. If f is a continuous function of bounded variation on $[0,1]$, then, as above

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{1}{6}\max_{t\in[0,1]}|G_1(t)|\cdot V_0^1(f).$$

Since $\max_{t\in[0,1]}|G_1(t)|=1$, we get

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{1}{6}\cdot V_0^1(f) \quad \text{and} \quad |R| \le \frac{1}{3}\cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{1}{12}\max_{t\in[0,1]}|F_2(t)|\cdot V_0^1(f').$$

Since $\max_{t\in[0,1]} |F_2(t)| = 1/8$, we get

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{96} \cdot V_0^1(f) \text{ and } |R| \le \frac{1}{48} \cdot V_0^1(f).$$

Remark 7. Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)} \in L_1[0,1]$ for some $n \ge 1$. In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, and we have

$$V_0^1\big(f^{(n-1)}\big) = \int_0^1 \left| f^{(n)}(t) \right| dt = \left\| f^{(n)} \right\|_1,$$

Therefore, the inequalities established in Theorem 5 hold with $\|f^{(n)}\|_1$ in place of $V_0^1(f^{(n-1)})$. However, a similar observation can be made for the results of Corollary 6 and Remark 6.

Theorem 6. *Assume that (p,q) is a pair of conjugate exponents, that is, $1 < p, q < \infty$, $1/p + 1/q = 1$ or $p = \infty$, $q = 1$. Let $f : [0,1] \to \mathbf{R}$ be an R-integrable function such that $f^{(n)} \in L_p[0,1]$ for some $n \ge 1$.*

If $n = 2r-1$, $r \ge 1$, then

$$\left| \int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f) \right| \le K(2r-1,p) \left\| f^{(2r-1)} \right\|_p. \tag{4.16}$$

If $n = 2r$, $r \ge 1$, then

$$\left| \int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f) \right| \le K^*(2r,p) \left\| f^{(2r)} \right\|_p. \tag{4.17}$$

Also, we have

$$\left| \int_0^1 f(t)\,dt - D(0,1) - T_r(f) \right| \le K(2r,p) \left\| f^{(2r)} \right\|_p. \tag{4.18}$$

Here

$$K(n,p) = \frac{1}{6n!} \left[\int_0^1 |G_n(t)|^q \, dt \right]^{1/q},$$

$$K^*(n,p) = \frac{1}{6n!} \left[\int_0^1 |F_n(t)|^q \, dt \right]^{1/q}.$$

Proof. Applying the Hölder inequality, we have

$$\begin{aligned}\left|\frac{1}{6(2r-1)!}\int_0^1 G_{2r-1}(t)\, f^{(2r-1)}(t)\, dt\right| \\ &\le \frac{1}{6(2r-1)!}\left[\int_0^1 |G_{2r-1}(t)|^q\, dt\right]^{1/q}\cdot \left\|f^{(2r-1)}\right\|_p \\ &= K(2r-1,p)\left\|f^{(2r-1)}\right\|_p\end{aligned}$$

The above estimate is just (4.16), by the identity (2.4). The inequalities (4.17) and (4.18) are obtained in the same manner from (2.3) and (2.2), respectively. □

Corollary 7. *Assume (p,q) is a pair of conjugate exponents, that is, $1 < p, q < \infty$, $1/p + 1/q = 1$ or $p = \infty$, $q = 1$. Let $f : [0,1] \to \mathbf{R}$ be an R-integrable function such that $f^{(n)} \in L_p[0,1]$ for some $n \ge 1$.*

If $n = 2r-1$, $r \ge 1$, then

$$|R - 2T_{r-1}(f)| \le 2K(2r-1,p)\left\|f^{(2r-1)}\right\|_p.$$

If $n = 2r$, $r \ge 1$, then

$$|R - 2T_{r-1}(f)| \le 2K^*(2r,p)\left\|f^{(2r)}\right\|_p.$$

Also, we have

$$|R - 2T_r(f)| \le 2K(2r,p)\left\|f^{(2r)}\right\|_p.$$

Proof. Follows from Theorem 6 and (4.1). □

Remark 8. Note that $K^*(1,p) = K(1,p)$ for $1 < p \le \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \le \infty$, we can easily calculate $K(1,p)$. We get

$$K(1,p) = \frac{1}{6}\left[\frac{2+2^{-q}}{3(1+q)}\right]^{1/q}, \quad 1 < p \le \infty.$$

At the end of this section we prove an interesting Grüss type inequality related to Bullen-Simpson's identity (2.2). To do this we use the following variant of the key technical result from the recent paper [**11**]:

Lemma 5. *Let $F, G : [0,1] \to \mathbf{R}$ be two integrable functions. If, for some constants $m, M \in \mathbf{R}$,*

$$m \le F(t) \le M, \quad 0 \le t \le 1 \quad \text{and} \quad \int_0^1 G(t)\,dt = 0,$$

then

$$\left| \int_0^1 F(t)G(t)\,dt \right| \le \frac{M-m}{2} \int_0^1 |G(t)|\,dt. \tag{4.19}$$

Theorem 7. *Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \ge 1$. Assume that*

$$m_n \le f^{(n)}(t) \le M_n, \quad 0 \le t \le 1,$$

for some constants m_n and M_n. Then

$$\left| \int_0^1 f(t)\,dt - D(0,1) - T_k(f) \right| \le \frac{1}{12\,(n!)}\, C_n (M_n - m_n), \tag{4.20}$$

where $k = [n/2]$ and

$$C_n = \int_0^1 |G_n(t)|\,dt, \quad n \ge 1.$$

Moreover, if $n = 2r-1$, $r \ge 2$, then

$$\left| \int_0^1 f(t)\,dt - D(0,1) - T_{r-1}(f) \right| \le \frac{4 \cdot 2^{-2r}\left(1 - 2^{-2r}\right)}{3\,(2r)!}\, |B_{2r}|\,(M_{2r-1} - m_{2r-1}). \tag{4.21}$$

Proof. We can rewrite the identity (2.2) in the form

$$\int_0^1 f(t)\,dt - D(0,1) - T_k(f) = \frac{1}{6(n!)} \int_0^1 F(t)G(t)\,dt, \tag{4.22}$$

where

$$F(t) = f^{(n)}(t), \quad G(t) = G_n(t), \quad 0 \le t \le 1.$$

In [**7**, Lemma 2] it was proved that, for all $n \ge 1$ and for every $\gamma \in \mathbf{R}$,

$$\int_0^1 B_n^*(\gamma - t)\,dt = 0,$$

so that we have

$$\begin{aligned}&\int_0^1 G(t)\,dt \\ &= \int_0^1 \left[B_n(1-t) + 2B_n^*\left(\frac{1}{4} - t\right) + B_n^*\left(\frac{1}{2} - t\right) + 2B_n^*\left(\frac{3}{4} - t\right)\right] dt \\ &= 0.\end{aligned}$$

Thus, we can apply (4.19) to the integral in the righthand side of (4.22) and (4.20) follows immediately. The inequality (4.21) follows from (4.20) and Corollary 2. □

Remark 9. For $n = 1$ and $n = 2$ we have already calculated

$$C_1 = \int_0^1 |G_1(t)|\,dt = \frac{5}{12}, \quad C_2 = \int_0^1 |G_2(t)|\,dt = \frac{1}{27},$$

so that we have

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{5}{144}\,(M_1 - m_1)$$

and

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le \frac{1}{648}\,(M_2 - m_2).$$

For $n = 3$ we apply (4.21) with $r = 2$ to get the inequality

$$\left| \int_0^1 f(t)\,dt - D(0,1) \right| \le \frac{1}{9216}\,(M_3 - m_3).$$

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