# A SHORT PROOF OF A CHARACTERIZATION
# OF INNER FUNCTIONS IN TERMS OF THE
# COMPOSITION OPERATORS THEY INDUCE

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ABSTRACT. The paper contains a new proof for the sufficiency in Joel H. Shapiro's recent characterization of inner functions saying that an analytic self-map $\varphi$ of the open unit disk is an inner function if and only if the essential norm of the composition operator of symbol $\varphi$ is equal to $\sqrt{(1 + |\varphi(0)|)/(1 - |\varphi(0)|)}$. The main ingredient in the proof is a formula for the essential norm of a composition operator in terms of Aleksandrov measures obtained by Cima and Matheson. The necessity was originally proved by Joel Shapiro in 1987. A short proof of the necessity, by Aleksandrov measure techniques, was obtained by Jonathan E. Shapiro in 1998.

For each holomorphic self-map $\varphi : \mathbf{U} \to \mathbf{U}$ of the open unit disk $\mathbf{U}$, the composition operator $C_\varphi$ of symbol $\varphi$ is defined as follows $C_\varphi f = f \circ \varphi$, $f \in H^2$. In this definition $H^2$ is the Hilbert Hardy space on $\mathbf{U}$, i.e., the set of all analytic functions on $\mathbf{U}$ with square summable Taylor coefficients. It is well known that each such $\varphi$ induces a bounded composition operator $C_\varphi$ on $H^2$. Recently Joel Shapiro obtained the following characterization of inner functions [**7**], (that is of self-maps $\varphi$ whose radial limit function is unimodular almost everywhere on the unit circle $\mathbf{T}$).

**Theorem 1.** *The function* $\varphi : \mathbf{U} \to \mathbf{U}$ *is inner if and only if*

$$(1) \qquad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}},$$

*where* $\|C_\varphi\|_e$ *denotes the essential norm of* $C_\varphi$.

The necessity in the previous equivalence was originally proved in [**6**]. Jonathan Shapiro [**5**] observed that Aleksandrov measures can be used

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to give a very short proof of the aforementioned implication. In this note we provide a new, short proof of the sufficiency in Theorem 1. Our approach is also based on Aleksandrov measures and a formula obtained by Cima and Matheson [**1**]. For each $\alpha \in \mathbf{T}$ the Aleksandrov measure of index $\alpha$ of $\varphi$ is the positive, finite measure $\tau_\alpha$ on $\mathbf{T}$ whose Poisson integral equals the positive harmonic function

$$(2) \qquad F_\alpha(z) = \operatorname{Re} \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2}, \quad z \in \mathbf{U}.$$

The fact that $F_\alpha$ is positive and harmonic is a direct consequence of its definition, while the existence and uniqueness of $\tau_\alpha$ is an immediate consequence of the well-known Herglotz theorem [**2**, Theorem 11.19]. Another well-known fact about the Poisson integral of a measure on $\mathbf{T}$ is that its radial limit function is the absolutely continuous part in its Lebesgue decomposition with respect to the normalized arc-length measure $m$ on $\mathbf{T}$. Therefore, if we denote by $\sigma_\alpha$ the singular part of $\tau_\alpha$, we obtain the equality

$$(3) \qquad d\tau_\alpha(\xi) = \frac{1 - |\varphi(\xi)|^2}{|\alpha - \varphi(\xi)|^2} \, dm(\xi) + d\sigma_\alpha(\xi).$$

The main result in [**1**] is the following formula

$$(4) \qquad \|C_\varphi\|_e = \sup_{\alpha \in \mathbf{T}} \sqrt{\|\sigma_\alpha\|}.$$

Based on it and the rather evident equality

$$(5) \qquad \sup_{\alpha \in \mathbf{T}} \operatorname{Re} \frac{\alpha + z}{\alpha - z} = \frac{1 + |z|}{1 - |z|},$$

valid for each $z \in \mathbf{U}$, we are able to give the following short proof to Theorem 1.

*Proof.* If $\varphi$ is inner, then by (3) $\tau_\alpha = \sigma_\alpha$ for any $\alpha \in \mathbf{T}$. Therefore,

$$\|\sigma_\alpha\| = \|\tau_\alpha\| = F_\alpha(0) = \operatorname{Re} \frac{\alpha + \varphi(0)}{\alpha - \varphi(0)}.$$

By formula (4) and equality (5) one deduces that (1) holds. This implication is proved by essentially this method in [**5**]. We included it for the sake of rendering this note self contained. Our original contribution is giving a short proof to the reverse implication, (which was only recently proved, [**7**]).

Assume (1) holds. For the beginning, we will work under the particular assumption $\varphi(0) = 0$. By (4) there exists a sequence $(\alpha_n)$ in $\mathbf{T}$ such that

$$\|\sigma_{\alpha_n}\|^2 \longrightarrow \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} = 1.$$

Since $\mathbf{T}$ is compact we will not reduce the generality by assuming that $(\alpha_n)$ is convergent to some $\alpha_0 \in \mathbf{T}$. Since

$$\|\tau_{\alpha_n}\| = \operatorname{Re} \frac{\alpha_n + \varphi(0)}{\alpha_n - \varphi(0)} = 1, \quad n = 1, 2, \dots,$$

one deduces by (3) that

$$\lim_{n \to \infty} \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \, dm(\xi) = 0.$$

Now, by applying the classical Fatou lemma in integration theory, [**2**, Lemma 1.28], to the sequence

$$\left( \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \right)$$

one obtains

$$0 \le \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_0 - \varphi(\xi)|^2} \, dm(\xi) \le \liminf_{n \to \infty} \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \, dm(\xi) = 0.$$

Hence $|\varphi(\xi)| = 1$ $m$-almost everywhere on $\mathbf{T}$, i.e., $\varphi$ is an inner function. To finish the proof, assume that

$$\varphi(0) = p \ne 0 \quad \text{and} \quad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

Denote by $\alpha_p$ the self-inverse, disk automorphism

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z},$$

and set $\psi = \alpha_p \circ \varphi$. Since $\alpha_p$ is inner we have that $\|C_{\alpha_p}\|_e = \sqrt{(1+|p|)/(1-|p|)} = \|C_\varphi\|_e$. Observe that $\psi(0) = 0$, $C_\psi = C_\varphi C_{\alpha_p}$ and $C_\psi C_{\alpha_p} = C_\varphi$, since $\alpha_p$ is equal to its inverse. Therefore one obtains

$$\|C_\varphi\|_e \leq \|C_\psi\|_e \|C_{\alpha_p}\|_e.$$

We deduce $\|C_\psi\|_e \geq 1$; hence, $\|C_\psi\|_e = 1$ because composition operators whose symbol fixes 0 are contractions. By what we proved for symbols fixing 0, it follows that $\psi$ is an inner function and hence $\varphi$ is also inner because $\alpha_p$ is a disk automorphism.    □

As a final comment we would like to note that, once the essential norm characterization of inner functions in Theorem 1 is proved, one can use it to obtain an operator norm characterization, see [**7**].

## REFERENCES

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