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ON THE GAUSS MAP OF RULED SURFACES IN MINKOWSKI SPACE

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ABSTRACT. In this paper, we study some characterization of ruled surfaces in Minkowski space in terms of the Gauss map. We give new examples of cylindrical and noncylindrical ruled surfaces in a 4-dimensional Minkowski space with the pointwise 1-type Gauss map.

1. Introduction. Since the late 1970's when B.-Y. Chen introduced the theory of finite type immersion, its study has been extended to the submanifolds of pseudo-Euclidean spaces, namely a pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbf{E}_s^m$ with signature $(s, m-s)$ is said to be of *finite type* if its position vector field x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is,

$$x = x_0 + x_1 + x_2 + \cdots + x_k,$$

where x_0 is a constant map, $x_1, \ldots, x_k$ nonconstant maps such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbf{R}$, $i = 1, 2, \ldots, k$, [**3, 7**]. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, then M is said to be of *k-type*. Similarly, we can apply this notion to a smooth map, for example, the Gauss map G that is one of the most natural smooth maps on an n-dimensional pseudo-Riemannian submanifold M of $\mathbf{E}_s^m$. Thus, the Gauss map G is said to be of *finite type* if G is a finite sum of $\mathbf{E}_s^m$- valued eigenfunctions of Δ [**2, 4**]. We also similarly define the notion of k-type Gauss map on M as usual.

There are many examples of submanifolds in the Minkowski space $\mathbf{E}_1^m$ with finite type Gauss map, for example, B-scrolls in $\mathbf{E}_1^3$, several kinds of cylinders and extended B-scrolls in $\mathbf{E}_1^4$ are those with 1-type

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Gauss map [**1, 10, 12**]. Recently, D.-S. Kim and the authors [**8, 9**] studied and characterized ruled surfaces with finite type Gauss map. Also, the many geometers tried to study the submanifolds of Euclidean or pseudo-Euclidean spaces satisfying a differential equation $\Delta G = \Lambda G$, where Δ denotes the Laplacian operator acting on the set of smooth functions on the submanifolds and Λ an endomorphism of the ambient manifold [**5**]. In particular, Alías et al. [**1**] proved that the so-called B-scrolls over light-like curves are the only null scrolls in $\mathbf{E}^3_1$ satisfying $\Delta G = \Lambda G$. Related to such matters, we may pose the following natural question: *"Can we completely characterize null scrolls in arbitrary dimensional Minkowski space with $\Delta G = \Lambda G$?"*

In this article, we study ruled surfaces including the null scrolls in an m-dimensional Minkowski space $\mathbf{E}^m_1$, and we give a complete classification theorem of null scrolls satisfying $\Delta G = \Lambda G$ in $\mathbf{E}^m_1$. Also, we characterize ruled surfaces in terms of the notion of pointwise 1-type Gauss map as well as we give some new examples of cylindrical and noncylindrical ruled surfaces. The authors proved the following theorems which will be used later.

Theorem A [**8**]. *Let M be a ruled surface with* 1*-type Gauss map in $\mathbf{E}^m_1$ if and only if M is an open part of one of the following*:

(1) *A Euclidean plane, a Minkowski plane, a hyperbolic cylinder $H^1 \times \mathbf{R}^1$, a Lorentz circular cylinder $\mathbf{R}^1_1 \times S^1$, a Lorentz hyperbolic cylinder $S^1_1 \times \mathbf{R}^1$ and a B-scroll if $m = 3$.*

(2) *A Euclidean plane, a Minkowski plane, $S^1 \times \mathbf{R}^1$, $H^1 \times \mathbf{R}^1$, $\mathbf{R}^1_1 \times S^1$, $S^1_1 \times \mathbf{R}^1$, helical cylinders and an extended B-scroll if $m \geqq 4$.*

Theorem B [**9**]. *Let M be a null scroll in an m-dimensional Minkowski space $\mathbf{E}^m_1$. Then, the following are equivalent.*

(1) *The Gauss map is of finite type.*

(2) *The Gauss map is of either* 1*-type or null* 2*-type.*

(3) *M is an open part of generalized B-scroll in $\mathbf{E}^m_1$.*

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless stated otherwise.

2. Preliminaries. Let $\mathbf{E}_s^m$ be an m-dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor $\tilde{g}$ in $\mathbf{E}_s^m$ has the form

$$\tilde{g} = -\sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{m} dx_i^2,$$

where $(x_1, x_2, \dots, x_m)$ is a standard rectangular coordinate system in $\mathbf{E}_s^m$. In particular, for $m \geq 2$, $\mathbf{E}_1^m$ is called *Minkowski m-space.*

Let $x : M \to \mathbf{E}_s^m$ be an isometric immersion of an oriented n-dimensional pseudo-Riemannian submanifold M into $\mathbf{E}_s^m$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection on $\mathbf{E}_s^m$ and ∇ the induced connection on M. Then, the Gauss and Weingarten formulas are given by respectively

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X V = -A_V X + D_X V \tag{2.2}$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V that is related with h by

$$\langle h(X, Y), V\rangle = \langle A_V X, Y\rangle.$$

If we define a covariant differentiation $\overline{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^{\perp}M$ of M by

$$(\overline{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

then we have the Codazzi equation

$$(\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla}_Y h)(X, Z) \tag{2.3}$$

for all tangent vector fields X, Y and Z of M.

From now on, a submanifold in $\mathbf{E}_s^m$ always means pseudo-Riemannian.

Let us now define the Gauss map G of a submanifold M into $G(n, m)$ in $\wedge^n \mathbf{E}_s^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbf{E}_s^m$ and $\wedge^n \mathbf{E}_s^m$ is the vector space obtained by the exterior product of n vectors in $\mathbf{E}^m_s$. In a natural way, for some positive integer k we can identify $\wedge^n\mathbf{E}^m_s$ with some pseudo-Euclidean space $\mathbf{E}^N_k$ where $N = \binom{m}{n}$. Let $e_1, e_2, \dots, e_m$ be an adapted local orthogonal frame in $\mathbf{E}^m_s$ such that $e_1, e_2, \dots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ normal to M. The map $G : M \to G(n,m) \subset \mathbf{E}^N_k$ defined by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ is called the *Gauss map* of M that is a smooth map which carries a point p in M into the oriented n-plane in $\mathbf{E}^m_s$ obtained from the parallel translation of the tangent space of M at p in $\mathbf{E}^m_s$.

For two vectors $e_{i_1} \wedge \cdots \wedge e_{i_m}$ and $f_{j_1} \wedge \cdots \wedge f_{j_m}$ of $\mathbf{E}^N_k$, we can define an *indefinite inner product* $\langle , \rangle$ by

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_m}, f_{j_1} \wedge \cdots \wedge f_{j_m} \rangle = \det(\langle e_{i_l}, f_{j_k} \rangle)$$

on $G(n,m) \subset \mathbf{E}^N_k$.

It is well known that in terms of local coordinates $\{x_i\}$ of M the Laplacian can be written as:

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|}\, g^{ij}\, \frac{\partial}{\partial x^j}\right), \tag{2.4}$$

where $\mathcal{G} = \det(g_{ij}), (g^{ij}) = (g_{ij})^{-1}$ and (g_{ij}) are the components of the metric of M with respect to $\{x_i\}$.

Now, we define a ruled surface M in $\mathbf{E}^m_1$. Let I and J be open intervals containing 0 in the real line $\mathbf{R}$. Let $\alpha = \alpha(s)$ be a curve on $\mathbf{E}^m_1$ defined on J and $\beta = \beta(s)$ a transversal vector field along α, that is, $\alpha'(s) \wedge \beta(s) \neq 0$ for every $s \in J$. Then, a ruled surface M is defined by the parametrization given as follows:

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I.$$

For such a ruled surface, α and β are called the *base curve* and β the *director curve*. In particular, if β is constant, the ruled surface is said to be *cylindrical*, and if it is not so, it is called *noncylindrical*. If the base curve α is nonnull, i.e., space-like or time-like, the director curve β can naturally be chosen so that it is orthogonal to α. Furthermore, in this case, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows.

If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is nonnull or null, respectively. When β is time-like, β' must be space-like by causal character. In this case, M said to be of type M_+^3. For the ruled surface of type M_-, it is also said to be of type M_-^1 or M_-^2 if β' is nonnull or null, respectively. Note that in the case of type M_- the director curve β is always space-like. The ruled surface of type M_+^1 or M_+^2, respectively M_+^3, M_-^1 or M_-^2, is clearly space-like, respectively time-like, [**10, 12**]. But, if the base curve α is a light-like curve and the vector field β along α is a light-like vector field, then the ruled surface M is called a *null scroll*. In particular, a null scroll with the Cartan frame in $\mathbf{E}_1^3$ is said to be a *B-scroll* [**1, 6**]. It is also a time-like surface. The authors [**8, 9**] defined the extended B-scrolls and the generalized B-scrolls in arbitrary dimensional Minkowski spaces.

3. Examples.

Example 1. Let M be a space-like ruled surface in $\mathbf{E}_1^4$ with parametrization $x(s,t) = (s^2, s^2, s, t)$. Then, it is cylindrical and the mean curvature vector field H is parallel satisfying $\Delta G = 0$. We call this ruled surface the *quadric ruled surface of the first kind*.

Example 2. Let M be a space-like ruled surface in $\mathbf{E}_1^4$ with parametrization $x(s,t) = ((s^2/2) + ts, (s^2/2) + ts, s, t)$. Then, it is noncylindrical with the light-like parallel mean curvature vector field H is parallel satisfying $\Delta G = fG$ for some function f. We call this ruled surface the *quadric ruled surface of the second kind*.

Example 3. Let M be a space-like ruled surface in $\mathbf{E}_1^4$ with parametrization $x(s,t) = (as, as, (t+u)\cos s, (t+u)\sin s)$ where $u^2 = a^2$ for some $a > 0$, which lies in $\mathbf{E}_1^4$. It is noncylindrical of type M_-^2. We call this ruled surface the *helicoid of the fourth kind* which is noncylindrical.

Example 4. Let M be a null scroll generated by a light-like curve $\alpha=\alpha(s)$ in $\mathbf{E}^m_1$ and $\beta=\beta(s)$ a light-like vector field along α, which is up to congruences parametrized by

$$x=x(s,t)=\alpha(s)+t\beta(s), \qquad s\in J, \quad t\in I$$

such that $\langle\alpha',\alpha'\rangle=0$, $\langle\beta,\beta\rangle=0$, $\langle\alpha',\beta\rangle=-1$, where I and J are some open intervals. Furthermore, by appropriate change of parameter, we may assume $\langle\alpha',\beta'\rangle=0$, which is equivalent to choose α as a light-like geodesic of M.

Let $\alpha=\alpha(s)$ be a light-like curve in $\mathbf{E}^m_1$ and let $A(s), B(s), C_1(s),\cdots,$ $C_{m-2}(s)$ be a null frame along α satisfying

$$\begin{aligned}&\langle A,A\rangle=\langle B,B\rangle=\langle A,C_i\rangle=\langle B,C_i\rangle=0,\\ &\langle A,B\rangle=-1, \quad \langle C_i,C_j\rangle=\delta_{ij}, \quad \alpha'(s)=A(s)\end{aligned}$$

for $1\le i,\, j\le m-2$. Let $X(s)$ be the matrix $\left(\, A(s)\ B(s)\ C_1(s)\ \cdots\ C_{m-2}(s)\,\right)$ consisting of column vectors of $A(s), B(s), C_1(s),\cdots,C_{m-2}(s)$ with respect to the standard coordinate system in $\mathbf{E}^m_1$. We then have

$$X^t(s)EX(s)=T,$$

where $E=\operatorname{diag}(-1,1,\cdots 1,1)$ and

$$T=\begin{pmatrix} 0 & -1 & 0 & & \\ -1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & & \\ & \dots & & \ddots & \\ 0 & & & & 1\end{pmatrix},$$

where $X^t(s)$ denotes the transpose of $X(s)$.

Consider a system of ordinary differential equations

$$X'(s)=X(s)M(s), \tag{3.1}$$

where

$$M(s)=\begin{pmatrix} 0 & 0 & -a & 0 & \cdots & 0\\ 0 & 0 & -k_1(s) & -k_2(s) & \cdots & -k_{m-2}(s)\\ -k_1(s) & -a & 0 & -w_2(s) & \cdots & -w_{m-2}(s)\\ -k_2(s) & 0 & w_2(s) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ -k_{m-2}(s) & 0 & w_{m-2}(s) & 0 & \cdots & 0\end{pmatrix},$$

where a is a constant and $k_1, k_2, \cdots, k_{m-2}$, $w_1, w_2, \cdots, w_{m-2}$ are some smooth functions.

For a given initial condition $X(0) = \left(\begin{smallmatrix} A(0) & B(0) & C_1(0) & \cdots & C_{m-2}(0) \end{smallmatrix} \right)$ satisfying $X^t(0)EX(0) = T$, there is a unique solution to equation (3.1). Since T is symmetric and MT is skew-symmetric, $d(X^t(s)EX(s))/ds = 0$ and hence we have

$$X^t(s)EX(s) = T.$$

Therefore, $A(s), B(s), C_1(s), \cdots, C_{m-2}(s)$ form a null frame along a light-like curve α in $\mathbf{E}_1^m$. Let $x(s,t) = \alpha(s) + tB(s)$. Then, it defines a time-like surface M in $\mathbf{E}_1^m$, which is called the *generalized B-scroll* in $\mathbf{E}_1^m$ [**9**]. In particular, if $w_1, w_2, \cdots, w_{m-2}$ are identically zero, then it is just an extended B-scroll defined in [**8**]. The Laplacian Δ of M is given by

$$\Delta f = 2f_{st} + 2ta^2 f_t + t^2 a^2 f_{tt}$$

for a function f defined on M. Let H be the mean curvature vector field of M defined by $H = \operatorname{tr} h/2$. The Beltrami equation $\Delta x = -2H$ gives

$$H(s,t) = -ta^2 B(s) + aC_1(s),$$

which implies

$$\Delta H = 2a^2 H.$$

And, the shape operator A_H associated with the mean curvature vector field H has the form

$$\begin{pmatrix} a^2 & 0 \\ ak_1 & a^2 \end{pmatrix}$$

in the coordinate frame $\{x_s, x_t\}$. The Gauss map G of M satisfies

$$\Delta^2 G = 2a^2 \Delta G.$$

Remark. 1. A generalized B-scroll cannot be of 1-type unless $w_j(s) \equiv 0$ for all $j = 2, \ldots, m-2$.

2. The ruled surfaces described in Examples 1, 2 and 3 are new examples of those with the pointwise 1-type Gauss map.

4. Ruled surfaces with $\Delta G = \Lambda G$ and generalized B-scrolls. In this section, we study ruled surfaces satisfying $\Delta G = \Lambda G$ and characterize generalized B-scrolls in term of it.

Theorem 4.1. *There are no noncylindrical ruled surfaces except of type M_+^2 with nonnull base curve in $\mathbf{E}_1^m$ whose Gauss map satisfies*

$$\Delta G = \Lambda G, \quad \Lambda \in \mathbf{R}^{N\times N} \quad \textit{and} \quad N = \binom{m}{2}. \tag{4.1}$$

Proof. In this case, we also split it into two cases.

Case 1. Let M be a noncylindrical ruled surface of one of three types M_+^1, M_+^3 or M_-^1 according to the character of the base curve α and the director curve β:

(1) $\alpha = \alpha(s)$ is space-like and $\beta = \beta(s)$ is space-like,

(2) $\alpha = \alpha(s)$ is space-like and $\beta = \beta(s)$ is time-like,

(3) $\alpha = \alpha(s)$ is time-like and $\beta' = \beta'(s)$ is nonnull,

where s is the arc-length of the director curve β. We also express the ruled surface M with the following parametrization

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I$$

such that $\langle \alpha', \beta\rangle = 0, \langle \beta, \beta\rangle = \varepsilon_2(=\pm 1)$ and $\langle \beta', \beta'\rangle = \varepsilon_3(=\pm 1)$. Then, the Gauss map G of M is easily obtained by $(1/\|x_s \wedge x_t\|)x_s \wedge x_t$. For later use we define smooth functions q, u, v and 2-planes A, B as follows:

$$q = \|x_s\|^2 = \varepsilon_4\langle x_s, x_s\rangle, \quad u = \langle \alpha', \beta'\rangle, \quad v = \langle \alpha', \alpha'\rangle,$$
$$A = \alpha' \wedge \beta, \quad B = \beta' \wedge \beta.$$

In turn, we have by the simple computation

$$q = \varepsilon_4(\varepsilon_3 t^2 + 2ut + v), \qquad G = \frac{1}{q^{1/2}}(A + tB). \tag{4.2}$$

It is easy to show that the Laplacian Δ of M can be expressed as

$$\Delta = -\varepsilon_4\left(\frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2}\frac{1}{q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s}\right) - \varepsilon_2\left(\frac{\partial^2}{\partial t^2} + \frac{1}{2}\frac{1}{q}\frac{\partial q}{\partial t}\frac{\partial}{\partial t}\right). \tag{4.3}$$

By a straightforward computation, the Laplacian ΔG of the Gauss map G with of help of (4.2) and (4.3) turns out to be

$$\begin{aligned}\Delta G = \Big\{ & 2\varepsilon_2\varepsilon_3\varepsilon_4 q^{-1} - 2\varepsilon_2(\varepsilon_3 t + u)^2 q^{-2} \\ & - \varepsilon_4(2u't + v')^2 q^{-3} + \frac{1}{2}(2u''t + v'')q^{-2}\Big\} G \\ & + \frac{1}{2} q^{-5/2}\{2\varepsilon_2\varepsilon_4 q(-\varepsilon_3 A + uB) \\ & - 2\varepsilon_4 q(A'' + tB'') + 3(2u't + v')(A' + tB')\}.\end{aligned}$$

Suppose that the ruled surface satisfy the condition (4.1). By the definition of the function q, (4.1) becomes the polynomial with the variable t whose coefficients are functions of the variable s. Then, by the coefficients of t^6 and t^7, we have

$$\Lambda A = 0, \quad \Lambda B = 0, \tag{4.5}$$

where Λ is the constant matrix and A and B are vectors. Suppose that Λ is nonsingular. Then, (4.5) means that $A = B = 0$, which implies that $G = 0$, a contradiction. Accordingly, we see that the matrix Λ is singular.

Next, considering the coefficients of the other powers of t in (4.1) and using (4.5), we obtain

$$B'' = 0, \tag{4.6}$$

$$\varepsilon_2 A'' + \varepsilon_3 A + 4\varepsilon_2\varepsilon_3 u B'' - 3\varepsilon_2\varepsilon_3 u'B' - (u + \varepsilon_2\varepsilon_3 u'')B = 0, \tag{4.7}$$

$$\begin{aligned} 8\varepsilon_2\varepsilon_3 uA'' - 6\varepsilon_2\varepsilon_3 u'A' + (8u - 2\varepsilon_2\varepsilon_3 u'')A & \\ + (4\varepsilon_2\varepsilon_3 v + 8\varepsilon_2 u^2)B'' - (12\varepsilon_2 uu' + 3\varepsilon_2\varepsilon_3 v')B' & \\ - (4v + 4\varepsilon_3 u^2 - 8\varepsilon_2 u'^2 + \varepsilon_2\varepsilon_3 v'' + 4\varepsilon_2 uu'')B = 0, & \end{aligned} \tag{4.8}$$

$$\begin{aligned} (4\varepsilon_2\varepsilon_3 v + 8\varepsilon_2 u^2)A'' - (12\varepsilon_2 uu' + 3\varepsilon_2\varepsilon_3 v')A' & \\ + (12\varepsilon_3 u^2 + 8\varepsilon_2 u'^2 + 2v - \varepsilon_2\varepsilon_3 v' - 4\varepsilon_2 uu'')A + 8\varepsilon_2 uvB'' & \\ - (6\varepsilon_2 u'v + 6\varepsilon_2 uv')B' & \\ - (16\varepsilon_3 uv - 8\varepsilon_2 u'v' + 2\varepsilon_2 uv'' + 2\varepsilon_2 u''v)B = 0, & \end{aligned} \tag{4.9}$$

$$(4.10)\quad 8\varepsilon_2 uvA'' - (6\varepsilon_2 u'v + 6\varepsilon_2 uv')A' + (8u^3 + 8\varepsilon_2 u'v' - 2\varepsilon_2 uv'' - 2\varepsilon_2 u''v)A + 2\varepsilon_2 v^2 B'' - 3\varepsilon_2 vv'B' + (2\varepsilon_2 v'^2 - \varepsilon_2 vv'' - 4u^2 v - 4\varepsilon_3 v^2)B = 0,$$

(4.11)

$$2\varepsilon_2 v^2 A'' - 3\varepsilon_2 vv'A' + (4u^2 v + 2\varepsilon_2 v'^2 - 2\varepsilon_3 v^2 - \varepsilon_2 vv'')A - 2uv^2 B = 0.$$

Using the above equations, we can eliminate A'', A' and B'. Consequently, we get

(4.12)

$$(4\varepsilon_2\varepsilon_3 u'^2 v - \varepsilon_2 v'^2 + \varepsilon_3 v^2)A + (4\varepsilon_2\varepsilon_3 u'vv' - 2uv^2 - 8\varepsilon_2 uu'^2 v)B = 0,$$

$$(4.13)\qquad (2uv'^2 - 4u'vv')A + (4\varepsilon_3 u'^2 v^2 - vv'^2)B = 0.$$

Consider an open subset $\mathcal{U} = \{p \in M \mid (A \wedge B)(p) \neq 0\}$. Suppose that $\mathcal{U}$ is not empty. On $\mathcal{U}$, (4.12) and (4.13) imply

$$(4.14)\qquad \begin{cases} 4\varepsilon_2\varepsilon_3 u'^2 v - \varepsilon_2 v'^2 + \varepsilon_3 v^2 = 0, \\ 4\varepsilon_2\varepsilon_3 u'vv' - 2uv^2 - 8\varepsilon_2 uu'^2 v = 0, \\ 2uv'^2 - 4u'vv' = 0, \\ 4\varepsilon_3 u'^2 v^2 - vv'^2 = 0. \end{cases}$$

It follows from (4.14) that

$$(4.15)\qquad u'vv'^2(v - \varepsilon_3 u^2) = 0 \quad \text{on} \quad \mathcal{U}.$$

If the open subset $\mathcal{U}_1 = \{p \in \mathcal{U} \mid (v - \varepsilon_3 u^2)(p) \neq 0\}$ is not empty, then $u'vv'^2(p) = 0$ on $\mathcal{U}_1$. Let $\mathcal{U}_2 = \{p \in \mathcal{U}_1 \mid u'(p) \neq 0\} \neq \phi$. Then $vv'^2(p) = 0$ on $\mathcal{U}_2$, that is, v is constant on each component of $\mathcal{U}_2$. But, the fourth equation in (4.14) implies $v = 0$ on $\mathcal{U}_2$, which is a contradiction. Therefore, the subset $\mathcal{U}_2$ is empty and u is constant on each component of $\mathcal{U}_1$. We now consider the matter on a component $\mathcal{O}$ of $\mathcal{U}_1$ for a while. From the second equation in (4.14), we have $u = 0$ on $\mathcal{O}$ because v is nonzero. Then, the fourth equation in (4.14) gives that v is a nonzero constant on $\mathcal{O}$. Furthermore, the first equation in (4.14) implies $v = 0$ on $\mathcal{O}$, which is a contradiction and hence the subset $\mathcal{U}_1$ is empty. Thus we have

$$v = \varepsilon_3 u^2, \qquad q = \varepsilon_3\varepsilon_4(t + \varepsilon_3 u)^2 \quad \text{on} \quad \mathcal{U}.$$

In this case, from the first equation in (4.14) we have $v = 0$, which is also a contradiction. Therefore, $\mathcal{U}$ must be empty. Hence, A and B are linearly dependent. Therefore, there exist some smooth functions λ and μ such that $\alpha' - \lambda\beta' = \mu\beta$. By using the properties of α and β, we can get $\mu = 0$, $u = \lambda\varepsilon_3$ and $v = \lambda^2\varepsilon_3$. From (4.12) we have $\lambda^5\beta' \wedge \beta = 0$, that is, $\lambda = 0$ and $v = 0$ because of $\beta \wedge \beta' \neq 0$. It contradicts the property of the curve α. Consequently, Case 1 never occurs.

Case 2. Let M be a noncylindrical ruled surface of type M^2_+ or M^2_-. We may also assume that M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta,\beta\rangle = 1$, $\langle\alpha',\beta\rangle = 0$, $\langle\alpha',\alpha'\rangle = \varepsilon_1(= \pm 1)$ and β' is light-like. It is easy to get the Gauss map of the surface M as

$$G = \frac{1}{||(\alpha' + t\beta') \wedge \beta||}\,(\alpha' + t\beta') \wedge \beta.$$

Similarly to Case 1, we also define functions q and u by

$$q = ||x_s||^2 = \varepsilon_4\langle x_s, x_s\rangle, \qquad u = \langle\alpha',\beta'\rangle,$$

which give

$$q = \varepsilon_4(2ut + \varepsilon_1), \qquad G = \frac{1}{q^{1/2}}\,(A + tB), \tag{4.16}$$

where we put $A = \alpha' \wedge \beta$ and $B = \beta' \wedge \beta$ and the region of t runs so that $q > 0$.

By the straightforward computation, we easily have the Laplacian Δ of M in the form of

$$\Delta = -\varepsilon_4\Big(-\frac{1}{2}\frac{1}{q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} + \frac{1}{q}\frac{\partial^2}{\partial s^2}\Big) - \Big(\frac{1}{2}\frac{1}{q}\frac{\partial q}{\partial t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\Big). \tag{4.17}$$

For the Gauss map G, we have

$$\Delta G = \frac{G_1(t)}{q^{(1/2)+3}}, \dots, \Delta^r G = \frac{G_r(t)}{q^{(1/2)+3r}}, \qquad \deg(G_r(t)) \le 1 + 2r$$

where $G_1(t), \cdots, G_r(t)$ are polynomials in t with 2-planes in s as coefficients.

Suppose the matrix Λ is not zero. Consider an open subset $\mathcal{U} = \{p \in M | u(p) \neq 0\}$. Suppose $\mathcal{U}$ is not empty. Then, by the Cayley-Hamilton theorem, we have

$$(\Lambda^N + c_0\Lambda^{N-1} + \cdots + c_{N-1}\Lambda + c_N I)G = 0$$

for some constants $c_0, \dots, c_N$, where I denotes the $N \times N$-identity matrix, in other words,

$$(\Delta^{N+1} + c_0\Delta^N + \cdots + c_N\Delta)G = 0.$$

As in the previous argument, by considering the degree of the polynomials obtained from $\Delta G, \dots, \Delta^r G$ $(r = 1, 2, \dots, N+1)$, we obtain

$$\Delta G = 0, \tag{4.18}$$

that is a contradiction to the assumption for a non-zero matrix Λ. Consequently, the matrix Λ satisfying (4.1) is zero. Using (4.16) and (4.17), we can obtain by a direct computation

$$\begin{aligned} \Delta G =& (-2u^2q^{-2} + u''tq^{-2} - 4\varepsilon_4 u'^2t^2q^{-3})G \\ &+ q^{-5/2}\{\varepsilon_4 uBq + 3u't(A' + tB') - \varepsilon_4(A'' + tB'')q\}. \end{aligned} \tag{4.19}$$

Similarly to Case 1, by using (4.18) and (4.19), we have the following equations:

$$u^2B'' - 3uu'B' + 2u'^2B - uu''B = 0, \tag{4.20}$$

$$\begin{aligned} &4u^2A'' - 6uu'A' - 2uu''A + 4u'^2A \\ &\qquad + 4\varepsilon_1 uB'' - 3\varepsilon_1 u'B' + \varepsilon_1 u''B = 0, \end{aligned} \tag{4.21}$$

$$4\varepsilon_1 uA'' - 3\varepsilon_1 u'A' + 4u^3A - \varepsilon_1 u''A + B'' - 2\varepsilon_1 u^2 B = 0, \tag{4.22}$$

$$A'' + 2\varepsilon_1 u^2 A - uB = 0. \tag{4.23}$$

Using the above equations, we can eliminate A'', B'', A' and B' so that

$$2\varepsilon_1 uu'^2 A = (u'^2 - uu'')B. \tag{4.24}$$

We now consider an open subset $\mathcal{U}_1 = \{p \in \mathcal{U} \mid (u^2)'(p) \neq 0\}$. Suppose that $\mathcal{U}_1$ is not empty. Then, from (4.24) we get $\alpha' = \rho\beta$ for some function ρ on $\mathcal{U}_1$, which is a contradiction. Therefore, the open subset $\mathcal{U}_1$ is empty, that is, the function u is constant on M. Suppose $u \neq 0$. Equations (4.20)–(4.23) imply that $uA - (1/2)\epsilon_1 B = 0$, which gives $u\alpha' = (1/2)\epsilon_1\beta'$, a contradiction. Thus, u is identically zero on M.

Let M be a surface of type M^2_-. It is impossible because there is no time-like vector orthogonal to a light-like vector in Minkowski space.

If M is a surface of type M^2_+, then it is easily seen that M is flat. □

Remark. Even in the case of ruled surfaces in $\mathbf{E}^4_1$ of type M^2_+ with the Gauss map of null 2-type, there are abundant examples of them [**12**].

Theorem 4.2. *Let M be a null scroll in Minkowski m-space $\mathbf{E}^m_1$. Then, the Gauss map G of M satisfies*

$$\Delta G = \Lambda G, \quad \Lambda \in \mathbf{R}^{N\times N} \quad \text{and} \quad N = \binom{m}{2} \tag{4.25}$$

if and only if M is an open part of generalized B-scroll.

Proof. Let $\alpha = \alpha(s)$ be a light-like curve in $\mathbf{E}^m_1$ and $\beta = \beta(s)$ a light-like vector field along α. Then, the null scroll M is up to congruences parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle \beta, \beta \rangle = 0$, $\langle \alpha', \beta \rangle = -1$, where I and J are some open intervals. Furthermore, without loss of generality we may choose α as a light-like geodesic of M. We then have $\langle \alpha'(s), \beta'(s) \rangle = 0$ for all s. Therefore, we have the natural frame $\{x_s, x_t\}$ given by

$$x_s = \alpha' + t\beta', \quad x_t = \beta.$$

We also define smooth functions q and v as follows:

$$q = \langle x_s, x_s \rangle, \quad v = \langle \beta', \beta' \rangle. \tag{4.26}$$

Similarly as before, the Laplacian Δ of M can be given as follows [**8**]:

$$\Delta = -2 \frac{\partial^2}{\partial s \partial t} + \frac{\partial q}{\partial t} \frac{\partial}{\partial t} + q \frac{\partial^2}{\partial t^2}. \tag{4.27}$$

Furthermore, the Gauss map G is determined by

$$G = \left(\frac{1}{\|x_s \times x_t\|} \right) (x_s \times x_t) = A + tB, \tag{4.28}$$

where we put $A = \alpha' \wedge \beta$ and $B = \beta' \wedge \beta$. Then, (4.27) and (4.28) imply

$$\Delta G = -2\beta'' \wedge \beta + 2vt\beta' \wedge \beta. \tag{4.29}$$

Together with (4.25) and (4.29), we have

$$-2\beta'' \wedge \beta + 2vt\beta' \wedge \beta = \Lambda(\alpha' \wedge \beta + t\beta' \wedge \beta). \tag{4.30}$$

Differentiating (4.30) with respect to t, we get

$$(2vI - \Lambda)\beta' \wedge \beta = 0, \tag{4.31}$$

where I denotes the $N \times N$-identity matrix. Let $\mathcal{V}$ be an open subset $\mathcal{V} = \{p \in M \mid \det(\Lambda - 2vI) \neq 0\}$. If $\mathcal{V} \neq \varnothing$, then $\beta' \wedge \beta = 0$, where det means the determinant. By the choice of β and the causal character of the light-like vector field β, we see that β is a constant light-like vector. Then, (4.30) with $\beta' = 0$ implies $\Lambda(\alpha' \wedge \beta) = 0$. Since $\det(\Lambda - 2vI) = \det \Lambda \neq 0$, $\alpha' \wedge \beta = 0$, which is a contradiction. Thus, the open subset $\mathcal{V}$ is the empty set and $\Lambda - 2vI$ is a singular matrix.

Thus, $2v$ is an eigenvalue of Λ and it is a constant. Since v is constant, it is easily obtained that

$$\Delta^2 G - 2v\Delta G = 0.$$

Hence, the Gauss map is of at most null 2-type, i.e., it is of null 2-type, 1-type or $\Delta G = 0$. If $\Delta G = 0$, it is easily obtained that M is an open part of the flat extended B-scroll which is a special case of the generalized B-scroll, see [**9**]. Using Theorem B in Section 1, we have the result. □

Theorem 4.3. *Let M be a cylindrical ruled surface in $\mathbf{E}_1^m$. Then, the mean curvature vector is parallel in the normal bundle if and only if M is locally a Euclidean plane, a Minkowski plane, a circular cylinder lying in $\mathbf{E}^3$, a circular cylinder of index* 1 *lying in $\mathbf{E}_1^3$, a hyperbolic cylinder, a Lorentz cylinder or a quadric ruled surface of the first kind.*

Proof. Let M be a cylindrical ruled surface. Then, we can take its parametrization as

$$x(s,t) = \alpha(s) + t\beta$$

where α is a unit speed curve and β a constant vector such that $\langle \alpha'(s), \beta\rangle = 0$ for all s. From this, we have the mean curvature vector field H of the form

$$H = \frac{1}{2}\,\varepsilon(\alpha')\alpha'',$$

from which,

$$\alpha'''(s) = -4\varepsilon(\alpha')\langle H, H\rangle \alpha'(s) \tag{4.32}$$

where $\varepsilon(\alpha')$ is the sign of α'. Suppose that H is parallel in the normal bundle. Then, the mean curvature $\langle H, H\rangle$ is constant. If $\langle H, H\rangle 0$, then α'' is either zero or light-like because of the causal character.

If α'' is zero, then M is an Euclidean plane or a Minkowski plane. Suppose α'' is light-like. Then, α may be expressed up to congruence as

$$\alpha(s) = \frac{s^2}{2}\,\mathbb{C} + s\mathbb{D}$$

where $\mathbb{D}$ is a unit constant vector orthogonal to $\mathbb{C}$ and β. From the character of the director curve β, $\mathbb{C}$ is also orthogonal to β. And, $\mathbb{D}$

turns out to be space-like no matter whether the base curve α may be space-like or time-like because the ambient space $\mathbf{E}_1^m$ has index 1. Therefore, up to congruence, the parametrization of ruled surface M is given by

$$x(s,t) = (s^2, s^2, s, t)$$

which is space-like and lies in $\mathbf{E}_1^4$. It is the quadric ruled surface of the first kind given in Section 3. Suppose $\langle H, H\rangle \neq 0$. By solving the differential equation (4.32), the ruled surface M is an open part of the ordinary circular cylinder, the circular cylinder of index 1, the hyperbolic cylinder or the Lorentz cylinder. The converse is obvious. □

Corollary 4.4. *Let M be a cylindrical ruled surface in $\mathbf{E}_1^m$, $m \geq 4$. If the mean curvature vector field is parallel in the normal bundle, then M lies in at most four-dimensional Minkowski space $\mathbf{E}_1^4$.*

5. Ruled surfaces with pointwise 1-type Gauss map. In this section, we study a ruled surface M in $\mathbf{E}_1^m$ with pointwise 1-type Gauss map, that is, the Gauss map G satisfies

$$\Delta G = fG \tag{5.1}$$

for some smooth function f on M.

In the first place, we prove

Lemma 5.1. *Let M be an n-dimensional submanifold of a pseudo-Euclidean space $\mathbf{E}_s^m$ with pointwise* 1*-type Gauss map G. Then, the mean curvature vector field H is parallel in the normal bundle.*

Proof. Let $e_1, \cdots, e_n$ be a local orthonormal frame on M which defines the Gauss map G by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ for each point $p \in M$. Let $\widetilde{\nabla}$ be the Levi-Civita connection defined on $\mathbf{E}_s^m$ and ∇ its induced connection on M. Then, we have

$$\widetilde{\nabla}_{e_i} G = \sum_{k=1}^{n} e_1 \wedge \cdots \wedge (\nabla_{e_i} e_k + h(e_i, e_k)) \wedge \cdots \wedge e_n. \tag{5.2}$$

For each point $p \in M$, we may choose a local orthonormal frame in such a way that $(\nabla_{e_i} e_j)(p) = 0$. From (5.2), we have

$$
\begin{aligned}
(\Delta G)(p) &= -\sum_i \varepsilon_i(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} G - \widetilde{\nabla}_{\nabla_{e_i}e_i} G)(p) \\
&= -\sum_i \sum_k \varepsilon_i(e_i \wedge \cdots \wedge \{-A_{h(e_i,e_k)}e_i + D_{e_i}h(e_i,e_k)\} \\
&\quad \wedge \cdots \wedge e_n)(p) \\
&\quad -\sum_i \sum_{h\neq k} \varepsilon_i(e_1 \wedge \cdots \wedge h(e_i,e_h) \wedge \cdots \wedge h(e_i,e_k) \\
&\quad \wedge \cdots \wedge e_n)(p),
\end{aligned}
\tag{5.3}
$$

where ε_i is the sign of e_i, A the shape operator, $h(e_i, e_j)$ the second fundamental form evaluated at e_i and e_j and D the normal connection defined on the normal bundle of M.

Suppose that the Gauss map G of M is of pointwise 1-type, that is, (5.1) is satisfied. By comparing the tangential part and the normal part in (5.3), we see that

$$\sum_i D_{e_i} h(e_i, e_k) = 0$$

for all $k = 1, 2, \ldots, n$, from which, by using the Codazzi equation (2.3), we get

$$\sum_i \big((\bar{\nabla}_{e_k} h)(e_i, e_i)\big)(p) = 0,$$

or equivalently, $D_{e_k} H = 0$ for all $k = 1, 2, \ldots, n$. Thus, the mean curvature vector field H is parallel. $\square$

Therefore, using Theorem 4.3 and Lemma 5.1, we have

Theorem 5.2. *Let M be a cylindrical ruled surfaces in $\mathbf{E}^m_1$. Then, M has pointwise 1-type Gauss map if and only if M is locally a Euclidean plane, a Minkowski plane, a circular cylinder lying in $\mathbf{E}^3$, a circular cylinder of index 1 lying in $\mathbf{E}^3_1$, a hyperbolic cylinder, a Lorentz cylinder or a quadric ruled surface of the first kind.*

Theorem 5.3. *Let M be a noncylindrical ruled surface in $\mathbf{E}_1^m$ of type M_+^1, M_+^3 or M_-^1 with pointwise 1-type Gauss map. Then, M is minimal.*

Proof. Suppose that M is one of the ruled surfaces of type M_+^1, M_+^3 and M_-^1 in $\mathbf{E}_1^m$ satisfying the condition (5.1). We will use the same notations used in Case 1 of Theorem 4.1.

By using (4.3), the Beltrami equation $\Delta x = -2H$ gives

(5.4)

$$H(s,t) = \varepsilon_4 \frac{1}{2q}(\alpha''+t\beta'') - \frac{1}{4q^2}(2u't+v')(\alpha'+t\beta') + \varepsilon_2\varepsilon_4 \frac{1}{2q}(\varepsilon_3 t+u)\beta.$$

By Lemma 5.1 we have

$$\widetilde{\nabla}_{x_t} H = -\varepsilon_4 q^{-1}\langle h(x_s, x_t), H\rangle x_s. \tag{5.5}$$

Differentiating (5.4) with respect to t and using (2.1), we can show that the coefficient of t^4 is obtained by

$$\beta'' = -\varepsilon_2\varepsilon_3\beta. \tag{5.6}$$

By (5.5) the coefficients of t^3, t^2, t^1 and t^0 are given by

$$2\alpha'' + 2\varepsilon_2 u\beta - \varepsilon_3 u'\beta' - \varepsilon_3\langle\alpha'', \beta'\rangle\beta' = 0. \tag{5.7}$$

(5.8)

$$6\varepsilon_3 u\alpha'' - 2\varepsilon_3 u'\alpha' - \varepsilon_3 v'\beta' + uu'\beta' + 6\varepsilon_2\varepsilon_3 u^2\beta - (\varepsilon_3\alpha' + 2u\beta')\langle\alpha'', \beta'\rangle = 0.$$

$$\begin{aligned}(4\varepsilon_3 v + 8u^2)\alpha'' - (3\varepsilon_3 v' + 2uu')\alpha' - (uv' - 4u'v)\beta' \\ + (4\varepsilon_2\varepsilon_3 uv + 8\varepsilon_2 u^3)\beta - (2v\beta' + 4u\alpha')\langle\alpha'', \beta'\rangle = 0.\end{aligned} \tag{5.9}$$

$$4uv\alpha'' - (3uv' - 2u'v)\alpha' + vv'\beta' + 4\varepsilon_2 u^2 v\beta - 2v\langle\alpha'', \beta'\rangle\alpha' = 0. \tag{5.10}$$

If we take the scalar product with α' and β' in equation (5.7), respectively, then we have

$$v' - \varepsilon_3 uu' - \varepsilon_3\langle\alpha'', \beta'\rangle u = 0, \quad \langle\alpha'', \beta'\rangle = u',$$

which imply

$$v' = 2\varepsilon_2 u u'. \tag{5.11}$$

Therefore, from (5.7), we obtain

$$\alpha'' = \varepsilon_3 u' \beta' - \varepsilon_2 u \beta. \tag{5.12}$$

Putting (5.12) into (5.8) implies

$$u'(\alpha' - \varepsilon_3 \beta') = 0. \tag{5.13}$$

Let's consider an open subset $\mathcal{O} = \{p \in M | u' \neq 0\}$ of M. Suppose $\mathcal{O} \neq \varnothing$. Then, (5.13) yields

$$\alpha' = \varepsilon_3 u \beta'. \tag{5.14}$$

Since $\beta'' = -\varepsilon_2 \varepsilon_3 \beta$, β is given by

$$\beta(s) = \cos s\, \mathbb{C} + \sin s\, \mathbb{D} \tag{5.15}$$

for some unit space-like constant vectors $\mathbb{C}$ and $\mathbb{D}$ which are orthogonal to each other, or

$$\beta(s) = \cosh s\, \mathbb{C} + \sinh s\, \mathbb{D} \tag{5.16}$$

where $\mathbb{C}$ and $\mathbb{D}$ satisfy $\langle \mathbb{C}, \mathbb{C} \rangle = -\langle \mathbb{D}, \mathbb{D} \rangle = 1$ and $\langle \mathbb{C}, \mathbb{D} \rangle = 0$. Together with (5.14), (5.15) and (5.16), α can be given by

$$\alpha(s) = -\varepsilon_3 \int^s u(\sigma) \sin \sigma \, d\sigma\, \mathbb{C} + \varepsilon_3 \int^s u(\sigma) \cos \sigma \, d\sigma\, \mathbb{D}$$

or

$$\alpha(s) = \varepsilon_3 \int^s u(\sigma) \sinh \sigma \, d\sigma\, \mathbb{C} + \varepsilon_3 \int^s u(\sigma) \cosh \sigma \, d\sigma\, \mathbb{D}.$$

Thus, each component of $\mathcal{O}$ is contained either an Euclidean plane or a Minkowski plane. By continuity, a component of $\mathcal{O}$ is either an Euclidean plane or a Minkowski plane.

Now, we suppose $u' = 0$ on M, that is, u is constant on M. By (5.11), v is also a constant. Then, (5.12) implies

$$\alpha'' = -\varepsilon_2 u\beta. \tag{5.17}$$

Making use of (5.6) and (5.17), we easily see that the mean curvature vector H given by (5.4) is identically zero, that is, the surface is minimal. $\square$

We now consider the case that the ruled surface M is noncylindrical of type M^2_+ or M^2_-. Then, the surface M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta,\beta\rangle = 1$, $\langle\alpha',\beta\rangle = 0$, $\langle\alpha',\alpha'\rangle = \varepsilon_1(=\pm 1)$ and β' is lightlike. By using (4.17) and the Beltrami equation, the mean curvature vector H is determined by

$$H(s,t) = \varepsilon_4 \frac{1}{2q}(\alpha'' + t\beta'') - \frac{1}{2q^2}u't(\alpha' + t\beta') + \varepsilon_4 \frac{1}{2q}u\beta. \tag{5.18}$$

By (5.18) we get

$$\langle H,H\rangle = \frac{1}{4}q^{-2}\{\langle\alpha'' + t\beta'', \alpha'' + t\beta''\rangle - u^2 - \varepsilon_4 q^{-1}{u'}^2 t^2\}. \tag{5.19}$$

Differentiating (5.19) with respect to t and using Lemma 5.1, we have

$$\begin{cases} 2\varepsilon_1 u\langle\beta'',\beta''\rangle - 4u^2\langle\alpha'',\beta''\rangle + u{u'}^2 = 0, \\ \langle\beta'',\beta''\rangle - 4u^2\langle\alpha'',\alpha''\rangle - \varepsilon_1 {u'}^2 + 4u^4 = 0, \\ \langle\alpha'',\beta''\rangle - 2\varepsilon_1 u\langle\alpha'',\alpha''\rangle + 2\varepsilon_1 u^3 = 0, \end{cases} \tag{5.20}$$

which imply

$$u' = 0. \tag{5.21}$$

On the other hand, by (2.2) and Lemma 5.1, we obtain

$$\widetilde{\nabla}_{x_s} H = -\frac{\varepsilon_4}{q}\langle h(x_s,x_s), H\rangle x_s - \langle h(x_s,x_t), H\rangle x_t, \tag{5.22}$$

and by (2.1) the second fundamental form h satisfies

$$\begin{cases} h(x_s, x_s) = \alpha'' + t\beta'' + u\beta, \\ h(x_s, x_t) = -\varepsilon_4(u/q)\alpha' + (1 - \varepsilon_4(u/q)t)\beta', \\ h(x_t, x_t) = 0. \end{cases} \tag{5.23}$$

Thus, from (5.18), (5.21), (5.22) and (5.23) we have

$$\begin{cases} u\beta'(u^2 - \langle \alpha'', \alpha'' \rangle) = 0, \\ \beta''' = (u^2 - \langle \alpha'', \alpha'' \rangle)(\varepsilon_1\beta' + 2u\alpha'), \\ \alpha''' + u\beta' = \varepsilon_1(u^2 - \langle \alpha'', \alpha'' \rangle)\alpha' - \langle \alpha'', \beta' \rangle\beta. \end{cases} \tag{5.24}$$

Thus, we have

Lemma 5.4. *Let M be a noncylindrical ruled surface in $\mathbf{E}_1^m$ of type M_+^2 or M_-^2 with pointwise 1-type Gauss map. Then, the function u is constant on M.*

We now prove

Theorem 5.5. *Let M be a noncylindrical ruled surface in $\mathbf{E}_1^m$ with pointwise* 1*-type Gauss map if and only if M is one of the ordinary helicoid, the helicoid of the first, the second kind, the third and the fourth kind, the Euclidean plane, the Minkowski plane, the quadric ruled surface of the second kind and the conjugate Enneper's surface of the second kind.*

Proof. Suppose the Gauss map G is of pointwise 1-type, i.e., (5.1) is satisfied. In the first place, we consider the ruled surface M in $\mathbf{E}_1^m$ is of type M_+^1, M_+^3 or M_-^1. In the proof of Theorem 5.3, the function u and v are constant if M is neither an Euclidean plane nor a Minkowski plane. We now consider that M is neither an Euclidean plane nor a Minkowski plane.

Case I. $\varepsilon_2\varepsilon_3 = 1$. In this case, because of the causal character, we have $(\varepsilon_2, \varepsilon_3) = (1, 1)$. So, (5.6) yields

$$\beta(s) = \cos s\,\mathbb{C} + \sin s\,\mathbb{D}.$$

Since $\langle \beta, \beta \rangle = \langle \beta', \beta' \rangle = 1$, we get $\langle \mathbb{C}, \mathbb{C} \rangle = \langle \mathbb{D}, \mathbb{D} \rangle = 1$ and $\langle \mathbb{C}, \mathbb{D} \rangle = 0$.

Suppose the given ruled surface M is space-like, that is, $\varepsilon_4 = 1$, and thus $q = (t^2 + 2ut + v) > 0$.

If $u^2 - v < 0$, then the function q is positive for all possible t. Since u is constant, (5.12) gives $\alpha'' = -u\beta$. Thus, up to congruence, we obtain

$$\alpha(s) = u \cos s\, \mathbb{C} + u \sin s\, \mathbb{D} + s\, \mathbb{E}$$

for some constant vector $\mathbb{E}$, where $\langle \mathbb{C}, \mathbb{E} \rangle = \langle \mathbb{D}, \mathbb{E} \rangle = 0$ and $\langle \mathbb{E}, \mathbb{E} \rangle = v - u^2$. Therefore, up to congruence, M is parametrized as

$$x(s,t) = ((t+u)\cos s, (t+u)\sin s, as) \tag{5.25}$$

for some $a > 0$, which is an ordinary helicoid in a three-dimensional Euclidean space $\mathbf{E}^3$.

If $u^2 - v > 0$, then $t > u + \sqrt{u^2 - v}$ or $t < u - \sqrt{u^2 - v}$. Then, $\langle \mathbb{E}, \mathbb{E} \rangle = v - u^2 < 0$, and thus up to congruence, M is given by

$$x(s,t) = (as, (t+u)\cos s, (t+u)\sin s) \tag{5.26}$$

for some $a > 0$, which lies in a three-dimensional Minkowski space $\mathbf{E}^3_1$, where $t > u + \sqrt{u^2 - v}$ or $t < u - \sqrt{u^2 - v}$. It is the helicoid of the first kind as a space-like surface, see [**11**].

We now suppose M is time-like, that is, $\varepsilon_4 = -1$. In this case, $\varepsilon_2 = 1$ and we must have $u - \sqrt{u^2 - v} < t < u + \sqrt{u^2 - v}$. By a similar argument to the previous case, we have up to congruence

$$x(s,t) = (as, (t+u)\cos s, (t+u)\sin s) \tag{5.27}$$

where $u - \sqrt{u^2 - v} < t < u + \sqrt{u^2 - v}$ for a positive number a, which lies in a 3-dimensional Minkowski space $\mathbf{E}^3_1$ that is the helicoid of the first kind as a time-like surface.

Suppose $v - u^2 = 0$. If M is space-like, $\varepsilon_4 = 1$ and $t \neq -u$. Up to congruences, the straightforward computation gives the parametrization of M by

$$x(s,t) = (as, as, (t+u)\cos s, (t+u)\sin s) \tag{5.28}$$

where $u^2 = a^2$, which lies in $\mathbf{E}^4_1$. This ruled surface is the helicoid of the fourth kind described in Section 3. If M is time-like, then $\varepsilon_4 = -1$ and $q = -(t^2 + 2ut + v) > 0$ that is impossible.

Case II. $\varepsilon_2\varepsilon_3 = -1$. Suppose $(\varepsilon_2, \varepsilon_3) = (1, -1)$. (5.6) implies

$$\beta(s) = \cosh s\,\mathbb{C} + \sinh s\,\mathbb{D} \tag{5.29}$$

for some constant vector $\mathbb{C}$ and $\mathbb{D}$. Since $\langle\beta, \beta\rangle = 1$ and $\langle\beta', \beta'\rangle = -1$, we have $\langle\mathbb{C}, \mathbb{C}\rangle = 1$, $\langle\mathbb{D}, \mathbb{D}\rangle = 1$ and $\langle\mathbb{C}, \mathbb{D}\rangle = 0$. Then, (5.12) together with $u' = 0$ and (5.29) gives the base curve α such that

$$\alpha(s) = -u\cosh s\,\mathbb{C} - u\sinh s\,\mathbb{D} + s\,\mathbb{E}$$

up to congruence for some space-like constant vector $\mathbb{E}$. Thus, according to the sign of ε_4, we have up to congruence

$$x(s,t) = ((t-u)\sinh s, (t-u)\cosh s, as) \tag{5.30}$$

for $u - \sqrt{u^2 + a^2} < t < u + \sqrt{u^2 + a^2}$ with $a^2 = v + u^2$, which defines M as a space-like ruled surface, or

$$x(s,t) = ((t-u)\sinh s, (t-u)\cosh s, as) \tag{5.31}$$

for $t > u - \sqrt{u^2 + a^2}$ or $t < u - \sqrt{u^2 + a^2}$ with $a^2 = v + u^2$ as a time-like surface. The ruled surfaces with parametrization (5.30) and (5.31) are the helicoid of the second kind, see [**11**].

Next, consider $(\varepsilon_2, \varepsilon_3) = (-1, 1)$. Then, (5.6) and (5.12) imply $\beta'' = \beta$ and $\alpha'' = u\beta$. We can easily solve these equations. By considering the causal character, we see that M is time-like and $\varepsilon_4 = 1$ of type M^3_+ with the parametrization

$$x(s,t) = ((t+u)\cosh s, (t+u)\sinh s, as) \tag{5.32}$$

up to congruence for all real t and $v - u^2 = a^2$, which lies in $\mathbf{E}^3_1$. It is the helicoid of the third kind described in [**11**].

Now, we consider the case that the ruled surface M in $\mathbf{E}^m_1$ is noncylindrical of type M^2_+ or M^2_-. By Lemma 5.4, the function u is constant on M.

Suppose that $u = 0$. By the causal character, the ruled surface M under consideration is of type M_+^2. From the second equation of (5.20), we get $\langle \beta'', \beta'' \rangle = 0$. If β'' is light-like, β'' is parallel with β'. From the causal character of β', we may assume that every component of β' is nonzero. Thus, by a direct computation, we obtain $\beta(s) = P(s)\mathbb{N}$ for some nonzero function $P(s)$ and a constant light-like vector $\mathbb{N}$, which contradicts the fact that β is a space-like unit vector. Hence, the vector field β'' must be zero. In turn, the director curve β is given by $\beta(s) = s\,\mathbb{N} + \mathbb{M}$ for some constant light-like vector $\mathbb{N}$ and a space-like unit vector $\mathbb{M}$ such that $\langle \mathbb{N}, \mathbb{M} \rangle = 0$. We also obtain $\langle \alpha'', \alpha'' \rangle = 0$ from the second equation of (5.24). Together with the third equation of (5.24), we have $\alpha''' = 0$. So, α'' is a constant vector $\mathbb{F}$. If $\mathbb{F}$ is the zero vector, then α' is a constant vector $\mathbb{G}$ orthogonal to $\mathbb{N}$ and $\mathbb{M}$. Thus, up to congruence, M is parametrized as

$$x(s,t) = (ts, ts, s, t) \tag{5.33}$$

which lies in $\mathbf{E}_1^4$, which is an Euclidean plane. If $\mathbb{F}$ is light-like, then up to congruence, $\alpha(s) = (s^2/2)\,\mathbb{F} + s\,\mathbb{G}$ for some space-like unit vector $\langle \mathbb{F}, \mathbb{G} \rangle = 0$. Since $\langle \alpha', \beta \rangle = 0$ and $\langle \alpha', \beta' \rangle = 0$, $\langle \mathbb{M}, \mathbb{F} \rangle = \langle \mathbb{N}, \mathbb{F} \rangle = \langle \mathbb{G}, \mathbb{N} \rangle = \langle \mathbb{G}, \mathbb{M} \rangle = 0$. Since $\mathbb{F}$ and $\mathbb{N}$ are both light-like, they are collinear. Thus, up to congruence, M can be parametrized as

$$x(s,t) = \big((s^2/2) + ts,\ (s^2/2) + ts,\ t,\ s\big) \tag{5.34}$$

which lies in $\mathbf{E}_1^4$. It is the quadric ruled surface of the second kind given in Section 3.

Next, consider the case of $u \neq 0$. From (5.24), we get $\langle \alpha'', \alpha'' \rangle = u^2$ and $\alpha''' = -\langle \alpha'', \beta' \rangle \beta - u\beta'$. Taking into account of the second equation of (5.20) and the causal character of β', we get $\beta'' \equiv 0$. Therefore, β' is a constant light-like vector $\mathbb{C}_1$ and thus we have

$$\beta(s) = s\,\mathbb{C}_1 + \mathbf{s}\,\mathbb{D}_1 \tag{5.35}$$

where $\mathbb{D}_1$ is a unit space-like constant vector orthogonal to $\mathbb{C}_1$. Since $\langle \alpha', \beta' \rangle = u$, $\alpha''' = -u\,\mathbb{C}_1$. By solving this differential equation, we obtain

$$\alpha(s) = -u(s^3/6)\,\mathbb{C}_1 + (s^2/2)\,\mathbb{D}_2 + s\,\mathbb{E}_1 \tag{5.36}$$

where $\langle \mathbb{C}_1, \mathbb{D}_2 \rangle = 0$, $\langle \mathbb{D}_2, \mathbb{D}_2 \rangle = u^2$, $\langle \mathbb{C}_1, \mathbb{E}_1 \rangle = u$, $\langle \mathbb{D}_2, \mathbb{E}_1 \rangle = 0$ and $\langle \mathbb{E}_1, \mathbb{E}_1 \rangle = \varepsilon_1$. Also, $\langle \alpha', \beta \rangle = 0$ yields $\langle \mathbb{D}_1, \mathbb{E}_1 \rangle = 0$ and $\langle \mathbb{D}_1, \mathbb{D}_2 \rangle = -u$. So, the vector $\mathbb{D}_1$ and $\mathbb{D}_2$ are collinear. Hence, if α is space-like, that is, $\varepsilon_1 = 1$, then up to congruence, the ruled surface M is given by

$$x(s,t) = \left(-\frac{s^3}{6} + ts,\ \frac{s^2}{2} - t,\ -\frac{s^3}{6} + s + ts \right) \tag{5.37}$$

which is of type M_+^2 and lies in $\mathbf{E}_1^3$. In case α is time-like, that is, $\varepsilon_1 = -1$, M is parametrized up to congruence by

$$x(s,t) = \left(-\frac{s^3}{6} + s + ts,\ \frac{s^2}{2} - t,\ -\frac{s^3}{6} + ts \right) \tag{5.38}$$

which is of type M_-^2 and lies in $\mathbf{E}_1^3$. The ruled surfaces represented by (5.37) and (5.38) are the so-called conjugate Enneper's surface of the second kind, see [**11**].

The converse is straightforward. Consequently, the proof is completed. $\square$

We now prove

Theorem 5.6. *Let M be a null scroll with pointwise 1-type Gauss map in an m-dimensional Minkowski space $\mathbf{E}_1^m$. Then, M is an open part of a Minkowski plane or an extended B-scroll.*

Proof. Let $\alpha = \alpha(s)$ be a light-like curve in $\mathbf{E}_1^m$ and $\beta = \beta(s)$ be a light-like vector field along α. Then, the null scroll M is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle \beta, \beta \rangle = 0$ and $\langle \alpha', \beta \rangle = -1$. We have the natural frame $\{x_s, x_t\}$ given by

$$x_s = \alpha' + t\beta', \quad x_t = \beta(s). \tag{5.39}$$

Furthermore, we may choose an appropriate parameter s in such a way that $u = \langle \alpha'(s), \beta'(s) \rangle = 0$, which is possible if the base curve α is chosen as a geodesic of M. Again, we define smooth functions q and v as follows:

$$q = ||x_s||^2 = \langle x_s, x_s \rangle, \quad v = \langle \beta', \beta' \rangle. \tag{5.40}$$

Similarly as before, the Laplacian Δ of M can be given as follows:

$$\Delta = -2 \frac{\partial^2}{\partial s \partial t} + \frac{\partial q}{\partial t} \frac{\partial}{\partial t} + q \frac{\partial^2}{\partial t^2}. \tag{5.41}$$

Since $\langle x_s \wedge x_t, x_s \wedge x_t \rangle = -1$, the Gauss map G of M is determined by

$$G = (x_s \wedge x_t) = (\alpha' + t\beta') \wedge \beta = \alpha' \wedge \beta + t\beta' \wedge \beta. \tag{5.42}$$

Using (5.41) and (5.42), we can compute

$$\Delta G = -2(\beta'' - vt\beta') \wedge \beta. \tag{5.43}$$

Since the Gauss map G is of pointwise 1-type,

$$\{f(\alpha' + t\beta') + 2(\beta'' - vt\beta')\} \wedge \beta = 0, \tag{5.44}$$

which implies

$$(f - 2v)\beta' \wedge \beta = 0.$$

Consider an open subset $\mathcal{O} = \{p \in M \mid f(p) \neq 2v(p)\}$. Suppose that $\mathcal{O}$ is not empty. Thus, $\beta' \wedge \beta = 0$ on $\mathcal{O}$. Since $\langle \alpha', \beta' \rangle = 0$ and $\langle \alpha', \beta \rangle = -1$, $\beta' = 0$ on $\mathcal{O}$. Thus, β is a constant vector and $\Delta G = 0$ on $\mathcal{O}$. Then, (5.44) implies $f(\alpha' \wedge \beta) = 0$. Since $f \neq 0$ on $\mathcal{O}$, we have $\alpha' \wedge \beta = 0$ that contradicts the fact $G = \alpha' \wedge \beta$ in this case. Consequently, the open subset $\mathcal{O}$ is empty. Hence, the function f depends only on s such that

$$f = 2v.$$

Together with (5.44), we get

$$(f\alpha' + 2\beta'') \wedge \beta = 0,$$

which implies the function v is constant on M by considering our setting for α and β. In the sequel, $f = 2v$ is constant, that is the

Gauss map is of ordinary 1-type. According to classification theorem of ruled surfaces in Minkowski space with 1-type Gauss map, we have the theorem. □

Remark. The extended B-scroll includes the Minkowski planes as one of flat extended B-scrolls, see [**8**].

REFERENCES

1. L.J. Alías, A. Ferrández, P. Lucas and M.A. Meroño, *On the Gauss map of B-scrolls*, Tsukuba J. Math. **22** (1998), 371–377.

2. C. Baikoussis, B.-Y. Chen and L. Verstraelen, *Ruled surfaces and tubes with finite type Gauss map*, Tokyo J. Math. **16** (1993), 341–348.

3. B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Ruled surfaces of finite type*, Bull. Austral. Math. Soc. **42** (1990), 447–453.

4. B.-Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc. **35** (1987), 161–186.

5. F. Dillen, J. Pas and L. Verstraelen, *On the Gauss map of surfaces of revolution*, Bull. Inst. Math. Acad. Sinica **18** (1990), 239–249.

6. L.K. Graves, *Codimension one isometric immersions between Lorentz spaces*, Trans. Amer. Math. Soc. **252** (1979), 367–392.

7. D.-S. Kim and Y.H. Kim, *Null* 2*-type surfaces in Minkowski* 4*-space*, Houston J. Math. **22** (1996), 279–296.

8. D.-S. Kim, Y.H. Kim and D.W. Yoon, *Extended B-scrolls and their Gauss maps*, Indian J. Pure Appl. Math. **33** (2002), 1031–1040.

9. ———, *Characterization of generalized B-scrolls and cylinders over finite type curves*, Indian J. Pure Appl. Math. **34** (2003), 1523–1532.

10. Y.H. Kim and D.W. Yoon, *Ruled surfaces with finite type Gauss map in Minkowski spaces*, Soochow J. Math. **26** (2000), 85–96.

11. ———, *Ruled surfaces with pointwise* 1*-type Gauss map*, J. Geom. Phys. **34** (2000), 191–205.

12. ———, *Correction to*: *Ruled surfaces with finite type Gauss map in Minkowski spaces*, Soochow J. Math. **31** (2005), 1–3.

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