ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 6, 2005

# EXACT NUMBER OF SOLUTIONS FOR SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper establishes the exact multiplicities and properties of positive solutions for singular Dirichlet boundary value problems of second order ordinary differential equations.

**1. Introduction.** In this paper we study the exact multiplicities and properties of positive solutions of the following singular boundary value problems

$$(1_\lambda) \qquad \begin{cases} -x''(t) = \lambda(x^q(t) + kx(t) + x^{-m}(t)) & t \in (0,1), \\ x(t) > 0 & t \in (0,1), \\ x(0) = x(1) = 0 \end{cases}$$

where $\lambda$ is a parameter, $k \geq 0$ is a constant and $q,\ m$ satisfy either

(H1) $0 \leq m \leq 1/3$, $1 < q < \infty$; or

(H2) $1/3 < m < 1$, and

$$1 < q < 1+\left[\frac{1+m}{2(3m-1)}\right]\left[(3-5m) + \sqrt{(3-5m)^2 + 8(3m-1)(1-m)}\right].$$

By singularity we mean that the function $f = \lambda(x^{-m}(t) + x^q(t) + kx(t))$ in $(1_\lambda)$ is unbounded at the end points $t = 0$ and $t = 1$. A function $x(t) \in C[0,1] \cap C^2(0,1)$ is called a $C[0,1]$ positive solution of $(1_\lambda)$ if it satisfies $(1_\lambda)$. A $C[0,1]$ positive solution of $(1_\lambda)$ is called a $C^1[0,1]$ positive solution if $x'(0^+)$ and $x'(1^-)$ both exist.

Problem $(1_\lambda)$ comes from a problem raised by Agarwal and O'Regan [**1**]. Agarwal and O'Regan proved the equation

$$\begin{cases} y''(t) + \delta(y^{-\alpha}(t) + y^\beta(t) + 1) = 0 & 0 < t < 1 \\ y(0) = y(1) = 0 & \delta > 0 \text{ a parameter} \end{cases}$$

2000 AMS *Mathematics Subject Classification.* Primary 34B15.
*Key words and phrases.* Singular boundary value problem, positive solution, exact structure of all solutions, properties of solutions.
Received by the editors on January 15, 2003, and in revised form on May 5, 2003.

with $0 \leq \alpha < 1 < \beta$, has a nonnegative solution for all $\delta > 0$ small enough.

Exact multiplicity results are usually difficult to establish, see, e.g., [**6**]. The exact number of solutions was studied earlier by many authors for both elliptic and ordinary differential equations involving only concave or convex nonlinearities or cubic polynomials, see [**2–5, 10, 12, 14, 15**]; the references [**2–5, 10, 12, 14, 15**] didn't consider some properties of solution. Liu in [**7**] and [**8**] considered the following two point boundary value problem

$$\begin{cases} -v''(t) = \mu(v^p(x) + v^q(x) + kv(x)) & a \leq x \leq b \\ v(x) > 0 & x \in (a,b) \\ v(a) = v(b) = 0, \end{cases} \tag{2}$$

where $0 < q < 1 < p$ and $k \geq 0$ are fixed given numbers and $\mu > 0$ is parameter. He not only gave the exact number of solutions of (2) but also many interesting properties of the solutions. The existence of positive solutions for singular boundary value problems has been investigated by many authors for both elliptic and ordinary differential equations, see [**9, 11, 13, 16**] and the references therein.

In this paper we first give the exact multiplicity results of solutions of $(1_\lambda)$ and some useful properties of the solutions. Then we will give some several important lemmas. Finally, we will give the proof of the main results.

**2. The main result.** Let us list the main results which are the following theorems.

**Theorem 1.** *Suppose either* (H1) *or* (H2) *holds. There exists a number* $\lambda^*$ *with* $0 < \lambda^* < +\infty$ *such that*

(i) *for* $\lambda > \lambda^*$, $(1_\lambda)$ *has no solution*;

(ii) *for* $\lambda = \lambda^*$, $(1_\lambda)$ *has exactly one solution* $x_{\lambda*}$;

(iii) *for* $0 < \lambda < \lambda^*$, $(1_\lambda)$ *has exactly two solutions* $x_{\lambda,\,1}$, $x_{\lambda,\,2}$ *with* $x_{\lambda,\,1}(t) < x_{\lambda,\,2}(t)$ *in* $t \in (0,1)$.

*Moreover, if we denote* $x_{\lambda^*,\,1} = x_{\lambda^*,\,2} = x_{\lambda^*}$, *then the solutions of* $(1_\lambda)$ *satisfy the following properties*:

(iv) *for* $0 < \lambda_1 < \lambda_2 \le \lambda^*$ *and* $0 < t < 1$, $x_{\lambda_1,\,1}(t) < x_{\lambda_2,\,1}(t)$ ;

(v) *for* $0 < \lambda_1 < \lambda_2 \le \lambda^*$ *and* $0 < t < 1$ , $x_{\lambda_1,\,2}(t) > \sqrt{\lambda_1/\lambda_2}\, x_{\lambda_2,\,2}(t)$; *for* $0 < \lambda_1 < \lambda_2 \le \lambda^*$, $\|x_{\lambda_1,\,2}\| > \|x_{\lambda_2,\,2}\|$;

(vi) $x_{\lambda,\,1}$ *and* $x_{\lambda,\,2}$ *are continuous from* $(0, \lambda^*]$ *to* $C^1[0,1]$;

(vii) *for* $0 < t < 1$, $\lim_{\lambda\to 0^+} x_{\lambda,\,1}(t) = 0$, $\lim_{\lambda\to 0^+} x_{\lambda,\,2}(t) = \infty$.

**Theorem 2.** *Suppose either* (H1) *or* (H2) *holds. For* $0 < \lambda_1 < \lambda_2 \le \lambda^*$ *and* $0 < t < 1$, *we have* $x_{\lambda_1,\,2}(t) > x_{\lambda_2,\,2}(t)$ *if one of the following conditions is satisfied.*

(i) $(m+q)^2 < 4 + 2(q-m) + 4\sqrt{(1+q)(1-m)}$;

(ii) $(m+q)^2 < 4 + 2(q-m) + [(2(1-m^2))/(q-1)]$.

**3. Several important lemmas.** Denote $f(t) = t^q + kt + t^{-m}$, $F(t) = [(t^{q+1})/(q+1)] + kt^2/2 + [(t^{1-m})/(1-m)]$. Define a function $g:\ (0,+\infty) \to (0,+\infty)$ as

$$g(s) = \int_0^s [F(s) - F(t)]^{-1/2}\, dt, \quad s > 0,$$

that is,

$$g(s) = s \int_0^1 [F(s) - F(st)]^{-1/2}\ dt, \quad s > 0. \tag{3}$$

where

$$F(s) - F(st) = \frac{s^{q+1}(1-t^{q+1})}{q+1} + \frac{1}{2}\, ks^2(1-t^2) + \frac{1}{1-m}\, s^{1-m}(1-t^{1-m}).$$

**Lemma 1.** *For* $0 \le m < 1 < q < +\infty$, $g(s)$ *has continuous derivatives up to the second order on* $(0, +\infty)$, *and*

$$g'(s) = \frac{1}{2}\int_0^1 G_1(s,t)[F(s) - F(st)]^{-3/2}\, dt, \tag{4}$$

$$\begin{aligned} g''(s) &= \frac{1}{2}\int_0^1 G_2(s,t)[F(s)-F(st)]^{-3/2}\,dt \\ &\quad - \frac{3}{4}\int_0^1 G_3(s,t)[F(s)-F(st)]^{-5/2}\,dt. \end{aligned} \tag{5}$$

*where*

$$\begin{aligned} G_1(s,t) &= \frac{1-q}{1+q}\,s^{1+q}(1-t^{1+q}) + \frac{1+m}{1-m}\,s^{1-m}(1-t^{1-m}) \\ G_2(s,t) &= (1-q)s^q(1-t^{1+q}) + (1+m)s^{-m}(1-t^{1-m}) \\ G_3(s,t) &= \left[s^q(1-t^{1+q}) + s^{-m}(1-t^{1-m})\right] \\ &\quad \times \left[\left(\frac{1-q}{1+q}\right)s^{1+q}\left(1-t^{1+q}+\left(\frac{1+m}{1-m}\right)s^{1-m}(1-t^{1-m})\right)\right]. \end{aligned}$$

*Proof.* For $0 \le t \le 1$, we have

$$\begin{aligned} 1-t \le 1-t^{q+1} &\le (q+1)(1-t), \\ 1-t \le 1-t^2 &\le 2(1-t), \\ (1-m)(1-t) \le 1-t^{1-m} &\le (1-t). \end{aligned}$$

It follows that, for any $0 < s_1 < s_2 < +\infty$, there exists a positive constant $C$ depending only on $p$, $q$, $s_1$ and $s_2$ such that, if $s_1 \le s \le s_2$ and $0 \le t \le 1$, the absolute value of each integrand in (3)–(5) is less than $C(1-t)^{-1/2}$. This implies that each singular integral in (3)–(5) converges uniformly with respect to $s_1 \le s \le s_2$. Therefore, $g(s)$ has continuous derivatives up to the second order on $[s_1, s_2]$ and $g'(s)$ and $g''(s)$ have expressions (4) and (5), respectively. In view of the fact that $s_1$ and $s_2$ are arbitrary, we get the result.

**Lemma 2.** *For $0 \le p < 1 < q < +\infty$, we have*

(i) $\lim_{s\to 0^+} g(s) = 0$, $\lim_{s\to+\infty} g(s) = 0$;

(ii) *There are two numbers $0 < s_1 < s_2 < +\infty$ such that $g'(s) > 0$ for $0 < s \le s_1$ and $g'(s) < 0$ for $s \ge s_2$.*

*Proof.* Since

$$F(s) - F(st) \ge s^{1-m}(1-t),$$

and

$$F(s) - F(st) \geq \frac{s^{1+q}(1-t)}{1+q},$$

for $s > 0$ and $0 \leq t \leq 1$, from the definition of $g(s)$, we get that

$$g(s) \leq 2s \cdot s^{-(1-m)/2} \longrightarrow 0, \quad s \to 0,$$

and

$$g(s) \leq 2(1+q)^{1/2} s^{(1-q)/2} \longrightarrow 0, \quad s \to \infty.$$

By $0 < p < 1 < q < +\infty$ and $0 \leq m < 1$ it is easy to see that, for $0 < t < 1$,

$$\begin{aligned} \frac{1-m}{1+q}\,(1-t^{1+q}) &< 1-t^{1-m} < 1-t^{1+q}, \\ \frac{2}{1+q}\,(1-t^{1+q}) &< 1-t^2 < 1-t^{1+q}. \end{aligned}$$

From (4) we see that, if we let $s_1 = [(1+m)/(q-1)]^{1/(q+m)}$, and

$$s_2 = \left[\left(\frac{1+m}{1-m}\right)\left(\frac{q+1}{q-1}\right)\right]^{1/(q+m)},$$

$g'(s) > 0$ for $0 < s \leq s_1$ and $g'(s) < 0$ for $s \geq s_2$.

**Lemma 3.** *For $0 \leq m < 1 < q < +\infty$, $s > 0$, and $0 < t < 1$, we have*

$$G(s,0) < G(s,t) < G(s,1). \tag{6}$$

*where*

$$G(s,0) = \left(s^{q+1} + ks^2 + s^{1-m}\right) \bigg/ \left(\frac{s^{1+q}}{1+q} + \frac{ks^2}{2} + \frac{s^{1-m}}{1-m}\right),$$

$$G(s,t) = \left[s^{q+1}(1-t^{1+q}) + ks^2(1-t^2) + s^{1-m}(1-t^{1-m})\right][F(s)-F(st)]^{-1},$$

$$G(s,1) = \frac{(1+q)s^{q+1} + 2ks^2 + (1-m)s^{1-m}}{s^{1+q} + ks^2 + s^{1-m}}.$$

*Proof.* A direct computation shows that

$$\frac{\partial G(s,t)}{\partial t} = \frac{H_1(s,t)}{[F(s)-F(st)]^2}$$

where

$$\begin{aligned} H_1(s,t) = &-[(q+1)s^{q+1}t^q + 2ks^2t + (1-m)s^{1-m}t^{-m}][F(s)-F(st)] \\ &+ [s^{q+1}(1-t^{q+1}) + ks^2(1-t^2) + s^{1-m}(1-t^{1-m})] \\ &\times [s^{q+1}t^q + ks^2t + s^{1-m}t^{-m}]. \end{aligned}$$

Compute $H_1(s,t)$ as

$$\begin{aligned} H_1(s,t) = &\,k\,\frac{1-q}{2}\,s^{q+3}(1-t^2)\,t^q - \frac{m+q}{1-m}\,s^{q+2-m}(1-t^{1-m})t^q \\ &+ k\,\frac{q-1}{q+1}\,s^{q+3}(1-t^{q+1})\,t - k\,\frac{m+1}{1-m}\,s^{3-m}(1-t^{1-m})t \\ &+ \frac{q+m}{q+1}\,s^{q+2-m}(1-t^{q+1})\,t^{-m} + k\,\frac{m+1}{2}\,s^{3-m}(1-t^2)t^{-m}, \end{aligned}$$

that is,

$$\begin{aligned} H_1(s,t) = &\,k\frac{(q-1)}{2(q+1)}\,s^{q+3}t\left[2-(q+1)\,t^{q-1}+(q-1)\,t^{q+1}\right] \\ &+ \frac{(q+m)}{(q+1)(1-m)}\,s^{q+2-m}t^{-m} \\ &\times \left[(1-m)-(q+1)\,t^{q+m}+(q+m)\,t^{q+1}\right] \\ &+ k\frac{(m+1)}{2(1-m)}\,s^{3-m}t^{-m}\left[-2t^{m+1}+(1-m)+(1+m)\,t^2\right]. \end{aligned}$$

Let

$$\begin{aligned} J_1(t) &= 2-(q+1)\,t^{q-1}+(q-1)\,t^{q+1}, \\ J_2(t) &= (1-m)-(q+1)\,t^{q+m}+(q+m)\,t^{q+1}, \\ J_3(t) &= -2t^{m+1}+(1-m)+(1+m)\,t^2. \end{aligned}$$

Then, for $0<t<1$,

$$\begin{aligned} J_1'(t) &= -(q+1)(q-1)\,t^{q-2}(1-t^2) < 0, \\ J_2'(t) &= -(q+1)(q+m)\,t^{q+m-1}(1-t^{1-m}) < 0, \\ J_3'(t) &= -2(m+1)\,t^m(1-t^{1-m}) < 0. \end{aligned}$$

It follows that $J_i(t) > J_i(1) = 0$, $0 \le t < 1$, $i = 1,\ 2,\ 3$. Therefore, $(\partial/\partial t)G(s,t) > 0$, and for $s > 0$ and $0 < t < 1$,

$$G(s,0) < G(s,t) < \lim_{t\to 1^-} G(s,t),$$

which is just (6).

**Lemma 4.** *If either* (H1) *or* (H2) *holds, and* $g'(s_*) = 0$, *then* $g''(s_*) < 0$.

*Proof*. Assume that $g'(s_*) = 0$ for $s_* > 0$. Then from (4) we see that

$$(7)\quad \left(\frac{1+m}{1-m}\right) s_*^{1-m} \int_0^1 \frac{(1-t^{1-m})}{[F(s_*)-F(s_*t)]^{3/2}}\,dt = \left(\frac{q-1}{1+q}\right) s_*^{1+q} \int_0^1 \frac{(1-t^{1+q})}{[F(s_*)-F(s_*t)]^{3/2}}\,dt,$$

that is,

$$(8)\quad s_*^{m+q} = \left(\frac{1+m}{1-m}\right)\left(\frac{q+1}{q-1}\right) \int_0^1 \frac{(1-t^{1-m})}{[F(s_*)-F(s_*t)]^{3/2}}\,dt \Bigg/ \int_0^1 \frac{(1-t^{1+q})}{[F(s_*)-F(s_*t)]^{3/2}}\,dt.$$

Since $[(1-m)/(1+q)](1-t^{1+q}) < 1-t^{1-m} < 1-t^{1+q}$, for $0 < t < 1$, it follows that

$$(9)\qquad s_1^{m+q} = \frac{1+m}{q-1} < s_*^{p+q} < \left(\frac{1+m}{1-m}\right)\left(\frac{q+1}{q-1}\right) = s_2^{m+q}.$$

For the sake of brevity, we denote $s_*$ by $s$ in the rest of the proof of this lemma. From (5)–(9) we can estimate as follows

(10)

$$\begin{aligned}
g''(s) &= \frac{1}{2}\int_0^1 \left[(1+m)s^{-m}(1-t^{1-m}) + (1-q)s^q(1-t^{1+q})\right] \\
&\quad \times [F(s)-F(st)]^{-3/2}\,dt \\
&\quad - \frac{3}{4}\int_0^1 \left[\left(\frac{1+m}{1-m}\right)s^{-m}(1-t^{1-m}) + \left(\frac{1-q}{1+q}\right)s^q(1-t^{1+q})\right] \\
&\quad \times [F(s)-F(st)]^{-3/2}G(s,t)\,dt \\
&< \frac{1}{2}\int_0^1 \left[\left(\frac{q-1}{q+1}\right)((1-m)-(1+q))\,s^q(1-t^{1+q})\right] \\
&\quad \times [F(s)-F(st)]^{-3/2}\,dt \\
&\quad - \frac{3}{4}\int_0^1 \left[\left(\frac{q-1}{q+1}\right)s^q(1-t^{1+q})\right][F(s)-F(st)]^{-3/2}G(s,0)\,dt \\
&\quad + \frac{3}{4}\int_0^1 \left[\left(\frac{q-1}{q+1}\right)s^q(1-t^{1+q})\right][F(s)-F(st)]^{-3/2}G(s,1)\,dt \\
&= s^q\int_0^1 (1-t^{1+q})[F(s)-F(st)]^{-3/2}\,dt \times \frac{1}{4}\left(\frac{q-1}{q+1}\right) \\
&\quad \times [-2(q+m) + 3(G(s,1)-G(s,0))] \quad \text{(use (7))}.
\end{aligned}$$

Let

$$K = [2(q+m) - 3(G(s,1)-G(s,0))]\,.$$

Then

(11)

$$\begin{aligned}
K = &\frac{k}{2}\,s^{q+1} \\
&\times \left[\frac{2(q+m)(q+3)-3(q-1)^2}{(q+1)} + \frac{2(q+m)(3-m)-3(m+1)^2}{(1-m)}s^{-(q+m)}\right] \\
&+ \frac{(q+m)}{(q+1)(1-m)}\,s^{q-m} \\
&\times \left[4-q-5m+2(1-m)s^{q+m} + 2(q+1)s^{-(q+m)}\right] + k^2(q+m)s^2.
\end{aligned}$$

Let

$$\begin{aligned}K_1 &= \frac{2(q+m)(q+3)-3(q-1)^2}{(q+1)}\\ &\quad + \frac{2(q+m)(3-m)-3(m+1)^2}{(1-m)}\, s^{-(q+m)},\\ K_2 &= 4-q-5m+2(1-m)s^{q+m}+2(q+1)s^{-(q+m)}.\end{aligned}$$

Then

$$\begin{aligned}K_1 &> \frac{2(q+m)(q+3)-3(q-1)^2}{(q+1)} + \frac{2(q+m)(3-m)-3(m+1)^2}{(1-m)}\\ &\quad \times \left(\frac{1-m}{1+m}\right)\left(\frac{q-1}{1+q}\right) = \frac{1}{(1+m)(q+1)}\\ &\quad \times \left[q^2(5-3m)+3q(1-m^2)+16qm+11m^2+3m\right] > 0.\\ K_2 &> 2(1-m)\left(\frac{m+1}{q-1}\right)\\ &\quad + 2(1+q)\frac{(1-m)(q-1)}{(1+m)(1+q)} + 4-q-5m\\ &= 2(1-m)\left(\frac{m+1}{q-1}+\frac{q-1}{1+m}\right)+4-q-5m = K_3.\end{aligned}$$

If (H1) holds, then $K_2 > K_3 = 3-5m+[(q-1)/(1+m)](1-3m)+[(2(1-m)(m+1))/(q-1)] > 0$.

If (H2) holds, let $q = 1+a$. Then

$$\begin{aligned}K_2 &> 2(1-m)\left(\frac{m+1}{a}+\frac{a}{1+m}\right)+3-5m-a = \frac{1}{a(1+m)}\\ &\quad \left(-(3m-1)a^2+(1+m)(3-5m)a+2(1-m)(1+m)^2\right) > 0.\end{aligned}$$

In either case, we have $K_2 > 0$. Equations (10) and (11) imply that $g''(s) < 0$.

**Lemma 5.** *If either* (H1) *or* (H2) *holds, there exists* $s_*$ *such that* $g'(s_*) = 0$, $g'(s) > 0$ *for* $0 < s < s_*$, $g'(s) < 0$ *for* $s > s_*$.

*Proof.* Combining Lemmas 2 and 4 gives the results of this lemma.

## 4. The proof of the main result.

*Proof of Theorem* 1. Let $x(t)$ be a solution of $(1_\lambda)$. Then it is well known that $x(t)$ takes its maximum at $1/2$, $x(t)$ is symmetric with respect to $1/2$, $x'(t) > 0$ for $0 \le t < 1/2$ and $x'(t) < 0$ for $1/2 < t \le 1$. Hence, $(1_\lambda)$ is equivalent to the following problem defined on $[0, 1/2]$

$$(11_\lambda) \qquad \begin{cases} -x''(t) = \lambda(x^{-m}(t) + x^q(t) + kx(t)) & 0 < t \le 1/2, \\ x(t) > 0 & 0 < t \le 1/2, \\ x(0) = x'(1/2) = 0. \end{cases}$$

Multiply the first equality with $x'(t)$ and integrate it from $t$ to $1/2$. Then

$$(12_\lambda) \qquad (x'(t))^2 = 2\lambda\,[F(x(1/2)) - F(x(t))].$$

Denote $x(1/2)$ by $s$, take the square root of $(12_\lambda)$, and then integrate it from 0 to $t$. It follows that

$$(13_\lambda) \qquad \int_0^{x(t)} [F(s) - F(\xi)]^{-1/2}\,d\xi = (2\lambda)^{1/2} t.$$

Choosing $t$ to be $1/2$ in $(13_\lambda)$, we get that

$$(14_\lambda) \qquad g(s) = \int_0^{s} [F(s) - F(\xi)]^{-1/2}\,d\xi = (\lambda/2)^{1/2}.$$

Conversely, for a given $\lambda$, if $s$ satisfies $(14_\lambda)$, then $(13_\lambda)$ defines a function $x(t)$ on $[0, 1/2]$ satisfying $(11_\lambda)$ with $x(1/2) = s$, and it is easy to see that $x(t)$ is a solution of $(11_\lambda)$. Therefore the number of solutions of $(11_\lambda)$ is equal to the number of $s$ satisfying $(14_\lambda)$. According to Lemma 5, there exists a number $s_* > 0$ such that $g'(s_*) = 0$, $g'(s) > 0$ for $0 < s < s_*$, and $g'(s) < 0$ for $s > s_*$. Denote $\lambda^* = 2(g(s_*))^2$. From (i) in Lemma 2, we see that there is no $s$ satisfying $(14_\lambda)$ for $\lambda > \lambda^*$; there is exactly one $s$ satisfying $(14_\lambda)$ for $\lambda = \lambda^*$ and there are exactly two $s$ satisfying$(14_\lambda)$ for $0 < \lambda < \lambda^*$. Therefore, there is no solution of $(1_\lambda)$ for $\lambda > \lambda^*$; there is exactly one solution $x_{\lambda^*}$ of $(1_\lambda)$ for $\lambda = \lambda^*$ and there are exactly two solutions, denoted by $x_{\lambda,\,1}$ and $x_{\lambda,\,2}$ of $(1_\lambda)$ for $0 < \lambda < \lambda^*$. Without loss of generality, we can assume that

$\|x_{\lambda,\,1}\| < \|x_{\lambda,\,2}\|$ for $0 < \lambda < \lambda^*$ and denote $x_{\lambda^*,\,1} = x_{\lambda^*,\,2} = x_{\lambda^*}$. Then these solutions have the properties that $\|x_{\lambda^*}\| = s_*$, $g'(\|x^*_\lambda\|) = 0$, $\|x_{\lambda,\,1}\| < s_* < \|x_{\lambda,\,2}\|$, $g'(\|x_{\lambda,\,1}\|) > 0$, and $g'(\|x_{\lambda,\,2}\|) < 0$ for all $0 < \lambda < \lambda^*$. From Lemma 5 and the fact that $s_{\lambda,\,1} = \|x_{\lambda,\,1}\|$ and $s_{\lambda,\,2} = \|x_{\lambda,\,2}\|$ are the two numbers satisfying $(14_\lambda)$, we see that $\|x_{\lambda,\,1}\|$ is a continuous and strictly increasing function on $(0, \lambda^*]$. While $\|x_{\lambda,\,2}\|$ is a continuous and strictly decreasing function on $(0, \lambda^*]$. By the condition $0 \le p < 1 < q < +\infty$, we have

$$0 < \int_0^1 \big((t(1-t))^{-m} + (t(1-t))^q + kt(1-t)\big)\, dt < +\infty.$$

In view of the main results in [**9, 11, 16**], we get $x_{\lambda,\,1}$ and $x_{\lambda,\,2} \in C^1[0,1]$. By means of the continuous dependence on initial values and parameters of the solutions of initial value problems, we see that $x_{\lambda,\,1}$ and $x_{\lambda,\,2}$ are continuous from $(0, \lambda^*]$ to $\in C^1[0,1]$. By now, we have obtained the results in (i)–(iii) and (vi).

We claim that, for $0<\lambda_1<\lambda_2\le\lambda^*$, $x_{\lambda_1,\,1}(t) < x_{\lambda_2,\,1}(t)$, $0 < t < 1$.

For $0 < \lambda_1 < \lambda_2 \le \lambda^*$, since $\|x_{\lambda_1,\,1}\| < \|x_{\lambda_2,\,1}\|$, taking $t = 0$ in $(12_\lambda)$, we have

$$\frac{1}{2\lambda_1}\left(x'_{\lambda_1,\,1}(0)\right)^2 = F(\|x_{\lambda_1,\,1}\|) < F(\|x_{\lambda_2,\,1}\|) = \frac{1}{2\lambda_2}\left(x'_{\lambda_2,\,1}(0)\right)^2.$$

This implies that there exists $\varepsilon > 0$ such that for $0 < t < \varepsilon$,

$$x_{\lambda_1,\,1}(t) < x_{\lambda_2,\,1}(t). \tag{15}$$

If (15) does not hold for all $t \in (0,1)$, then there exists $t^* \in (0, 1/2)$ such that $x_{\lambda_1,\,1}(t^*) = x_{\lambda_2,\,1}(t^*)$. From $(13_\lambda)$, it follows that

$$\begin{aligned}\sqrt{2\lambda_1}\, t^* &= \int_0^{x_{\lambda_1,\,1}(t^*)} \left[F(\|x_{\lambda_1,\,1}\|) - F(\xi)\right]^{-1/2} d\xi\\ &> \int_0^{x_{\lambda_2,\,1}(t^*)} \left[F(\|x_{\lambda_2,\,1}\|) - F(\xi)\right]^{-1/2} d\xi\\ &= \sqrt{2\lambda_2}\, t^*.\end{aligned}$$

This implies $\lambda_1 > \lambda_2$, which is a contradiction. This proves (iv).

We claim that, for $0 < \lambda_1 < \lambda_2 \le \lambda^*$,

$$x_{\lambda_1,\,2}(t) > \sqrt{\frac{\lambda_1}{\lambda_2}}\, x_{\lambda_2,\,2}(t), \quad 0 < t < 1. \tag{16}$$

Indeed, for $0 < \lambda_1 < \lambda_2 \le \lambda^*$, since $\|x_{\lambda_1,\,2}\| > \|x_{\lambda_2,\,2}\|$, taking $t = 0$ in $(12_\lambda)$, we have

$$\frac{1}{2\lambda_1}\left(x'_{\lambda_1,\,2}(0)\right)^2 = F(\|x_{\lambda_1,\,2}\|) > F(\|x_{\lambda_2,\,2}\|) = \frac{1}{2\lambda_2}\left(x'_{\lambda_2,\,2}(0)\right)^2.$$

This implies that there exists $\varepsilon > 0$ such that (16) is valid for $0 < t < \varepsilon$. If (16) is not true for all $t \in (0,1)$, then there exists $t^* \in (0,1/2)$ such that $x_{\lambda_1,\,2}(t^*) = \sqrt{\lambda_1/\lambda_2}\, x_{\lambda_2,\,2}(t^*)$. From $(13_\lambda)$, it follows that

$$\begin{aligned}
\sqrt{2\lambda_1}\, t^* &= \int_0^{x_{\lambda_1,2}(t^*)} [F(\|x_{\lambda_1,\,2}\|) - F(\xi)]^{-1/2}\, d\xi \\
&= \sqrt{\frac{\lambda_1}{\lambda_2}} \int_0^{x_{\lambda_2,2}(t^*)} \left[F(\|x_{\lambda_1,\,2}\|) - F\left(\sqrt{\frac{\lambda_1}{\lambda_2}}\,\xi\right)\right]^{-1/2} d\xi \\
&< \sqrt{\frac{\lambda_1}{\lambda_2}} \int_0^{x_{\lambda_2,2}(t^*)} [F(\|x_{\lambda_2,\,2}\|) - F(\xi)]^{-1/2}\, d\xi \\
&= \sqrt{\frac{\lambda_1}{\lambda_2}}\, \sqrt{2\lambda_2}\, t^* = \sqrt{2\lambda_1}\, t^*,
\end{aligned}$$

which is contradiction. This proves (v).

Now we prove (vii). By Lemmas 2(i) and 5, it is easy to prove that

$$\lim_{\lambda\to 0^+} \|x_{\lambda,\,1}\| = 0, \qquad \lim_{\lambda\to 0^+} \|x_{\lambda,\,2}\| = +\infty. \tag{17}$$

The first limit in (vii) is true according to the first limit in (17).

Note that every solution of $(1)_\lambda$ is a concave function. Then we have

$$\begin{cases} x_{\lambda,\,2}(t) \ge [t/(1/2)]\|x_{\lambda,\,2}\| & 0 \le t \le 1/2, \\ x_{\lambda,\,2}(t) \ge [(1-t)/(1/2)]\|x_{\lambda,\,2}\| & 1/2 \le t \le 1, \end{cases} \tag{18}$$

which, combined with (17), implies the second inequality in (vii).

*Proof of Theorem* 2. In order to prove the conclusion: $x_{\lambda_1,\,2}(t) > x_{\lambda_2,\,2}(t)$ for $0 < \lambda_1 < \lambda_2 \le \lambda^*$ and $0 < t < 1$, we study the monotonicity of the function $\lambda F(\|x_{\lambda,\,2}\|)$.

From $(12)_\lambda$, we see that

$$\left(x'_{\lambda,\,2}(0)\right)^2 = 2\lambda F(x_{\lambda,\,2}(1/2)) = 2\lambda F(\|x_{\lambda,\,2}\|). \tag{19}$$

From $(14)_\lambda$, we have

$$(g(\|x_{\lambda,\,2}\|))^2 = \lambda/2. \tag{20}$$

Denoting $\|x_{\lambda,\,2}\|$ by $s_{\lambda,\,2}$ and differentiating (20) with respect to $\lambda$, we get that

$$\frac{ds_{\lambda,\,2}}{d\lambda} = \frac{1}{4g(s_{\lambda,\,2})g'(s_{\lambda,\,2})} = \frac{g(s_{\lambda,\,2})}{2\lambda g'(s_{\lambda,\,2})}. \tag{21}$$

Then, by (20) and (21),

(22)

$$\begin{aligned}\frac{d}{d\lambda}\,(\lambda F(s_{\lambda,\,2})) &= F(s_{\lambda,\,2}) + \lambda f(s_{\lambda,\,2})\,\frac{ds_{\lambda,\,2}}{d\lambda}\\ &= F(s_{\lambda,\,2}) + f(s_{\lambda,\,2})\cdot\frac{g(s_{\lambda,\,2})}{2g'(s_{\lambda,\,2})}\\ &= \frac{1}{g'(s_{\lambda,\,2})}\left[F(s_{\lambda,\,2})g'(s_{\lambda,\,2}) + f(s_{\lambda,\,2})g(s_{\lambda,\,2})/2\right],\end{aligned}$$

which can be rewritten as, by (3) and (4),

$$\frac{d}{d\lambda}\,(\lambda F(s_{\lambda,\,2})) = \frac{s^2_{\lambda,\,2}}{2g'(s_{\lambda,\,2})}\int_0^1 \frac{H(s_{\lambda,\,2},t)}{\left[F(s_{\lambda,\,2}) - F(s_{\lambda,\,2}t)\right]^{3/2}}\,dt, \tag{23}$$

where

$$\begin{aligned}H(s,t) = {} & \frac{1}{(1+q)}\,s^q(1-t^{1+q})\\ & \times\left(\frac{2}{1+q}\,s^q + \frac{k}{2}\,(3-q)s + \frac{2-(q+m)}{1-m}\,s^{-m}\right)\\ & + \frac{1}{(1-m)}\,s^{-m}(1-t^{1-m})\\ & \times\left(\frac{2+q+m}{1+q}\,s^q + \frac{k}{2}\,(3+m)s + \frac{2}{1-m}\,s^{-m}\right)\\ & + \frac{k}{2}\,(1-t^2)s\left(s^q + s + s^{-m}\right).\end{aligned} \tag{24}$$

For $0 < t < 1$, since $1 - t^{1-m} > [(1-m)/(1+q)](1 - t^{q+1})$, $1 - t^2 > [2/(1+q)](1 - t^{q+1})$,

$$\begin{aligned} H(s,t) > \frac{1}{(1+q)}\, s^q (1 - t^{q+1}) \bigg[ & \frac{2}{1+q}\, s^{2q} + \frac{2}{1-m}\, s^{-2m} \\ & + \left( \frac{2-(q+m)}{1-m} + \frac{2+q+m}{1+q} \right) s^{q-m} \\ & + \frac{k}{2}\,(5-q) s^{1+q} + \frac{k}{2}\,(5+m) s^{1-m} + k^2 s^2 \bigg]. \end{aligned} \tag{25}$$

Let

(26)
$$f_1(s) = \frac{k}{2}\,(5-q) s^{1+q} + \frac{k}{2}\,(5+m) s^{1-m} + k^2 s^2,$$

(27)
$$f_2(s) = \frac{2}{1+q}\, s^{2q} + \frac{2}{1-m}\, s^{-2m} + \left( \frac{2-(q+m)}{1-m} + \frac{2+q+m}{1+q} \right) s^{q-m}.$$

Then, from (9), we have

(28)
$$f_1(s) \geq \frac{k}{2}\, s^{1-m} \left[ (5-q) \left( \frac{1+m}{q-1} \right) + (5+m) \right] = \frac{2k(q+m)}{q-1}\, s^{1-m} \geq 0,$$

(29)
$$\begin{aligned} f_2(s) &= \frac{1}{(1+q)(1-m)}\, s^{q-m} \Big[ (q+1)(2-m-q) + (1-m)(2+q+m) \\ &\qquad + 2(1-m) s^{q+m} + 2(q+1) s^{-(q+m)} \Big] \\ &\geq \frac{1}{(1+q)(1-m)}\, s^{q-m} \\ &\qquad \times \left[ 4 + 2(1-m) - (q+m)^2 + 4\sqrt{(q+1)(1-m)} \right]. \end{aligned}$$

For case (i), $(m+q)^2 < 4 + 2(q-m) + 4\sqrt{(1+q)(1-m)}$ and from (24)–(29) we have $H(s,t) > 0$ for all $s > 0$ and $0 < t < 1$. Note that $g'(s_{\lambda,\,2}) < 0$ for $0 < \lambda < \lambda^*$. From (23) and (19), we see that, for $0 < \lambda < \lambda^*$,

$$\frac{d}{d\lambda}\,(\lambda F(s_{\lambda,\,2})) < 0, \qquad \frac{d}{d\lambda}\,(x'_{\lambda,\,2}(0)) < 0;$$

therefore $x'_{\lambda_1,\,2}(0) > x'_{\lambda_2,\,2}(0)$ for $0 < \lambda_1 < \lambda_2 \le \lambda^*$. If the result of Theorem 2 were not true, then there would be $0 < \lambda_1 < \lambda_2 \le \lambda^*$ and $t^* \in (0,1)$ which is the minimum of all $t \in (0,1)$ satisfying $x_{\lambda_1,\,2}(t) = x_{\lambda_2,\,2}(t)$. At the point $t^*$, we have on one hand

$$0 < x'_{\lambda_1,\,2}(t^*) \le x'_{\lambda_2,\,2}(t^*).$$

But on the other hand, by $(12)_\lambda$,

$$\begin{aligned}\left[x'_{\lambda_1,\,2}(t^*)\right]^2 &= 2\lambda_1[F(s_{\lambda_1,\,2}) - F(x_{\lambda_1,\,2}(t^*))],\\ \left[x'_{\lambda_2,\,2}(t^*)\right]^2 &= 2\lambda_2[F(s_{\lambda_2,\,2}) - F(x_{\lambda_2,\,2}(t^*))].\end{aligned}$$

Since $\lambda_1 F(s_{\lambda_1,\,2}) > \lambda_2 F(s_{\lambda_2,\,2})$ and $x_{\lambda_1,\,2}(t^*) = x_{\lambda_2,\,2}(t^*)$, we have $[x'_{\lambda_1,\,2}(t^*)]^2 > [x'_{\lambda_2,\,2}(t^*)]^2$. This is a contradiction.

For case (ii), $(m+q)^2 < 4 + 2(q-m) + [(2(1-m^2))/(q-1)]$ and $0 < \lambda_1 < \lambda_2 \le \lambda^*$. In this case (9) implies that, if $s > s_*$, then $H(s,t) > 0$. By (23)–(29) and the fact that $s_{\lambda,\,2} \ge s_*$, $(d/d\lambda)(\lambda F(s_{\lambda,\,2})) < 0$ for $0 < \lambda \le \lambda^*$. Then by the same argument as in case (i), we get that $x_{\lambda_1,\,2}(t) > x_{\lambda_2,\,2}(t)$ for $0 < \lambda_1 < \lambda_2 \le \lambda^*$ and $0 < t < 1$. The proof of Theorem 2 is complete. □

**Acknowledgments.** The work was done while the author was visiting at the Institute of Mathematics, Academia Sinica, Beijing. The author would like to thank Professor Sujie Li for his guidance and encouragement and the referee for some helpful comments and suggestions.

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