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A BEST APPROXIMATION THEOREM FOR NONEXPANSIVE SET-VALUED MAPPINGS IN HYPERCONVEX METRIC SPACES

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1. Introduction. Recent results have shown that many fixed point and best approximation theorems previously established for Banach spaces have analogues in hyperconvex metric spaces, see, for example, [**2, 4–6**]. In [**4**] the authors gave a hyperconvex version of the Fan best approximation theorem for set-valued mappings on compact sets. It is the purpose of this paper to show that a best approximation theorem can be obtained in hyperconvex spaces for set-valued mappings without compactness assumptions, under the additional requirement that the mappings are nonexpansive. This result is applied to obtain some fixed point theorems.

2. Preliminaries. Using $B(x,r)$ to denote the closed ball with center $x \in M$ and radius r, a metric space (M,d) is hyperconvex if, given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, it is the case that $\cap B(x_\alpha, r_\alpha) \neq \varnothing$. Hyperconvex metric spaces were introduced and their basic properties elaborated in [**1**].

The *externally hyperconvex* subsets (relative to M), denoted by $E(M)$, are those subsets S such that, given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $d(x_\alpha, S) \leq r_\alpha$, it follows that $S \cap (\cap B(x_\alpha, r_\alpha)) \neq \varnothing$. Throughout, $d(x,S) = \inf_{y \in S} d(x,y)$ for any subset S.

The admissible subsets of M, denoted by $A(M)$, are sets of the form $\cap B(x_\alpha, r_\alpha)$, i.e., the family of ball intersections in M. Admissible subsets are externally hyperconvex [**1**]. A subset S is *proximinal* provided for each $x \in M$ there is an $s \in S$ such that $d(x,s) = d(x,S)$.

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Externally hyperconvex subsets are proximinal [**1**].

For a subset S of M, $N_\varepsilon(S)$ denotes the closed ε-neighborhood of S, i.e., $N_\varepsilon(S) = \{x \in M : d(x, S) \le \varepsilon\}$. If S is externally hyperconvex (admissible), then $N_\varepsilon(S)$ is externally hyperconvex (admissible) [**3,6**].

If U, V are closed bounded subsets of M, let D be the *Hausdorff metric* defined as $D(U, V) = \inf\{\varepsilon > 0 : U \subseteq N_\varepsilon(V) \text{ and } V \subseteq N_\varepsilon(U)\}$. For any subset S of M, a set-valued mapping $F : S \to E(M)$ is *nonexpansive* if $D(F(x), F(y)) \le d(x, y)$ for any $x, y \in S$.

Lemma. *Let M be a hyperconvex metric space, X an admissible subset, and U and V externally hyperconvex subsets of M. Then, $D(N_\alpha(X) \cap U, N_\beta(X) \cap V) \le D(U, V)$, where*

$$\alpha = \inf\{\varepsilon > 0 : N_\varepsilon(X) \cap U \ne \varnothing\}$$

and

$$\beta = \inf\{\varepsilon > 0 : N_\varepsilon(X) \cap V \ne \varnothing\}.$$

Proof. We define the sets $U_0 = N_\alpha(X) \cap U$ and $V_0 = N_\beta(X) \cap V$ and observe that they are nonempty since U and V are externally hyperconvex sets. Assume $\alpha \ge \beta$. We prove the lemma by showing that $d(u, V_0) \le D(U, V)$, for each $u \in U_0$ and that $d(v, U_0)) \le D(U, V)$, for each $v \in V_0$.

By the definition of U_0,

$$d(u, N_\beta(X)) = \inf_{y \in U} d(y, N_\beta(X)) \quad \text{for any } u \in U_0. \tag{1}$$

Since $N_{D(U,V)}(U) \cap V_0 \ne \varnothing$, we have $N_{D(U,V)}(U) \cap N_\beta(X) \ne \varnothing$. In view of (1), for any $u \in U_0$, $d(u, N_\beta(X)) \le D(U, V)$, and therefore, $B(u, D(U, V)) \cap N_\beta(X)) \ne \varnothing$.

Since $N_\beta(X)$ is admissible, $N_\beta(X) = \cap B_i$, where each B_i is a ball in M. Clearly, $B_i \cap V \ne \varnothing$, and $B(u, D(U, V)) \cap B_i \ne \varnothing$, for each $u \in U_0$ and each i. Because V is externally hyperconvex, it follows that

$$B(u, D(U, V)) \cap N_\beta(X) \cap V \ne \varnothing, \quad \text{for } u \in U_0.$$

Thus for $u \in U_0$, $d(u, V_0) \le D(U, V)$.

Assume $v \in V_0$. Then $B(v, D(U,V)) \cap U \neq \varnothing$, $N_\alpha(X) \cap U \neq \varnothing$ by definition, and $B(v, D(U,V)) \cap N_\alpha(X) \neq \varnothing$ since $v \in N_\alpha(X)$. Because $N_\alpha(X)$ is admissible and U is externally hyperconvex, the same argument as above implies $B(v, D(U,V)) \cap N_\alpha(X) \cap U \neq \varnothing$. Hence, for any $v \in V_0$, $d(v,U) \leq D(U,V)$. It follows that $D(U_0, V_0) \leq D(U,V)$.

3. A best approximation theorem. The following Theorem 1 gives a set-valued version of the Ky Fan best approximation theorem for nonexpansive mappings defined on an admissible set with values in $E(M)$. The same result was obtained in [**4**] under the assumption that the domain is a compact admissible set and the mapping is continuous with values in $A(M)$. A point valued version of Theorem 1 appears in [**6**].

Theorem 1. *Let M be a bounded hyperconvex metric space, X an admissible subset and $F : X \to E(M) \setminus \{\varnothing\}$ a nonexpansive mapping. Then either there is an $x_0 \in X$ such that $x_0 \in F(x_0)$ or there is an x_0 in the boundary of X such that $0 < d(x_0, F(x_0)) \leq \inf_{x \in X} d(x, F(xo))$.*

Proof. For each $x \in X$ define the mapping $F_0 : X \to E(M)$ by $F_0(x) = N_\alpha(X) \cap F(x)$, where $\alpha = \inf_{z \in F(x)} d(z, X)$. The set $F_0(x)$ is a nonempty externally hyperconvex subset of M since it is the intersection of an admissible set and an externally hyperconvex set [**3**]. By the lemma F_0 is a nonexpansive mapping. The selection theorem in [**3**] implies the existence of a nonexpansive point-valued mapping $f_0 : X \to M$ such that

$$f_0(x) \in F_0(x) \quad \text{for } x \in X.$$

Define the mapping $P : M \to A(X)$ by $P(y) = \{x \in X : d(y,x) = \inf_{z \in X} d(y,z)\}$. Then, by a result in [**6**], there is a nonexpansive set-valued selection $P_0 : M \to A(X)$, where $P_0(x) \subseteq P(x)$ for $x \in M$.

Consider the mapping $P_0(f_0(\cdot)) : X \to A(X)$, which by definition is a nonexpansive set-valued mapping of X into itself. Since admissible subsets are externally hyperconvex, the fixed point existence theorem of [**3**] implies there is an $x_0 \in X$ such that $x_0 \in P_0(f_0(x_0))$. Thus, $d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0))$.

The remainder of the proof follows an idea of Park [**5**]. If $d(x_0, F(x_0)) = 0$, then x_0 is a fixed point of F. Otherwise, we have $0 < d(x_0, F(x_0)) \leq d(x, F(x_0))$ for each $x \in X$. To show that x_0 is in the boundary of X, assume that x_0 is in the interior of X. Then there is an $r > 0$ such that $B(x_0, r) \subseteq X$ and $r < d(x_0, F(x_0)) \leq d(x, F(x_0))$ for each $x \in B(x_0, r)$. By hyperconvexity there is a $y_0 \in B(x_0, r) \cap B(z, d(x_0, F(x_0)) - r)$, where $z \in F(x_0)$ and $d(x_0, z) = d(x_0, F(x_0))$. Hence, $d(y_0, F(x_0)) \leq d(x_0, F(x_0)) - r < d(x_0, F(x_0))$, which is a contradiction. Therefore, x_0 is in the boundary of X. □

4. Fixed point theorems. In this section we apply Theorem 1 to obtain some fixed point theorems for set-valued nonexpansive mappings with domain an admissible subset and values in $E(M)$. Theorem 2 and its corollary were obtained in [**4**] under the assumption that the domain of the mappings is a compact admissible subset and the mappings are continuous with values in $A(M)$. A point valued version of Theorem 3 appears in [**6**]

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Theorem 2. *Let M be a bounded hyperconvex metric space, X an admissible subset and $F : X \to E(M) \setminus \{\varnothing\}$ a nonexpansive set-valued mapping such that, for each $x \in X$ with $x \notin F(x)$, there exists $z \in X$ such that $d(z, F(x)) < d(x, F(x))$. Then there is an $x_0 \in X$ such that $x_0 \in F(x_0)$.*

Proof. By Theorem 1, there is an $x_0 \in X$ such that $d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0))$. We claim that x_0 is a fixed point of F. If not, then $x_0 \notin F(x_0)$ and, by assumption, there is a $z \in X$ such that $d(z, F(x_0)) < d(x_0, F(x_0))$. But this contradicts the fact that x_0 is a best approximation to $F(x_0)$ in X. Therefore, there is an $x_0 \in X$ such that $x_0 \in F(x_0)$. □

Corollary. *Let M be a bounded hyperconvex metric space, X an admissible subset and $F : X \to E(M) \setminus \{\varnothing\}$ a nonexpansive set-valued mapping such that for each $x \in X$, $F(x) \cap X \neq \varnothing$. Then there is an $x_0 \in X$ such that $x_0 \in F(x_0)$.*

Proof. Because $F(x) \cap X \neq \varnothing$ for each $x \in X$, it follows that for each $x \in X$ with $x \notin F(x)$, we can choose a $z \in F(x) \cap X$ such that $d(z, F(x)) = 0 < d(x, F(x)$. Thus, the conditions of Theorem 2 are satisfied and the conclusion follows.

Theorem 3. *Let M be a bounded hyperconvex metric space, X an admissible subset and $F : X \to E(M) \setminus \{\varnothing\}$ a nonexpansive set-valued mapping. If $F(x) \subseteq X$ for each x in the boundary of X, then there is an $x_0 \in X$ such that $x_0 \in F(x_0)$.*

Proof. Assume that F does not have a fixed point. By Theorem 1, there is an x_0 in the boundary of X such that $0 < d(x_0, F(x_0)) \leq \inf_{x \in X} d(x, F(x_0))$. However, since x_0 is in the boundary of X, $F(x_0) \subseteq X$, and therefore $\inf_{x \in X} d(x, F(x_0)) = 0$. This is a contradiction, implying the theorem. $\square$

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