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NON-EXISTENCE OF CERTAIN 3-STRUCTURES

T. KASHIWADA, F. MARTIN CABRERA AND MUKUT MANI TRIPATHI

ABSTRACT. We introduce the notion of an ε-framed 3-structure. This is a general structure which includes many widely studied 3-structures (almost quaternion, almost contact, hyper f-structure, almost product, etc.). We prove the existence of Riemannian metrics compatible with such a structure. We also study particular cases of ε-framed 3-structures showing the non-existence of certain remarkable types of such structures. First, we prove the non-existence of P-Sasakian almost r-paracontact 3-structures. Then, we show the non-existence of almost r-contact $\mathcal{S}$-3-structures (with $r > 1$). Finally, we establish the non-existence of proper trans-Sasakian almost contact 3-structures. A consequence of this last result is that any b-Kenmotsu almost contact 3-structure must be hypercosymplectic.

1. Introduction. In 1963, Yano [**33**] introduced the notion of an f-structure on a manifold, which is defined by a non-null $(1,1)$ tensor field f satisfying $f^3 + f = 0$. The concept of an f-structure includes the notions of almost complex and almost contact structures and it is well known that it is genuinely a more general structure. For instance, hypersurfaces of almost contact manifolds are not in general almost complex manifolds, but they have always f-structures associated to them.

Almost product structures are another type of structure widely studied by several authors, see [**34, 21**]. Analogously to the situation for almost complex and almost contact structures, almost paracontact structures are closely related to almost product structures. The concept of an $f(3,\varepsilon)$-structure was introduced in [**30**] as a uniform way of treating all the above geometries and several others. An $f(3,\varepsilon)$-structure, $\varepsilon \in \{\pm 1\}$, is defined by a non-null $(1,1)$ tensor field f satisfying $f^3 - \varepsilon f = 0$. It turns out that f is of constant rank and there are two complementary distributions associated with the $f(3,\varepsilon)$-structure, as happens with f-structures and some other known cases.

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The quaternionic analog of almost complex structure is the almost quaternion (hypercomplex) geometry which is defined by 3 local (global) almost complex structures which satisfy the same relations as the unit imaginary quaternions [**12**]. Quaternion Kähler manifolds and hyper-Kähler manifolds are special and interesting cases of Riemannian manifolds with almost quaternion and almost hypercomplex structures, respectively. Quaternion Kähler manifolds are Einstein, hyper-Kähler manifolds are Ricci flat and their respective holonomy groups are included in the Berger list [**1**]. Hypersurfaces of manifolds with almost hypercomplex structure inherit naturally three almost contact structures which constitute an almost contact 3-structure. This last type of geometric structure was defined by Kuo [**17**] and it is closely related to both almost quaternion and almost hypercomplex structures.

A particular and interesting class of almost contact 3-structure is the Sasakian 3-structure. Riemannian manifolds with Sasakian 3-structure are called 3-Sasakian manifolds. They are Einstein [**14**] and have many links with quaternion Kähler and hyper-Kähler manifolds. In fact, a 3-Sasakian manifold, with some regularity conditions, fibers over a quaternion Kähler manifold [**12**] and can be imbedded in a hyper-Kähler manifold [**5**].

In [**11**], Hernández studied quaternionic and hyper f-structures which are the natural extension of almost quaternion and almost contact 3-structures. He proves some interesting results, which relate f-structures and hyper f-structures. For instance, it is shown that compact manifolds with regular normal hyper f-structure of corank 3 fiber over hypercomplex manifolds. Moreover, if the hyper f-structure is 3-contact (in such a case, the manifold is called PS*-Sasakian*), then the fibration is over a hyper-Kähler manifold. This is one of the motivations for Hernández' study of PS-structures.

In the present paper, we begin by giving diverse preliminary definitions and related concepts which we need to introduce the notion of an ε-framed 3-structure. This is a general structure which includes the structures mentioned in the last two paragraphs and others (almost product 3-structures, almost paracontact 3-structures, etc.) and, as it happens for all these structures, it is proved that ε-framed 3-structures always admit Riemannian metrics compatible with them.

Next, we study particular cases of ε-framed 3-structures showing the non-existence of certain remarkable types of such structures. First, we prove the non-existence of almost r-paracontact 3-structures (1-framed 3-structures) of P-Sasakian type. Second, we show the non-existence for $r > 1$ of almost r-contact 3-structures (-1-framed 3-structures) where each one is an $\mathcal{S}$-structure (when $r = 1$, an $\mathcal{S}$-structure is a Sasakian structure). It is also proved that a manifold equipped with an almost r-contact structure ($r > 1$) of $\mathcal{S}$-structure type cannot be Einstein. Finally, we establish the non-existence of proper trans-Sasakian almost contact 3-structures (a particular case of $r = 1$ and -1-framed 3-structures). Namely, we prove that any trans-Sasakian 3-structure must be a-Sasakian, a type of structure whose metric is the constant multiple a^2 of a 3-Sasakian structure. Then, as a consequence of this last result, any almost contact 3-structure of b-Kenmotsu type [**16**] must be hypercosymplectic.

2. ε-framed f-structure. Let M be an n-dimensional differentiable manifold, and let there be given a nowhere zero tensor field f of type $(1,1)$ satisfying

$$f^3 - \varepsilon f = 0, \quad \varepsilon^2 = 1. \tag{1}$$

We call such a structure a *$f\,(3,\varepsilon)$-structure*. If M is connected, following [**26**], we know that the rank of f is constant. Let rank $(f) = k$. If we put

$$\mathfrak{l} = \varepsilon f^2, \quad \mathfrak{m} = I - \varepsilon f^2,$$

then the tensors $\mathfrak{l}$, $\mathfrak{m}$ acting in the tangent space at each point of M are commuting projection operators which define complementary distributions $\mathcal{L}$ and $\mathcal{M}$. The dimension of the distribution $\mathcal{L}$ is k and $\mathcal{M}$ has dimension $(n-k)$. For $\varepsilon = -1$, $f\,(3,\varepsilon)$-structures are f-structures [**33**]; and in this case rank (f) is always even.

Let $n - k = r$. Suppose M admits r linearly independent vector fields $\xi_1, \dots, \xi_r$ spanning the distribution $\mathcal{M}$ at each point of M. If in addition, there are r 1-forms $\eta^1, \dots, \eta^r$ such that

$$f(\xi_\alpha) = 0, \tag{2}$$

$$f^2 = \varepsilon(I - \eta^\alpha \otimes \xi_\alpha), \tag{3}$$

then the structure $\Sigma = (f, \xi_\alpha, \eta^\alpha)$ is called an *ε-framed structure* on M, and the pair (M, Σ) or simply M is called an *ε-framed manifold.*

From the above two equations it follows that

$$\eta^\alpha \circ f = 0, \tag{4}$$

$$\eta^\alpha (\xi_\beta) = \delta^\alpha_\beta. \tag{5}$$

If M is an ε-framed manifold, there always exists a positive definite Riemannian metric g on M with respect to which $\mathcal{L}$ and $\mathcal{M}$ are orthogonal and

$$g(X, \xi_\alpha) = \eta^\alpha(X), \tag{6}$$

$$g(fX, fY) = g(X, Y) - \sum_\alpha \eta^\alpha (X) \eta^\alpha (Y). \tag{7}$$

The set$(\Sigma, g) = (f, \xi_\alpha, \eta^\alpha, g)$ is said to be an *ε-framed metric structure*, [**30**], on M and M equipped with this structure is called an *ε-framed metric manifold*. The above metric is said to be a metric associated to the ε-framed structure on M.

In view of (6) and (7), on an ε-framed metric manifold M we always have

$$g(\xi_\alpha, fX) = 0, \tag{8}$$

$$F(X, Y) := g(X, fY) = \varepsilon F(Y, X). \tag{9}$$

The ε-framed metric structure, respectively manifold, is a general structure, respectively manifold, which in special cases reduces to several known structures, respectively manifolds, shown below which have been widely studied in the past.

ε	**r**	**Structure/manifold**
-1		framed metric [**33**] or
-1		almost r-contact metric [**32**]
-1	1	almost contact metric [**3**]
-1	0	almost Hermitian [**34**]
1		almost r-paracontact Riemannian [**7**]
1	1	almost paracontact Riemannian [**23**]
1	0	almost product Riemannian [**34, 21**]

Notation. Throughout the paper the following notations will be followed:

(a) X, Y, Z are vector fields on M.

(b) $\mathcal{C}\,(1,2,3)$ is the set of all cyclic permutations of $(1,2,3)$.

(c) $\alpha, \beta, \gamma, \varepsilon$ run over $\{1, \dots, r\}$.

(d) an ε-framed 3-structure will have the constituent structures

$\Sigma_{(\lambda)} = \left(f_{(\lambda)}\,,\ \xi_{(\lambda)_\alpha},\ \eta^\alpha_{(\lambda)}\right)$, $\lambda = 1,2,3$ satisfying the relations (2), (3) and (15).

3. Non-uniqueness of ε-framed structures. In view of (2)–(5), we are able to state the following theorem.

Theorem 3.1. *Let $(f, \xi_\alpha, \eta^\alpha)$ and $(f, \xi_\alpha, \bar{\eta}^\alpha)$, respectively $(f, \bar{\xi}_\alpha, \eta^\alpha)$, be two ε-framed structures on a manifold M. Then we have $\eta^\alpha = \bar{\eta}^\alpha$, respectively $\xi_\alpha = \bar{\xi}_\alpha$.*

Thus we see that two ε-framed structures having the same f and the same ξ_α, respectively η^α, on a manifold are always identical. However, an ε-framed structure on a manifold M always induces another ε-framed structure on M. This is proved in the following

Theorem 3.2. *An ε-framed structure on a manifold is not unique.*

Proof. Let $(f, \xi_\alpha, \eta^\alpha)$ be an ε-framed structure on a manifold M. Let ψ be a non-singular $(1,1)$ tensor on M. Defining

$$\bar{f} = \psi^{-1} f \psi, \quad \bar{\xi}_\alpha = \psi^{-1}\xi_\alpha, \quad \bar{\eta}^\alpha = \eta^\alpha \circ \psi, \tag{10}$$

it is easy to verify that $\left(\bar{f}, \bar{\xi}_\alpha, \bar{\eta}^\alpha\right)$ is also an ε-framed structure on a manifold M. □

Moreover, if g is an associated metric to the structure $(f, \xi_\alpha, \eta^\alpha)$ of M, then a metric $\bar{g}$ on M defined by

$$\bar{g}\,(X, Y) = g\,(\psi X, \psi Y) \tag{11}$$

provides an associated metric to the structure $\left(\bar{f}, \bar{\xi}_\alpha, \bar{\eta}^\alpha\right)$ of M. We may state this fact as the following

Theorem 3.3. *An ε-framed metric structure on a manifold is not unique.*

4. An ε-framed 3-structure. Let $\Sigma_{(\lambda)} = \left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^\alpha_{(\lambda)}\right)$, $\lambda = 1, 2$ be two ε-framed structures on an n-dimensional manifold M which satisfy

$$\left.\begin{aligned} \eta^\alpha_{(1)}\left(\xi_{(2)_\beta}\right) = 0 = \eta^\alpha_{(2)}\left(\xi_{(1)_\beta}\right), \\ f_{(1)}\left(\xi_{(2)_\beta}\right) = \varepsilon f_{(2)}\left(\xi_{(1)_\beta}\right), \quad \eta^\alpha_{(1)} \circ f_{(2)} = \varepsilon \eta^\alpha_{(2)} \circ f_{(1)}, \\ f_{(1)} f_{(2)} + \varepsilon \eta^\alpha_{(2)} \otimes \xi_{(1)_\alpha} = \varepsilon\left(f_{(2)} f_{(1)} + \epsilon \eta^\alpha_{(1)} \otimes \xi_{(2)_\alpha}\right). \end{aligned}\right\} \tag{12}$$

Defining $\Sigma_{(3)} = \left(f_{(3)}, \xi_{(3)_\alpha}, \eta^\alpha_{(3)}\right)$ on M by

$$\left.\begin{aligned} f_{(3)} = f_{(1)} f_{(2)} + \varepsilon \eta^\alpha_{(2)} \otimes \xi_{(1)_\alpha} = \varepsilon\left(f_{(2)} f_{(1)} + \varepsilon \eta^\alpha_{(1)} \otimes \xi_{(2)_\alpha}\right), \\ \xi_{(3)_\beta} = f_{(1)}\left(\xi_{(2)_\beta}\right) = \varepsilon f_{(2)}\left(\xi_{(1)_\beta}\right), \\ \eta^\alpha_{(3)} = \eta^\alpha_{(1)} \circ f_{(2)} = \varepsilon \eta^\alpha_{(2)} \circ f_{(1)}, \end{aligned}\right\} \tag{13}$$

it is easy to verify that $\Sigma_{(3)}$ defines an ε-framed structure on M. We can also verify the following relations:

(14)

$$\left.\begin{aligned} f_{(1)} = f_{(2)} f_{(3)} + \varepsilon \eta^\alpha_{(3)} \otimes \xi_{(2)_\alpha} = \varepsilon\left(f_{(3)} f_{(2)} + \varepsilon \eta^\alpha_{(2)} \otimes \xi_{(3)_\alpha}\right), \\ f_{(2)} = f_{(3)} f_{(1)} + \varepsilon \eta^\alpha_{(1)} \otimes \xi_{(3)_\alpha} = \varepsilon\left(f_{(1)} f_{(3)} + \varepsilon \eta^\alpha_{(3)} \otimes \xi_{(1)_\alpha}\right), \\ \xi_{(1)_\beta} = f_{(2)}\left(\xi_{(3)_\beta}\right) = \varepsilon f_{(3)}\left(\xi_{(2)_\beta}\right), \quad \xi_{(2)_\beta} = f_{(3)}\left(\xi_{(1)_\beta}\right) = \varepsilon f_{(1)}\left(\xi_{(3)_\beta}\right), \\ \eta^\alpha_{(1)} = \eta^\alpha_{(2)} \circ f_{(3)} = \varepsilon \eta^\alpha_{(3)} \circ f_{(2)}, \quad \eta^\alpha_{(2)} = \eta^\alpha_{(3)} \circ f_{(1)} = \varepsilon \eta^\alpha_{(1)} \circ f_{(3)}, \\ \eta^\alpha_{(2)}\left(\xi_{(3)_\beta}\right) = 0 = \eta^\alpha_{(3)}\left(\xi_{(2)_\beta}\right), \quad \eta^\alpha_{(3)}\left(\xi_{(1)_\beta}\right) = 0 = \eta^\alpha_{(1)}\left(\xi_{(3)_\beta}\right). \end{aligned}\right\}$$

Equations (13) and (14) together are invariant under a cyclic permutation of subindices $1, 2, 3$ enclosed by parenthesis. Now, we make formally the following

Definition 4.1. Let M be a manifold with a rank 3 subbundle $\mathcal{F} \subset End\,(TM)$ and a rank $3r$ subbundle $\mathcal{E} \subset TM$. Suppose $\mathcal{E}$ has a global basis $\bigcup_{\lambda=1}^{3}\{\xi_{(\lambda)_1}, \dots, \xi_{(\lambda)_r}\}$ and that $\mathcal{F}$ has a local basis $\{f_{(1)}, f_{(2)}, f_{(3)}\}$. If each $\left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}\right)$ extends to an ε-framed structure $\left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^{\alpha}_{(\lambda)}\right)$ and these structures are compatible in the sense that

$$\left.\begin{aligned} \eta^{\alpha}_{(\lambda)}\left(\xi_{(\mu)_\beta}\right) = 0 = \eta^{\alpha}_{(\mu)}\left(\xi_{(\lambda)_\beta}\right), \quad \lambda \neq \mu, \\ f_{(\lambda)}\left(\xi_{(\mu)_\beta}\right) = \varepsilon f_{(\mu)}\left(\xi_{(\lambda)_\beta}\right) = \xi_{(\nu)_\beta}, \\ \eta^{\alpha}_{(\lambda)} \circ f_{(\mu)} = \varepsilon \eta^{\alpha}_{(\mu)} \circ f_{(\lambda)}, = \eta^{\alpha}_{(\nu)}, \\ f_{(\lambda)} f_{(\mu)} + \varepsilon \eta^{\alpha}_{(\mu)} \otimes \xi_{(\lambda)_\alpha} = \varepsilon\left(f_{(\mu)} f_{(\lambda)} + \varepsilon \eta^{\alpha}_{(\lambda)} \otimes \xi_{(\mu)_\alpha}\right) = f_{(\nu)} \end{aligned}\right\} \tag{15}$$

for $(\lambda, \mu, \nu) \in \mathcal{C}\,(1,2,3)$, then we say that M is equipped with an *ε-framed* 3*-structure*. If $\mathcal{F}$ admits a global basis $\{f_{(1)}, f_{(2)}, f_{(3)}\}$ with the above properties, we say that M admits *a hyper ε-framed* 3*-structure*.

Hyper ε-framed 3-structures and ε-framed 3-structures are generalized structures, which in special cases reduce to following structures.

ε	r	**Basis of $\mathcal{F}$**	**Structure**
-1	1	global	almost contact 3-structure [**31, 17**]
-1	0	local	almost quaternion structure [**12**]
-1	0	global	almost hypercomplex structure [**12**]
-1	r	local	almost quaternionic f-structure [**11**]
-1	r	global	hyper f-structure [**11**]
1	1	global	almost paracontact 3-structure [**8**]
1	0	global	almost product 3-structure

Note that if we consider the local tensor fields

$$f_{(\lambda)} - \eta^{\alpha}_{(\mu)} \otimes \xi_{(\nu)_\alpha} - \varepsilon \eta^{\alpha}_{(\nu)} \otimes \xi_{(\mu)_\alpha},$$

for $(\lambda, \mu, \nu) \in \mathcal{C}\,(1,2,3)$, we will have another local basis of the bundle $\mathcal{F}$ which, for $\varepsilon = -1$, satisfies the conditions of the definition of almost quaternionic f-structure given by Hernández [**11**].

Theorem 4.2. *If* $\Sigma_{(\lambda)} = \left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^\alpha_{(\lambda)}\right)$, $\lambda = 1, 2, 3$, *are three* ε*-framed structures on a manifold* M *satisfying the conditions given by* (15), *then*

(i) *the* $3r$ *vector fields* $\xi_{(1)_\beta}$, $\xi_{(2)_\beta}$, $\xi_{(3)_\beta}$ *are linearly independent*,

(ii) *the* $3r$ 1*-forms* $\eta^\alpha_{(1)}$, $\eta^\alpha_{(2)}$, $\eta^\alpha_{(3)}$ *are linearly independent, and*

(iii) *the* 3 *tensor fields* $f_{(1)}$, $f_{(2)}$, $f_{(3)}$ *are linearly independent.*

Proof. Let $h^\alpha_{(\lambda)}$ be $3r$ real-valued smooth functions on M such that

$$h^\alpha_{(1)}\xi_{(1)_\alpha} + h^\beta_{(2)}\xi_{(2)_\beta} + h^\gamma_{(3)}\xi_{(3)_\gamma} = 0. \tag{16}$$

Operating by $\eta^\varepsilon_{(\lambda)}$, $\lambda = 1, 2, 3$, we get

$$0 = h^\alpha_{(\lambda)}\delta^\varepsilon_\alpha = h^\varepsilon_{(\lambda)}.$$

Thus part (i) is proved. Similarly (ii) can be proved. Finally, let

$$h_1 f_{(1)} + h_2 f_{(2)} + h_3 f_{(3)} = 0,$$

where h_1, h_2, h_3 are real valued smooth functions on M. Operating the above equation by $\eta^\alpha_{(1)}$ and using (4) and (15) we obtain

$$h_2\eta^\alpha_{(3)} + \varepsilon h_3\eta^\alpha_{(2)} = 0,$$

which in view of (ii), gives $h_2 = h_3 = 0$, and ultimately $h_1 = 0$. This proves (iii). □

5. Existence of an associated metric. An associated metric for an ε-framed 3-structure in a manifold M is a Riemannian metric which is associated to each of the three constituent local structures; in such a case we say that we have an ε-framed metric 3-structure. In this section we establish the existence of such a metric.

First, we give some lemmas.

Lemma 5.1. *On an* ε*-framed metric manifold* M *we always have*

$$g(\xi_\alpha, fX) = 0, \tag{17}$$

$$g(X, fY) = \varepsilon g(fX, Y). \tag{18}$$

The proof follows from (6) and (7).

Lemma 5.2. *If a hyper ε-framed 3-structure manifold admits a Riemannian metric g which is associated to any two of the constituent structures, then g is also associated to the third constituent structure.*

Proof. Let g be a Riemannian metric associated to the structures $\left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^\alpha_{(\lambda)}\right)$ and $\left(f_{(\mu)}, \xi_{(\mu)_\alpha}, \eta^\alpha_{(\mu)}\right)$. Then, we show that g is also associated to $\left(f_{(\nu)}, \xi_{(\nu)_\alpha}, \eta^\alpha_{(\nu)}\right)$, where $(\lambda, \mu, \nu) \in \mathcal{C}(1,2,3)$. In view of $(15)_2$, (18), (6) and $(15)_3$ we have

$$\begin{aligned} g\left(\xi_{(\nu)_\alpha}, X\right) &= g\left(f_{(\lambda)}\left(\xi_{(\mu)_\alpha}\right), X\right) = \varepsilon g\left(\xi_{(\mu)_\alpha}, f_{(\lambda)}X\right) \\ &= \varepsilon \eta^\alpha_{(\mu)}\left(f_{(\lambda)}X\right) = \eta^\alpha_{(\nu)}(X). \end{aligned}$$

Similarly,

$$\begin{aligned} & g\left(f_{(\nu)}X, f_{(\nu)}Y\right) \\ &= g\left(f_{(\lambda)}f_{(\mu)}X + \varepsilon\eta^\alpha_{(\mu)}(X)\,\xi_{(\lambda)_\alpha}, f_{(\lambda)}f_{(\mu)}X + \varepsilon\eta^\beta_{(\mu)}(X)\,\xi_{(\lambda)_\beta}\right) \\ &= g\left(f_{(\lambda)}f_{(\mu)}X, f_{(\lambda)}f_{(\mu)}Y\right) + \varepsilon\eta^\alpha_{(\mu)}(X)\,g\left(\xi_{(\lambda)_\alpha}, f_{(\lambda)}f_{(\mu)}Y\right) \\ &\quad + \varepsilon\eta^\beta_{(\mu)}(Y)\,g\left(f_{(\lambda)}f_{(\mu)}X, \xi_{(\lambda)_\beta}\right) + \eta^\alpha_{(\mu)}(X)\,\eta^\beta_{(\mu)}(Y)\,g\left(\xi_{(\lambda)_\alpha}, \xi_{(\lambda)_\beta}\right) \\ &= g\left(f_{(\mu)}X, f_{(\mu)}Y\right) - \sum_\alpha \eta^\alpha_{(\lambda)}\left(f_{(\mu)}X\right)\eta^\alpha_{(\lambda)}\left(f_{(\mu)}Y\right) + \sum_\alpha \eta^\alpha_{(\mu)}(X)\,\eta^\alpha_{(\mu)}(Y) \\ &= g(X,Y) - \sum_\alpha \eta^\alpha_{(\nu)}(X)\,\eta^\alpha_{(\nu)}(Y), \end{aligned}$$

where $(15)_2$, (17), (6) and (7) are used. Thus the lemma is proved. □

Lemma 5.3. *Let a manifold M admit a hyper ε-framed 3-structure. Let G be a Riemannian metric associated to one of the constituent structures, say $\Sigma_{(\lambda)}$; then there exists a Riemannian metric G' on M*

which satisfies for $(\lambda, \mu, \nu) \in \mathcal{C}(1,2,3)$ *the following relations*

(19)

$$\left.\begin{array}{c} G'\left(\xi_{(\lambda)\alpha}, X\right) = \eta^{\alpha}_{(\lambda)}(X), \quad G'\left(\xi_{(\mu)\alpha}, X\right) = \eta^{\alpha}_{(\mu)}(X), \\ G'\left(\xi_{(\nu)\alpha}, X\right) = G\left(\xi_{(\nu)\alpha}, X\right) - \eta^{\beta}_{(\mu)}(X)\, G\left(\xi_{(\mu)\beta}, \xi_{(\nu)\alpha}\right), \\ G'\left(\xi_{(\lambda)\alpha}, \xi_{(\mu)\beta}\right) = 0,\ G'\left(\xi_{(\mu)\alpha}, \xi_{(\nu)\beta}\right) = 0,\ G'\left(\xi_{(\nu)\alpha}, \xi_{(\lambda)\beta}\right) = 0. \end{array}\right\}$$

Proof. Let G' be defined on M by

$$\begin{aligned} G'(X,Y) = G\Big(X - \eta^{\alpha}_{(\mu)}(X)\,\xi_{(\mu)\alpha}, Y - \eta^{\beta}_{(\mu)}(Y)\,\xi_{(\mu)\beta}\Big) \\ + \sum_{\alpha} \eta^{\alpha}_{(\mu)}(X)\,\eta^{\alpha}_{(\mu)}(Y). \end{aligned} \tag{20}$$

It is easy to verify that G' is a Riemannian metric on M. Using (20), $(15)_1$, (3), (6) and (7) one can easily prove the results of Lemma 5.3. □

Lemma 5.4. *A hyper* ε*-framed* 3*-structure manifold* M *always admits a Riemannian metric* g' *such that*

$$g'\left(\xi_{(\lambda)\alpha}, X\right) = \eta^{\alpha}_{(\lambda)}(X), \quad \lambda = 1,2,3. \tag{21}$$

Proof. Defining g' on M by

$$\begin{aligned} g'(X,Y) = G'\Big(X - \eta^{\alpha}_{(\nu)}(X)\,\xi_{(\nu)\alpha}, Y - \eta^{\beta}_{(\nu)}(Y)\,\xi_{(\nu)\beta}\Big) \\ + \sum_{\alpha} \eta^{\alpha}_{(\nu)}(X)\,\eta^{\alpha}_{(\nu)}(Y), \end{aligned} \tag{22}$$

where G' is defined by (20), we see that g' is a Riemannian metric on M. Again using (21), $(15)_1$, (3), (6) and (7) along with Lemma 5.3, we can easily prove Lemma 5.4. □

In view of the above four lemmas, we have

(23)
$$\left.\begin{aligned} g'\left(\left(f_{(\lambda)}\right)^2X,\left(f_{(\lambda)}\right)^2Y\right) &= g'(X,Y)-\sum_\alpha \eta^\alpha_{(\lambda)}(X)\,\eta^\alpha_{(\lambda)}(Y),\\ g'\left(f_{(\lambda)}f_{(\mu)}X,f_{(\lambda)}f_{(\mu)}Y\right) &= g'\left(f_{(\nu)}X,f_{(\nu)}Y\right)-\sum_\alpha \eta^\alpha_{(\mu)}(X)\,\eta^\alpha_{(\mu)}(Y),\\ g'\left(f_{(\mu)}f_{(\lambda)}X,f_{(\mu)}f_{(\lambda)}Y\right) &= g'\left(f_{(\nu)}X,f_{(\nu)}Y\right)-\sum_\alpha \eta^\alpha_{(\lambda)}(X)\,\eta^\alpha_{(\lambda)}(Y).\end{aligned}\right\}$$

Now, we are in a position to prove the main result of this section as follows.

Theorem 5.5. *A hyper ε-framed 3-structure manifold M always admits an associated Riemannian metric.*

Proof. We define a $(0,2)$ tensor field g on M by

$$g(X,Y)=\frac{1}{4}\Bigg[g'(X,Y)+\sum_{\lambda=1}^{3}\Bigg(g'\left(f_{(\lambda)}X,f_{(\lambda)}Y\right) + \sum_\alpha \eta^\alpha_{(\lambda)}(X)\,\eta^\alpha_{(\lambda)}(Y)\Bigg)\Bigg], \tag{24}$$

where g' is the Riemannian metric defined by (22). It is easy to verify that g is a Riemannian metric on M. Putting $X={\xi_{(\lambda)}}_\alpha$, $\lambda=1,2,3$ in (24) and using Lemma 5.4, (4), (3) and (15), we get

$$\begin{aligned} g\left({\xi_{(\lambda)}}_\alpha,Y\right) &= \frac{1}{4}\Bigg[g'\left({\xi_{(\lambda)}}_\alpha,Y\right)+g'\left(f_{(\mu)}{\xi_{(\lambda)}}_\alpha,f_{(\mu)}Y\right)\\ &\qquad + g'\left(f_{(\nu)}{\xi_{(\lambda)}}_\alpha,f_{(\nu)}Y\right)+\sum_\beta \eta^\beta_{(\lambda)}\left({\xi_{(\lambda)}}_\alpha\right)\eta^\beta_{(\lambda)}(Y)\Bigg]\\ &= \frac{1}{4}\left(\eta^\alpha_{(\lambda)}(Y)+\varepsilon g'\left({\xi_{(\lambda)}}_\alpha,f_{(\mu)}Y\right)+g'\left({\xi_{(\mu)}}_\alpha,f_{(\nu)}Y\right)+\eta^\alpha_{(\lambda)}(Y)\right)\\ &= \eta^\alpha_{(\lambda)}(Y).\end{aligned}$$

Replacing X and Y by $f_{(\lambda)}X$ and $f_{(\lambda)}Y$, $\lambda=1,2,3$, respectively in (24) and using (5), (23), (15) and (24), we get

$$g\left(f_{(\lambda)}X,f_{(\lambda)}Y\right)=g(X,Y)-\sum_\alpha \eta^\alpha_{(\lambda)}(X)\,\eta^\alpha_{(\lambda)}(Y).$$

This completes the proof. □

If we have an ε-framed 3-structure manifold, we can take into account Theorem 5.5 to claim the existence of local Riemannian metrics associated with the local hyper ε-framed 3-structures. Then, by using partitions of unity, we can construct a global metric compatible with the ε-framed 3-structure of the manifold. Hence, we are able to state the following

Theorem 5.6. *An ε-framed 3-structure manifold always admits an associated Riemannian metric.*

Example 5.7. Taking $r = 2$, we construct an example of an ε-framed metric 3-structure in the Euclidean space $\mathbf{R}^6$. We define $\left(f_{(\lambda)}, \xi_{(\lambda)1}, \xi_{(\lambda)2}, \eta^1_{(\lambda)}, \eta^2_{(\lambda)}\right)$, $\lambda = 1, 2, 3$ and a metric g in $\mathbf{R}^6$ by their matrices as follows:

$$f_{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 & 0 \end{bmatrix}, \qquad f_{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$f_{(3)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \xi_{(1)1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_{(1)2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \xi_{(2)1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_{(2)_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \xi_{(3)_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_{(3)_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\begin{aligned} \eta^1_{(1)} &= [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0], & \eta^2_{(1)} &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0], \\ \eta^1_{(2)} &= [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], & \eta^2_{(2)} &= [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0], \\ \eta^1_{(3)} &= [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0], & \eta^2_{(3)} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \end{aligned}$$

and

$$g = I_6.$$

By direct computation, we find that the above set provides the required structure on $\mathbf{R}^6$ and g is its associated metric.

6. Non-existence of an almost r-paracontact metric 3-structure of P-Sasakian type. Taking $\varepsilon = 1$, the ε-framed metric structure becomes an almost r-paracontact metric structure [**7**], that is,

$$\left.\begin{aligned} f^2 = I - \eta^\alpha \otimes \xi_\alpha, \\ F(X,Y) = g(X, fY) = F(Y,X). \end{aligned}\right\} \tag{26}$$

An almost r-paracontact metric structure is [**6**] of *paracontact type* if

$$2F(X,Y) = (\nabla_X \eta^\alpha) Y + (\nabla_Y \eta^\alpha) X, \tag{27}$$

of *s-paracontact type* if

$$fX = \nabla_X \xi_\alpha \quad \text{or equivalently} \quad F(X,Y) = (\nabla_X \eta^\alpha) Y, \tag{28}$$

of *P-Sasakian type* if it is of s-paracontact type and

$$(29)\qquad \begin{aligned}(\nabla_Z F)(X,Y) = &-\sum_\beta \eta^\beta(X)\left(g(Y,Z) - \sum_\alpha \eta^\alpha(Y)\eta^\alpha(Z)\right)\\ &-\sum_\beta \eta^\beta(Y)\left(g(X,Z) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Z)\right),\end{aligned}$$

of *SP-Sasakian type* if it is of s-paracontact type and

$$(30)\qquad F(X,Y) = e\left(g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\right), \quad e^2 = 1.$$

An almost r-paracontact metric structure of SP-Sasakian type is always of P-Sasakian type.

Setting $\varepsilon = 1$, in Definition 4.1, we can define

Definition 6.1. A manifold M equipped with three almost r-paracontact structures $\Sigma_{(\lambda)} = \left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^\alpha_{(\lambda)}\right)$, $\lambda = 1,2,3$ satisfying

$$(31)\qquad \left.\begin{aligned}\eta^\alpha_{(\lambda)}\left(\xi_{(\mu)_\beta}\right) = 0 = \eta^\alpha_{(\mu)}\left(\xi_{(\lambda)_\beta}\right), \quad \lambda \neq \mu,\\ f_{(\lambda)}\left(\xi_{(\mu)_\beta}\right) = f_{(\mu)}\left(\xi_{(\lambda)_\beta}\right) = \xi_{(\nu)_\beta},\\ \eta^\alpha_{(\lambda)} \circ f_{(\mu)} = \eta^\alpha_{(\mu)} \circ f_{(\lambda)}, = \eta^\alpha_{(\nu)},\\ f_{(\lambda)}f_{(\mu)} + \eta^\alpha_{(\mu)} \otimes \xi_{(\lambda)_\alpha} = f_{(\mu)}f_{(\lambda)} + \eta^\alpha_{(\lambda)} \otimes \xi_{(\mu)_\alpha} = f_{(\nu)},\end{aligned}\right\}$$

where $(\lambda,\mu,\nu) \in \mathcal{C}(1,2,3)$, will be called an *almost r-paracontact* 3-*structure*.

Moreover, from Theorem 5.5, there is an associated Riemannian metric g on M such that for $\lambda = 1,2,3$ we have

$$(32)\qquad g\left(\xi_{(\lambda)_\alpha}, X\right) = \eta^\alpha_{(\lambda)}(X),$$

$$(33)\qquad g\left(f_{(\lambda)}X, f_{(\lambda)}Y\right) = g(X,Y) - \sum_\alpha \eta^\alpha_{(\lambda)}(X)\eta^\alpha_{(\lambda)}(Y).$$

Now, we need a lemma.

Lemma 6.2. *If M admits an almost r-paracontact metric 3-structure, such that each of the constituent structures is of s-paracontact type, then*

$$\left.\begin{aligned} \left(\nabla_X f_{(\lambda)}\right) \xi_{(\lambda)_\alpha} &= -\left(f_{(\lambda)}\right)^2 X, \\ \left(\nabla_X f_{(\lambda)}\right) \xi_{(\mu)_\alpha} &= \eta^{\beta}_{(\mu)}\left(X\right) \xi_{(\lambda)_\beta}, \\ \left(\nabla_X f_{(\lambda)}\right) \xi_{(\nu)_\alpha} &= \eta^{\beta}_{(\nu)}\left(X\right) \xi_{(\lambda)_\beta}, \end{aligned}\right\} \tag{34}$$

where $(\lambda, \mu, \nu) \in \mathcal{C}\,(1, 2, 3)$.

Proof. The proof follows from Definition 6.1 and (28). □

The above lemma implies the following proposition.

Proposition 6.3. *On an almost r-paracontact metric 3-structure manifold, such that each of the constituent structures is of s-paracontact type, we have*

$$\left.\begin{gathered} \left(\nabla_{\xi_{(\lambda)_\beta}} f_{(\lambda)}\right) \xi_{(\lambda)_\alpha} = 0, \\ \left(\nabla_{\xi_{(\lambda)_\beta}} f_{(\lambda)}\right) \xi_{(\mu)_\alpha} = 0, \qquad \left(\nabla_{\xi_{(\lambda)_\beta}} f_{(\lambda)}\right) \xi_{(\nu)_\alpha} = 0, \\ \left(\nabla_{\xi_{(\nu)_\beta}} f_{(\lambda)}\right) \xi_{(\mu)_\alpha} = 0, \qquad \left(\nabla_{\xi_{(\mu)_\beta}} f_{(\lambda)}\right) \xi_{(\nu)_\alpha} = 0, \\ \left(\nabla_{\xi_{(\mu)_\beta}} f_{(\lambda)}\right) \xi_{(\mu)_\alpha} = \xi_{(\lambda)_\beta}, \qquad \left(\nabla_{\xi_{(\nu)_\beta}} f_{(\lambda)}\right) \xi_{(\nu)_\alpha} = \xi_{(\lambda)_\beta}, \end{gathered}\right\} \tag{35}$$

where $(\lambda, \mu, \nu) \in \mathcal{C}\,(1, 2, 3)$.

Now, we prove the main result of this section.

Theorem 6.4. *If M is a manifold equipped with an almost r-paracontact metric 3-structure $(\Sigma_{(\lambda)}, g) = \left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^{\alpha}_{(\lambda)}, g\right)$, $\lambda = 1, 2, 3$, then all of the constituent structures cannot be of P-Sasakian type simultaneously.*

Proof. For $\lambda = 1, 2, 3$, we have

$$\left(\nabla_Z F_{(\lambda)}\right)(X, Y) = g\left(\left(\nabla_Z f_{(\lambda)}\right) X, Y\right), \tag{36}$$

where $F_{(\lambda)}(X,Y) = g\left(X, f_{(\lambda)}Y\right)$. Suppose all of the constituent structures are of P-Sasakian type. Now, putting $Z = \xi_{(\mu)_\beta}$, $X = \xi_{(\mu)_\alpha}$ in (36) and using (35), we get

$$\left(\nabla_{\xi_{(\mu)_\beta}} F_{(\lambda)}\right)\left(\xi_{(\mu)_\alpha}, Y\right) = g\left(\xi_{(\lambda)_\beta}, Y\right) = \eta^\beta_{(\lambda)}(Y), \tag{37}$$

while, from (29), we obtain

$$\begin{aligned}\left(\nabla_{\xi_{(\mu)_\beta}} F_{(\lambda)}\right)\left(\xi_{(\mu)_\alpha}, Y\right) &= -\sum_\gamma \eta^\gamma_{(\lambda)}(Y)\, g\left(\xi_{(\nu)_\alpha}, \xi_{(\nu)_\beta}\right) \\ &= -\sum_\gamma \eta^\gamma_{(\lambda)}(Y)\, \delta^\alpha_\beta.\end{aligned} \tag{38}$$

When $\alpha \neq \beta$, from (37) and (38) we have $\eta^\beta_{(\lambda)}(Y) = 0$ for all Y, which is a contradiction. □

Since an almost r-paracontact metric structure of SP-Sasakian type is always of P-Sasakian type, in view of Theorem 6.4, we have the following corollary.

Corollary 6.5. *Not all the constituent structures of an almost r-paracontact metric 3-structure manifold can be of SP-Sasakian type.*

In case of $r = 1$, we have the following corollary.

Corollary 6.6 [**8**]. *Not all the constituent structures of an almost paracontact metric 3-structure manifold can be of P-Sasakian type or SP-Sasakian type.*

7. Non-existence of $\mathcal{S}$-3-structure. Taking $\varepsilon = -1$, the ε-framed metric structure becomes a framed metric structure [**34**] (or almost r-contact metric structure [**32**]), that is,

$$\left.\begin{aligned} f^2 &= -I + \eta^\alpha \otimes \xi_\alpha, \\ F(X,Y) &= g(X, fY) = -F(Y,X). \end{aligned}\right\} \tag{39}$$

A framed metric structure is called *normal* if

$$[f, f] + 2d\eta^\alpha \otimes \xi_\alpha = 0,$$

and an *S-structure* [**2**] if it is normal and $F = d\eta^\alpha$, $\alpha \in \{1, \dots, r\}$. When $r = 1$, a framed metric structure is an almost contact metric structure, while an $\mathcal{S}$-structure is a Sasakian structure.

If a framed metric structure on M is an $\mathcal{S}$-structure then it is known (Blair [**2**]) that

$$(\nabla_X f)Y = \sum_\alpha \left(g(fX, fY)\, \xi_\alpha + \eta^\alpha(Y) f^2 X\right), \tag{40}$$

$$f = -\nabla \xi_\alpha, \quad \alpha \in \{1, \dots, r\}. \tag{41}$$

The converse may also be proved. In case of Sasakian structure, that is, $r = 1$, (40) implies (41).

For the sake of simplicity, we will write as

$$\sum \tilde{\xi} \otimes \tilde{\xi} := \sum_\alpha \xi_\alpha \otimes \xi_\alpha,$$
$$\Big(\text{in local coordinates,} \quad \sum \tilde{\xi}^i \tilde{\xi}^j := \sum_\alpha \xi^i_\alpha \xi^j_\alpha\Big).$$

We will also write

$$\sum \tilde{\tilde{\xi}}^i \tilde{\tilde{\xi}}^j := \sum_\beta \xi^i_\beta \xi^j_\beta.$$

Differentiating (40) covariantly, we have

$$\begin{aligned}(\nabla_Z \nabla f)(X, Y) = &- r\left(g(X, Y) - \sum \tilde{\eta}(X)\, \tilde{\eta}(Y)\right) fZ \\ &+ \left(g(fZ, X) \sum \tilde{\eta}(Y) + g(fZ, Y) \sum \tilde{\eta}(X)\right) \sum \tilde{\xi} \\ &+ r\left(X - \sum \tilde{\eta}(X) \tilde{\xi}\right) g(fZ, Y) \\ &- \left(g(fZ, X) \sum \tilde{\xi} + fZ \sum \tilde{\eta}(X)\right) \sum \tilde{\eta}(Y),\end{aligned}$$

that is,

$$\begin{aligned}\nabla_m \nabla_i f = &-r\left(g_{ij} - \sum \tilde{\eta}_i \tilde{\eta}_j\right) {f_m}^k \\ &+ \left(f_{mi} \sum \tilde{\eta}_j + f_{mj} \sum \tilde{\eta}_i\right) \sum \tilde{\xi}^k \\ &+ r\left(\delta_i^k - \sum \tilde{\eta}_i\, \tilde{\xi}^k\right) f_{mj} \\ &- \left(f_{mi} \sum \tilde{\tilde{\xi}}^k + {f_m}^k \sum \tilde{\eta}_i\right) \sum \tilde{\eta}_j\end{aligned}$$

by noticing

$$\sum \nabla \tilde{\xi} = -rf, \qquad \nabla_m (\sum \tilde{\eta}_i\, \tilde{\eta}_j) = -f_{mi} \sum \tilde{\eta}_j - f_{mj} \sum \tilde{\eta}_i.$$

Making use of the Ricci identity (with respect to Z, X in the above), we have

$$\begin{aligned}(42)\quad & g(R(Z,X)fY,W) + g(R(Z,X)Y,fW) \\ &= -\left(rg(X,Y) - r\sum \tilde{\eta}(X)\,\tilde{\eta}(Y) + \sum \tilde{\eta}(Y) \sum \tilde{\eta}(X)\right) g(fZ,W) \\ &\quad - \left(rg(X,W) - r\sum \tilde{\eta}(X)\,\tilde{\eta}(W) + \sum \tilde{\eta}(W) \sum \tilde{\eta}(X)\right) g(fZ,Y) \\ &\quad + \left(rg(Z,Y) - r\sum \tilde{\eta}(Z)\,\tilde{\eta}(Y) + \sum \tilde{\eta}(Y) \sum \tilde{\eta}(Z)\right) g(fX,W) \\ &\quad - \left(rg(Z,W) - r\sum \tilde{\eta}(Z)\,\tilde{\eta}(W) + \sum \tilde{\eta}(W) \sum \tilde{\eta}(Z)\right) g(fX,Y),\end{aligned}$$

that is,

$$\begin{aligned}R_{misk}{f_j}^s + R_{mijs}{f_k}^s = &-\left(rg_{ij} - r\sum \tilde{\eta}_i\, \tilde{\eta}_j + \sum \tilde{\eta}_j \sum \tilde{\eta}_i\right) f_{mk} \\ &+ \left(rg_{ik} - r\sum \tilde{\eta}_i\, \tilde{\eta}_k + \sum \tilde{\eta}_k \sum \tilde{\eta}_i\right) f_{mj} \\ &+ \left(rg_{mj} - r\sum \tilde{\eta}_m\, \tilde{\eta}_j + \sum \tilde{\eta}_j \sum \tilde{\eta}_m\right) f_{ik} \\ &- \left(rg_{mk} - r\sum \tilde{\eta}_m\, \tilde{\eta}_k + \sum \tilde{\eta}_k \sum \tilde{\eta}_m\right) f_{ij},\end{aligned}$$

cf. Blair [**2**, Lemma 2.2]. Contracting (42) with respect to Z, W, and by using $\sum_{c=1}^{n}\sum_{\alpha,\beta}\eta^{\alpha}(e_c)\,\eta^{\beta}(e_c)=r$, we have

$$(43)\quad (-R_{mijs}f^{ms}+R_i{}^sf_{sj}=)\;\frac{1}{2}\,R_{ijms}f^{ms}+R_i{}^sf_{sj}=r(n-r-1)f_{ij},$$

where $\{e_c\}$is an orthonormal basis of TM.

Example 7.1. Every n-dimensional Lie group G admits a framed f-structure of rank $2k$, where k is any positive integer less than $(n+1)/2$, cf. [**15**].

Theorem 7.2. *An $\mathcal{S}$-structure is not Einstein if $r>1$.*

Proof. Let $(f,\xi_\alpha,\eta^\alpha,g)$ be an $\mathcal{S}$-structure on an n-dimensional manifold M. From (40) and (41) for $\gamma=1,\dots,r(=n-2k)$, we have

$$\begin{aligned}(\nabla_X\nabla\xi_\gamma)\,Y=-&\left(g\,(X,Y)-\sum_\beta\eta^\beta\,(X)\,\eta^\beta\,(Y)\right)\sum_\alpha\xi_\alpha\\&+\left(X-\sum_\beta\eta^\beta\,(X)\,\xi_\beta\right)\sum_\alpha\eta^\alpha\,(Y)\,.\end{aligned}$$

In an $\mathcal{S}$-manifold each ξ_α is a Killing vector. Since a Killing vector ξ satisfies

$$(\nabla_X\nabla\xi)\,Y=-R\,(\xi,X)\,Y$$

where $R\,(X,Y)\,Z=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$; therefore, for $\gamma=1,\dots,r$, we have

$$(44)\quad\begin{aligned}R(\xi_\gamma,X)Y=&\left(g\,(X,Y)-\sum_\beta\eta^\beta\,(X)\,\eta^\beta\,(Y)\right)\sum_\alpha\xi_\alpha\\&-\left(X-\sum_\beta\eta^\beta\,(X)\,\xi_\beta\right)\sum_\alpha\eta^\alpha\,(Y)\end{aligned}$$

or

$$(45)\quad\begin{aligned}&\nabla_if_j^k\,(\;=\;-\nabla_i\nabla_j\xi_\gamma^k=\xi_\gamma^sR_{sijk}\,)\\&\quad=\left(g_{ij}-\sum_\beta\eta_i^\beta\eta_j^\beta\right)\sum_\alpha\xi_\alpha^k-\left(\delta_i^k-\sum_\beta\eta_i^\beta\xi_\beta^k\right)\sum_\alpha\eta_j^\alpha.\end{aligned}$$

Now, let S be the Ricci operator given by

$$SX = \sum_{c=1}^{n} R(X, e_c)e_c,$$

where $\{e_c\}$ is an orthonormal basis of TM. Contracting (44) in X, Y, we have for $\gamma = 1, \dots, r$,

$$S\xi_\gamma = (n-r)\sum_\alpha \xi_\alpha \tag{46}$$

because of

$$\sum_{c=1}^{n}\sum_\beta \eta^\beta(e_c)\,\eta^\beta(e_c) = r, \qquad \sum_{c=1}^{n}\eta^\alpha(e_c)e_c = \xi_\alpha,$$

and

$$\sum_{c=1}^{n}\left(\left(\sum_\beta \eta^\beta\left(e_c\right)\xi_\beta\right)\sum_\alpha \eta^\alpha\left(e_c\right)\right) = \sum_\beta \xi_\beta.$$

Now, assume that g is Einstein. By (46), we obtain

$$\frac{\tau}{n}\,\xi_\gamma = (n-r)\sum_\alpha \xi_\alpha,$$

where τ is the scalar curvature. As $\{\xi_1, \dots, \xi_r\}$ are linearly independent, the above relation implies $n-r=0$ if $r>1$, which means f vanishes. Hence, we know that an $\mathcal{S}$-structure is not Einstein if $r>1$. □

Setting $\varepsilon = -1$, in Definition 4.1, we have the following definition.

Definition 7.3 [**15**]. An n-dimensional manifold M equipped with three-framed f-structures $\Sigma_{(\lambda)} = \left(f_{(\lambda)}, {\xi_{(\lambda)}}_\alpha, \eta^\alpha_{(\lambda)}\right)$, $\lambda = 1, 2, 3$ of the same rank $2k$ satisfying

$$\left.\begin{aligned} \eta^\alpha_{(\lambda)}\left({\xi_{(\mu)}}_\beta\right) = 0, \quad \eta^\alpha_{(\mu)}\left({\xi_{(\lambda)}}_\beta\right) = 0, \quad \lambda \neq \mu, \\ f_{(\lambda)}\left({\xi_{(\mu)}}_\beta\right) = -f_{(\mu)}\left({\xi_{(\lambda)}}_\beta\right) = {\xi_{(\nu)}}_\beta, \\ \eta^\alpha_{(\lambda)} \circ f_{(\mu)} = -\eta^\alpha_{(\mu)} \circ f_{(\lambda)} = \eta^\alpha_{(\nu)}, \\ f_{(\lambda)} \circ f_{(\mu)} - \eta^\alpha_{(\mu)} \otimes {\xi_{(\lambda)}}_\alpha = -f_{(\mu)} \circ f_{(\lambda)} + \eta^\alpha_{(\lambda)} \otimes {\xi_{(\mu)}}_\alpha = f_{(\nu)}, \end{aligned}\right\} \tag{47}$$

where $(\lambda, \mu, \nu) \in \mathcal{C}\,(1,2,3)$, is said to have a framed f-3-structure.

An associated metric to a framed 3-structure in a manifold M is a Riemannian metric which is associated to each of the three constituent structures. In fact, there always exists such an associated Riemannian metric g on M satisfying

$$g\left(\xi_{(\lambda)_\alpha}, X\right) = \eta^{\alpha}_{(\lambda)}\left(X\right), \tag{48}$$

$$g\left(f_{(\lambda)}X, f_{(\lambda)}Y\right) = g\left(X, Y\right) - \sum_{\alpha} \eta^{\alpha}_{(\lambda)}\left(X\right) \eta^{\alpha}_{(\lambda)}\left(Y\right). \tag{49}$$

If each of the constituent structures is an $\mathcal{S}$-structure, the framed metric 3-structure will be called an $\mathcal{S}$-3-*structure*. An $\mathcal{S}$-3-structure with $r = 1$ is a *Sasakian* 3-*structure* [**17**].

Example 7.4 [**15**]. Every n-dimensional Lie group G admits a framed f-3-structure of rank $2k$, where k is even or odd according to whether n is even or odd.

Now, we prove the main result of this section.

Theorem 7.5. *If M is an n-dimensional manifold equipped with a framed metric 3-structure $\left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^{\alpha}_{(\lambda)}, g\right)$, $\lambda = 1, 2, 3$ of rank $2k$, then all of the constituent structures cannot be $\mathcal{S}$-structures simultaneously provided $n - 2k = r > 1$.*

Proof. Let $\left(f_{(\lambda)}, \xi_{(\lambda)_\alpha}, \eta^{\alpha}_{(\lambda)}\right)$, $\lambda = 1, 2, 3$, be an $\mathcal{S}$-3-structure. Then

$$\begin{aligned} f_{(3)} &= -f_{(2)} \circ f_{(1)} + \sum \eta^{\tilde{}}_{(1)} \otimes \xi^{\tilde{}}_{(2)}, \\ \eta^{\alpha}_{(\lambda)}(\xi_{(\mu)\beta}) &= 0 \quad \text{for } \lambda \neq \mu; \\ \xi_{(3)\alpha} &= f_{(1)}\left(\xi_{(2)\alpha}\right), \quad {\eta_{(3)}}^{\alpha} = -\eta_{(2)\alpha} \circ f_{(1)}, \end{aligned}$$

that is,

$$\begin{aligned} {f_{(3)}}_i^{\ j} &= -{f_{(1)}}_i^{\ s} {f_{(2)}}_s^{\ j} + \sum {\eta^{\tilde{}}_{(1)}}_i {\xi^{\tilde{}}_{(2)}}^{j}, \\ {\xi_{(3)}}^{j}_{\alpha} &= {f_{(1)}}^{j}_{s} {\xi_{(2)}}^{s}_{\alpha}, \quad {\eta_{(3)}}^{j}_{\alpha} = -{\eta_{(2)}}^{\alpha}_{s} {f_{(1)}}^{s}_{j}. \end{aligned}$$

Now, $f_{(1)}$ satisfies the relation (43):

$$\frac{1}{2}R_{ijms}f_{(1)}^{ms} + R_i{}^s f_{(1)}{}_{sj} = r(n-r-1)f_{(1)}{}_{ij}.$$

So, let us calculate the interior product with $\xi_{(2)\varepsilon}$:

$$\xi_{(2)\varepsilon}{}^i\left(\frac{1}{2}R_{ijms}f_{(1)}^{ms} + R_i{}^s f_{(1)}{}_{sj}\right) = \xi_{(2)\varepsilon}{}^i\left(r(n-r-1)f_{(1)}{}_{ij}\right).$$

Making use of (45), for $\xi_{(2)\varepsilon}$, the left-hand side of the equation is

$$\begin{aligned}
&\frac{1}{2}\Big\{\Big(g_{jm} - \sum \eta^{\tilde{\tilde{}}}_{(2)}{}_j\eta^{\tilde{\tilde{}}}_{(2)}{}_m\Big)\sum \eta^{\tilde{}}_{(2)}{}_s \\
&\qquad - \Big(g_{js} - \sum \eta^{\tilde{\tilde{}}}_{(2)}{}_j\eta^{\tilde{\tilde{}}}_{(2)}{}_s\Big)\sum \eta^{\tilde{}}_{(2)}{}_m\Big\} f_{(1)}^{ms} + (n-r)\sum \xi^{\tilde{}s}_{(2)}f_{sj} \\
&\quad = \Big(g_{jm} - \sum \eta^{\tilde{\tilde{}}}_{(2)}{}_j\eta^{\tilde{\tilde{}}}_{(2)}{}_m\Big)\Big(-\sum \xi^{\tilde{}}_{(3)}{}^m\Big) + (n-r)\sum \eta^{\tilde{}}_{(3)}{}_j \\
&\quad = (n-r-1)\sum \eta^{\tilde{}}_{(3)}{}_j,
\end{aligned}$$

where (46), $\eta^{\alpha}_{(2)}{}_s f_{(1)}^{ms} = -\xi^m_{(3)\alpha}$, and $\left(\sum \eta^{\tilde{\tilde{}}}_{(2)}{}_j\eta^{\tilde{\tilde{}}}_{(2)}{}_m\right)\sum \xi^{\tilde{}}_{(3)}{}^m = 0$ are used; while the right-hand side is equal to

$$r(n-r-1)\eta^{\varepsilon}_{(3)}{}_j.$$

Then

$$\sum \xi^{\tilde{}}_{(3)} = r\xi_{(3)\varepsilon}, \quad \text{for each } \varepsilon = 1, \dots, r.$$

Hence, there exists no linearly independent set $\{\xi_{(3)1}, \dots, \xi_{(3)r}\}$ if $r > 1$. □

8. Non-existence of proper trans-Sasakian 3-structure. An almost contact metric structure (f, ξ, η, g) is a special case of an ε-framed metric structure when $\varepsilon = -1$ and $r = 1$. Let M be an almost contact metric manifold ([**3**]) with an almost contact metric structure (f, ξ, η, g), that is, f is a $(1,1)$ tensor field, ξ is a vector field; η is a

1-form and g is a compatible Riemannian metric such that

$$f^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{50}$$

$$F(X,Y) = g(X, fY) = -F(Y,X), \quad g\,(X,\xi) = \eta(X) \tag{51}$$

for all $X, Y \in TM$.

In [**27**], Tanno gave a classification for connected almost contact metric manifolds whose automorphism groups have the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: **(1)** homogeneous normal contact Riemannian manifolds with $c > 0$, **(2)** global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and **(3)** warped product spaces $\mathbf{R} \times_f \mathbf{C}^n$ if $c < 0$. It is known that the manifolds of class **(1)** are characterized by some tensorial relations admitting a Sasakian structure. Kenmotsu [**16**] characterized the differential geometric properties of the third case by tensor equation $(\nabla_X f)Y = g(fX, Y)\xi - \eta(Y) fX$. The structure so obtained is now known as a Kenmotsu structure. In general, this structure is not Sasakian [**16**].

In the Gray-Hervella classification of almost Hermitian manifolds [**10**], there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kähler manifolds (for geometry of locally conformal Kähler manifolds we refer to the book of Dragomir and Ornea [**9**]). An almost contact metric structure (f, ξ, η, g) on M is called a *trans-Sasakian structure* (Oubina [**22**]) if $(M \times \mathbf{R}, J, G)$ belongs to the class $\mathcal{W}_4$, where J is the almost complex structure on $M \times \mathbf{R}$ defined by

$$J(X, cd/dt) = (fX - c\,\xi, \eta(X) d/dt)$$

for all vector fields X on $\bar{M}$ and smooth functions c on $M \times \mathbf{R}$ and G is the product metric on $M \times \mathbf{R}$. This may be expressed by the condition (Blair and Oubina [**4**])

$$(\nabla_X f)Y = a(g(X,Y)\xi - \eta(Y)X) + b(g(X, fY)\xi + \eta(Y) fX) \tag{52}$$

for some smooth functions a and b on M, and we call such a trans-Sasakian structure as (a, b)-trans-Sasakian structure. From the formula (52) it follows that [**4**]

$$\nabla_X \xi = -afX - bX + b\eta(X)\xi. \tag{53}$$

In [**29**], it is proved that trans-Sasakian manifolds are *generalized quasi-Sasakian* (Mishra [**20**]). It is also proved that certain Legendre curves of a Kenmotsu manifold are circles [**28**].

The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ [**18**] coincides with the class of (a, b)-trans-Sasakian structures. We note that $(0, 0)$-trans-Sasakian structures are cosymplectic [**3**], $(0, b)$-trans-Sasakian structures are b-Kenmotsu [**13**] and $(a, 0)$-trans-Sasakian structures are a-Sasakian [**13**].

If we have an almost contact metric 3-structure $(f_\lambda, \xi_\lambda, \eta_\lambda, g)$, $\lambda = 1, 2, 3$, [**17**] on a connected manifold M of dimension $4n + 3$, then we have the following

Theorem 8.1. *If $(f_\lambda, \xi_\lambda, \eta_\lambda, g)$, $\lambda = 1, 2, 3$, are (a_λ, b_λ)-trans-Sasakian, then $b_1 = b_2 = b_3 = 0$ and $a_1 = a_2 = a_3 = a$, where a is constant. Therefore, we have an a-Sasakian 3-structure.*

Proof. In fact, since (f_3, ξ_3, η_3, g) is (a_3, b_3)-trans-Sasakian, by (53) we have

$$\nabla_X \xi_3 = b_3 \eta_3(X) \xi_3 - b_3 X - a_3 f X. \tag{54}$$

On the other hand, from the defining conditions of 3-structure we have

$$\nabla_X \xi_3 = (\nabla_X f_1)(\xi_2) + f_1 (\nabla_X \xi_2).$$

Taking (52) and (53) into account, we get

(55)

$$\nabla_X \xi_3 = b_1 \eta_3(X) \xi_1 + b_2 \eta_2(X) \xi_3 - b_2 f_1 X + (a_1 - a_2) \eta_2(X) \xi_1 - a_2 f_3 X.$$

Now from (54) and (55) it follows

$$\nabla_{\xi_1} \xi_3 = -a_2 \xi_2 = -a_3 \xi_2 - b_3 \xi_1.$$

Therefore, $a_2 = a_3$ and $b_3 = 0$. Analogously, we get

$$\nabla_{\xi_2} \xi_3 = a_1 \xi_1 = a_3 \xi_1.$$

Then $a_1 = a_3$. Finally,

$$\nabla_{\xi_3} \xi_3 = b_1 \xi_1 + b_2 \xi_2 = 0.$$

Therefore, $b_1 = b_2 = b_3 = 0$ and $a_1 = a_2 = a_3 = a$. Thus, we have an a-Sasakian 3-structure and, in such a case, it follows that

$$dF_\lambda = 0, \qquad d\eta_\lambda = 2aF_\lambda(X, Y).$$

where $F_\lambda(X, Y) = g(X, f_\lambda Y)$ and $\lambda = 1, 2, 3$. Since, $0 = da \wedge F_\lambda$, for $\lambda = 1, 2, 3$, we get

$$0 = da(\xi_\lambda)F_\lambda(\xi_\mu, \xi_\nu) = -da(\xi_\lambda),$$

where $(\lambda, \mu, \nu) \in \mathcal{C}(1, 2, 3)$. If E is a unitary vector orthogonal to ξ_1, ξ_2, and ξ_3, we have

$$0 = da(E)F_\lambda(\xi_\mu, \xi_\nu) = -da(E).$$

It follows that $da = 0$. □

Remark 8.2. We also note that, even in a more general context, a is constant. For instance, for a trans-Sasakian 3-structure in the sense of Martin Cabrera, cf. [**19**, Corollary 4.15], a is constant. Here, M has a trans-Sasakian almost contact 3-structure, if $M \times \mathbf{R}$ has a locally conformal quaternionic Kähler structure. However, each almost complex structure of $M \times \mathbf{R}$ is not necessarily $\mathcal{W}_4$ and each almost contact structure of M is not necessarily trans-Sasakian.

From Theorem 8.1 we have the following corollaries.

Corollary 8.3. *If* $(f_\lambda, \xi_\lambda, \eta_\lambda, g)$, $\lambda = 1, 2, 3$, *are* b_λ*-Kenmotsu, then* $b_1 = b_2 = b_3 = 0$. *Therefore, we have a hypercosymplectic* 3*-structure* [**19**].

Corollary 8.4. *If* $(f_\lambda, \xi_\lambda, \eta_\lambda, g)$, $\lambda = 1, 2, 3$, *are* a_λ*-Sasakian, then* $a_1 = a_2 = a_3 = a$, *where* a *is constant. Therefore, we have an* a*-Sasakian* 3*-structure.*

Remark 8.5. In [**24, 25**], submanifolds of a manifold equipped with a Kenmotsu almost contact 3-structure are studied. In view of the above discussion, such structure cannot exist. However, the results of [**24, 25**] may be true when the ambient manifold carries a hypercosymplectic 3-structure [**19**]. A $(4n+3)$-dimensional torus $\mathbf{T}^{4n+3}$ $(n \geq 1)$ is a typical example carrying a hypercosymplectic 3-structure.

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Department of Information Science, Saitama College, Kazo-shi, Saitama, 347-0032, Japan
E-mail address: tkashiwa@h6.dion.ne.jp

Department of Fundamental Mathematics, University of La Laguna, Tenerife, Canary Islands, Spain
E-mail address: fmartin@ull.es

Department of Mathematics and Astronomy, Lucknow University, Lucknow 226 007, India and Department of Mathematics, Pohang University of Science and Technology (POSTECH), Pohang 790 784, South Korea
E-mail address: mmt66@satyam.net.in