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CARATHÉODORY SOLUTIONS TO HYPERBOLIC FUNCTIONAL DIFFERENTIAL SYSTEMS WITH STATE DEPENDENT DELAYS

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ABSTRACT. The paper is concerned with initial problems for quasilinear systems of first order partial functional differential equations. The unknown function is the functional argument in equations, the partial derivatives appear in a classical sense. A theorem on the existence of a solution and continuous dependence upon initial data is proved. The Cauchy problem is transformed into a system of functional integral equations. The existence of a solution of this system is proved by using integral inequalities and the method of bicharacteristics.

Differential systems with deviated variables and differential integral systems can be derived from a general model by specializing given operators.

1. Introduction. For any metric spaces U and V, let $C(U,V)$ denote the class of all continuous functions from U into V. Let $L([0,c],R_+)$ where $c>0$ and $R_+=[0,+\infty)$ is the set of all functions $\eta:[0,a]\to R_+$ which are integrable on $[0,c]$. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Denote by $M_{k\times n}$ the set of all matrices

$$X=[x_{ij}]_{i=1,\dots,k,\,j=1,\dots,n}$$

with real elements. For $x=(x_1,\dots,x_n)\in R^n$, $p=(p_1,\dots,p_k)\in R^k$ and $X\in M_{k\times n}$, we write

$$\|x\|=|x_1|+\dots+|x_n|,\quad \|p\|=\max\,\{|p_i|:\ 1\le i\le k\},$$
$$\|X\|=\max\,\left\{\sum_{j=1}^{n}|x_{ij}|:\ 1\le i\le k\right\}.$$

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Let $B = [-b_0, 0] \times [-b, b]$ and $E = [-b_0, a] \times R^n$, where $a > 0$, $b_0 \in R_+$, $b = (b_1, \dots, b_n) \in R_+^n$. For a given function $z : E \to R^k$ and a point $(t, x) \in [0, a] \times R^n$, we define the function $z_{(t,x)} : B \to R^k$ by $z_{(t,x)}(\tau, y) = z(t + \tau, x + y)$, $(\tau, y) \in B$.

Put $\Omega = [0, a] \times R^n \times C(B, R^k)$, $E_0 = [-b_0, 0] \times R^n$ and assume that

$$\begin{gathered} \varrho : \Omega \to M_{k\times n}, \ \ \varrho = [\, \varrho_{ij} \,]_{i=1,\dots,k, j=1,\dots,n}\,, \\ f : \Omega \to R^k, \ f = (f_1, \dots, f_k), \\ \psi_0 : [0, a] \to R, \ \psi_* : \Omega \to R^n, \ \psi_* = (\psi_1, \dots, \psi_n) \end{gathered}$$

are given functions. Write

$$\varrho_i = (\varrho_{i1}, \dots, \varrho_{in}) \quad \text{for} \quad 1 \le i \le k$$

and $\psi(t, x, w) = (\psi_0(t), \psi_*(t, x, w))$ for $(t, x, w) \in \Omega$.

For a given initial function $\varphi : E_0 \to R^k$, $\varphi = (\varphi_1, \dots, \varphi_k)$, consider the quasilinear system of functional differential equations with state dependent delays

$$\begin{aligned} (1) \quad \partial_t z_i(t, x) + \sum_{j=1}^{n} \varrho_{ij}(t, x, z_{\psi(t,x,z_{(t,x)})})\, \partial_{x_j} z_i(t, x) \\ = f_i(t, x, z_{\psi(t,x,z_{(t,x)})}), \quad i = 1, \dots, k, \end{aligned}$$

with the initial condition

$$(2) \qquad z(t, x) = \varphi(t, x) \quad \text{for} \quad (t, x) \in E_0,$$

where $z = (z_1, \dots, z_k)$. Note that the symbol $z_{\psi(t,x,z_{(t,x)})}$ denotes the restriction of z to the set

$$[\psi_0(t) - b_0, \psi_0(t)] \times [\psi_*(t, x, z_{(t,x)}) - b,\ \psi_*(t, x, z_{(t,x)}) + b],$$

and this restriction is shifted to the set B.

A function $u : E_c \to R^k$, where $E_c = [-b_0, c] \times R^n$, $0 < c \le a$, is the Carathéodory solution of problem (1), (2) if

(i) u is continuous and the derivatives $\partial_t u_i$, $\partial_x u_i = (\partial_{x_1} u_i, \dots, \partial_{x_n} u_i)$, $1 \le i \le k$, exist almost everywhere on $[0, c] \times R^n$;

(ii) u satisfies (1) almost everywhere on $[0, c] \times R^n$ and condition (2) holds.

In recent years a number of papers concerned with first order partial functional differential equations were published. The following questions were discussed: functional differential inequalities and their applications, existence theory of classical and generalized solutions, numerical methods for initial or mixed problems. It is not our aim to show a full review of papers concerning the above problems. We consider the questions of the existence of solutions only.

Classical solutions of initial problems have been considered in [**2, 3, 14, 20, 21**]. Existence results presented in these papers are based on a method of successive approximations which were introduced by Wazewski for first order partial differential systems without functional dependence [**25**]. Classical solutions of nonlinear functional differential equations exist locally with respect to t. This leads in a natural way to generalized or weak solutions. Distributional solutions of initial problems for quasilinear equations have been considered in [**22**]. Existence results to nonlinear equations and global Carathéodory solutions can be found in [**6**]. The proofs presented in these papers are constructive, and they are based on difference methods.

Generalized solutions of nonlinear equations are also investigated in the case that assumptions for given functions are extended. This leads to Cinquini Cibrario solutions. This class of solutions is placed between classical solutions and solutions in the Carathéodory sense. Existence results for initial or mixed problems, [**7, 18**] are obtained by a linearization procedure and by a construction of integral functional systems for unknown functions and for their partial derivatives with respect to spatial variables. Under natural assumptions on given functions, solutions of integral functional systems generate weak solutions of original problems. These papers deal with weakly coupled systems. This means that every equation consists of the vector on unknown functions and the derivatives of only one function. Carathéodory solutions of quasilinear differential functional equations have been considered in [**17, 23, 24**]. Existence results are obtained by investigations of adequate integral equations and by the bicharacteristics theory. The Barabshin type functional differential problems have been discussed [**15**]. For further bibliography concerning existence results, see the monograph [**16**].

Hyperbolic functional differential equations have applications in different branches of knowledge. We give a few examples. Quasilinear first order partial differential equations perturbed by a dissipative integral term of Volterra type arise from laser problems in nonlinear optics [**1**]. In the theory of the distribution of wealth, a differential equation with a deviated argument is used, [**8**]. Differential integral equations describing the dynamic of muscle contraction was studied in [**10**]. The paper [**11**] discusses, using differential integral equations, optimal harvesting policies for age structured populations harvested with effort independent of age. A system on nonlinear differential integral equations which mode an age dependent epidemic of a disease with vertical transmission is investigated in [**9**]. Almost linear differential integral equations are used in [**4**] to describe a model of proliferating cell populations.

Ordinary functional differential equations with state dependent delays, also called iterative functional differential equations, have attracted the attention of several authors in recent years, see e.g., [**5, 12, 13**].

Delay systems with state dependent delays occur as models for the dynamics of diseases when the mechanism of infection is such that the infectious dosage received by an individual has to reach a threshold value before the resistance of the individual is broken down and as a result the individual becomes infectious. A prototype of such a model was proposed in [**5**].

In this paper we initiate the study of the existence theory for first order functional partial differential equations with state dependent delays.

We will consider existence and uniqueness of local generalized solutions of problem (1), (2) in the "almost everywhere" sense. Our results are based on the method of bicharacteristics. The Cauchy problem will be transformed into integral functional equations. The existence and uniqueness of solutions of this system will be proved by using the Banach fixed point theorem.

2. Bicharacteristics of quasilinear systems. The following function spaces will be needed throughout the paper.

Let $\omega_0 \in L([-b_0, 0], R_+)$, $p = (p_0, p_1) \in R_+^2$. Denote by $J[\omega_0, p]$ the class of all functions $\varphi \in C(E_0, R^k)$ such that $\|\varphi(t,x)\| \le p_0$ on E_0 and

$$\|\varphi(t,x) - \varphi(\bar{t},\bar{x})\| \le \left| \int_t^{\bar{t}} \omega_0(s)\, ds \right| + p_1 \|x - \bar{x}\|$$

on E_0. The space $J[\omega_0, p]$ is the set of initial functions for problem (1), (2).

Suppose that

$$c \in (0,a], \quad q = (q_0, q_1) \in R_+^2, \quad q_0 \ge p_0, \quad q_1 \ge p_1,$$

and

$$\omega \in L([-b_0, c], R_+), \quad \omega(t) \ge \omega_0(t)$$

for almost all $t \in [-b_0, 0]$. Let $K_{\varphi.c}[\omega, q]$ be the class of all functions $z \in C(E_c, R^k)$ such that

(i) $z(t,x) = \varphi(t,x)$ on E_0;

(ii) for $(t,x), (\bar{t},\bar{x}) \in [0,c] \times R^n$ we have $\|z(t,x)\| \le q_0$ and

$$\|z(t,x) - z(\bar{t},\bar{x})\| \le \left| \int_t^{\bar{t}} \omega(s)\, ds \right| + q_1 \|x - \bar{x}\|.$$

Put $|q| = q_0 + q_1$. We will prove that, under suitable assumptions on ϱ, f and φ and for sufficiently small c with $0 < c \le a$, there exists a solution u of problem (1), (2) such that $u \in K_{\varphi.c}[\omega, q]$.

In force of bicharacteristic approach we cannot expect that the solutions present the same regularities with respect to t and x. In fact they are more regular with respect to x.

We will need the following spaces in the formulation of assumptions on ϱ and f. Let $C_{0.L}(B, R^k)$ be the class of all $w \in C(B, R^k)$ such that

$$\|w\|_L = \sup \left\{ \|w(t,x) - w(t,y)\| \cdot \|x - y\|^{-1} \; : \; (t,x), (t,y) \in B, \; x \ne y \right\} < +\infty.$$

Let us denote by $\|\cdot\|_0$ the supremum norm in the space $C(B, R^k)$ and

$$\|w\|_{0.L} = \|w\|_0 + \|w\|_L \quad \text{for} \quad w \in C_{0.L}(B, R^k).$$

Write

$$C(B,R^k;\kappa)=\{\,w\in C(B,R^k):\ \|w\|_0\le\kappa\,\}$$
$$C_{0.L}(B,R^k;\kappa)=\{\,w\in C_{0.L}(B,R^k):\ \|w\|_{0.L}\le\kappa\,\}$$

where $\kappa\in R_+$. We will denote by Θ the set of all functions $\alpha:[0,a]\times R_+\to R_+$ such that $\alpha(\,\cdot\,,t)\in L([0,a],R_+)$ for $t\in R_+$ and the function $\alpha(t,\,\cdot\,):R_+\to R_+$ is continuous and nondecreasing on R_+ for almost all $t\in[0,a]$ and $\alpha(t,0)=0$. Let $\Theta^\star$ denote the set of all functions $\alpha^\star\in C(R_+,R_+)$ such that $\alpha^\star(0)=0$ and $\alpha^\star$ is nondecreasing on R_+.

Assumption H $[\varrho]$. Suppose that 1) the function $\varrho(\cdot,x,w):[0,a]\to M_{k\times n}$ is measurable for $(x,w)\in R^n\times C(B,R^k)$ and $\varrho(t,\,\cdot\,):R^n\times C(B,R^k)\to M_{k\times n}$ is continuous for almost all $t\in[0,a]$;

2) there exist $\alpha_0,\alpha\in\Theta$ such that

$$\|\varrho(t,x,w)\|\le\alpha_0(t,\kappa)$$

for $(x,w)\in R^n\times C(B,R^k;\kappa)$ almost everywhere on $[0,a]$ and

$$\|\varrho(t,x,w)-\varrho(t,\bar x,\bar w)\|\le\alpha(t,\kappa)[\,\|x-\bar x\|+\|w-\bar w\|_0\,]$$

for $(x,w),(\bar x,\bar w)\in R^n\times C(B,R^k;\kappa)$ almost everywhere on $[0,a]$;

Assumption H $[\psi]$. Suppose that 1) the function $\psi_*(\,\cdot\,,x,w):[0,a]\to R^n$ is measurable for $(x,w)\in R^n\times C(B,R^k)$ and $\psi_*(t,\,\cdot\,):R^n\times C(B,R^k)\to R^n$ is continuous for almost all $t\in[0,a]$;

2) $\psi_0\in L([0,a],R)$, $-b_0\le\psi_0(t)\le t$ for almost all $t\in[0,a]$ and there is $\beta\in\Theta^\star$ such that

$$\|\,\psi_*(t,x,w)-\psi_*(t,\bar x,\bar w)\,\|\le\beta(\kappa)\,[\,\|x-\bar x\|+\|w-\bar w\|_0\,]\tag{3}$$

for $(x,w),(\bar x,\bar w)\in R^n\times C_{0.L}(B,R^k;\kappa)$ almost everywhere on $[0,a]$.

Suppose that $\varphi\in J[\omega_0,p]$, $c\in[0,a]$ and $z\in K_{\varphi.c}[\omega,q]$. Consider the Cauchy problem

$$\eta'(\tau)=\varrho_i(\tau,\eta(\tau),z_{\psi(\tau,\eta(\tau),z_{(\tau,\eta(\tau))})}),\quad \eta(t)=x,\tag{4}$$

where $(t,x)\in[0,c]\times R^n$ and $1\le i\le k$. Denote by $g_i[z](\,\cdot\,,t,x)$ the Carathéodory solution of (4). The function $g_i[z]$ is the ith bicharacteristic of system (1) corresponding to $z\in K_{\varphi.c}[\omega,q]$.

For $z \in C(E_c, R^k)$ and $t \in [0, a]$ we put

$$\|z\|_t = \sup\{\|z(\tau, y)\| : \ (\tau, y) \in [-b_0, t] \times R^n\}.$$

Lemma 2.1. *Suppose that Assumptions* H $[\varrho]$, H$[\psi]$ *are satisfied and*

$$c \in [0, a], \ \varphi, \bar{\varphi} \in J[\omega_0, p],$$
$$z \in K_{\varphi.c}[\omega, q], \ \bar{z} \in K_{\bar{\varphi}.c}[\omega, q].$$

Then for each i, $1 \leq i \leq k$, the bicharacteristics $g_i[z](\,\cdot\,, t, x)$ and $g_i[\bar{z}](\,\cdot\,, t, x)$ are defined on $[0, c]$ and they are unique. Moreover, we have the estimates

$$(5) \ \ \| g_i[z](\tau, t, x) - g_i[z](\tau, \bar{t}, \bar{x})\| \leq \Lambda(t, \tau)\left[\|x - \bar{x}\| + \left|\int_t^{\bar{t}} \alpha_0(s, q_0)\, ds\right|\right]$$

for $(t, x), (\bar{t}, \bar{x}) \in [0, c] \times R^n$, $\tau \in [0, c]$, where

$$\Lambda(t, \tau) = \exp\left[\delta(q)\left|\int_t^{\tau} \alpha(s, q_0)\, ds\right|\right], \quad \delta(q) = 1 + q_1(1 + q_1)\beta(|q|)$$

and

$$(6) \quad \| g_i[z](\tau, t, x) - g_i[\bar{z}](\tau, t, x)\| \leq \tilde{\Lambda}(t, \tau)\left|\int_t^{\bar{t}} \alpha(s, q_0)\, \|z - \bar{z}\|_s\, ds\right|$$

for $(\tau, t, x) \in [0, c]^2 \times R^n$, where

$$\tilde{\Lambda}(t, \tau) = (1 + q_1\beta(|q|))\ \Lambda(t, \tau).$$

Proof. Our proof starts with the observation that

$$\|z_{(\tau,y)}\|_{0.L} \leq |q| \quad \text{and} \quad \|z_{(\tau,y)} - z_{(\tau,\bar{y})}\|_0 \leq q_1\|y - \bar{y}\|$$

where $(\tau, y), (\tau, \bar{y}) \in [0, c] \times R^n$.

The existence and uniqueness of solutions of (4) follows from classical theorems. Note that the righthand side of the system satisfies the Carathéodory conditions and the Lipschitz estimate

$$\|\varrho_i(\tau, y, z_{\psi(\tau,y,z_{(t,y)})}) - \varrho_i(\tau, \bar{y}, z_{\psi(\tau,\bar{y},z_{(\tau,\bar{y})})})\| \leq \alpha(\tau, q_0)\,\delta(q)\,\|y - \bar{y}\|,$$

for $(\tau, y), (\tau, \bar{y}) \in [0, c] \times R^n$, holds.

The function $g_i[z](\cdot, t, x)$ satisfies the integral equation

$$g_i[z](\,\tau\,, t, x) = x + \int_t^\tau \varrho_i(s, g_i[z](s, t, x), z_{\psi(s,g_i[z](s,t,x),z_{(s,g_i[z](s,t,x))})})\,ds,$$

for $(t, x) \in [0, c] \times R^n,\ \tau \in [0, c]$.

From Assumptions H $[\varrho]$ and H $[\psi]$ we get the following integral inequality

$$\begin{aligned}
&\|g_i[z](\tau, t, x) - g_i[z](\tau, \bar{t}, \bar{x})\| \\
&\qquad \leq \|x - \bar{x}\| + \left| \int_t^{\bar{t}} \alpha_0(s, q_0)\,ds \right| \\
&\qquad\quad + \delta(q) \left| \int_t^\tau \alpha(s, q_0)\,\|g_i[z](s, t, x) - g_i[z](s, \bar{t}, \bar{x})\|\,ds \right|,
\end{aligned}$$

for $(t, x),\ (\bar{t}, \bar{x}) \in [0, c] \times R^n,\ \tau \in [0, c]$. Hence, and by the Gronwall inequality we obtain estimate (5).

For $z \in K_{\varphi.c}[\omega, q]$, $\bar{z} \in K_{\bar{\varphi}.c}[\omega, q]$, we have the integral inequality

$$\begin{aligned}
&\|g_i[z](\tau, t, x) - g_i[\bar{z}](\tau, t, x)\| \\
&\qquad \leq (1 + q_1\beta(|q|)) \left| \int_t^\tau \alpha(s, q_0)\,\|z - \bar{z}\|_s\,ds \right| \\
&\qquad\quad + \delta(q) \left| \int_t^\tau \alpha(s, q_0)\,\|g_i[z](s, t, x) - g_i[\bar{z}](s, t, x)\,\|\,ds \right|,
\end{aligned}$$

for $(t, x) \in [0, c] \times R^n,\ \tau \in [0, c]$. Now, we get (6) by the Gronwall inequality. This completes the proof of Lemma 2.1.

Remark 2.2. It is important in our considerations that we have assumed the Lipschitz condition for $\psi_\star$ in some special function spaces.

We have assumed that $\psi_\star(t,\,\cdot\,)$ satisfies the Lipschitz condition on the space $R^n \times C(B,R^k)$ for almost all $t \in [0,a]$, and this condition is local with respect to the functional variable. The Lipschitz coefficient depends on the space $C_{0.L}(B,R^k;\kappa)$. Let us consider the simplest assumption on $\psi_\star$. Suppose that there is $\tilde{L} \in R_+$ such that for almost all $t \in [0,a]$ we have

$$\|\psi_\star(t,x,w) - \psi_\star(t,\bar{x},\bar{w})\| \le \tilde{L}\,[\,\|x-\bar{x}\| + \|w-\bar{w}\|_0\,], \tag{7}$$

on $R^n \times C(B,R^k)$. Of course, our results are true if we assume (7) instead of (3). Now we show that there is a class of function $\psi_\star$ satisfying Assumption H $[\psi]$ and not satisfying (7). Consider the function $\psi_\star$ given by

$$\psi_\star(t,x,w) = \tilde{\psi}(\,t,x,w(\eta_0(t)-t,\eta(t,x)-x)\,) \tag{8}$$

where $\eta_0 \in C([0,a],R)$, $\eta \in C([0,a]\times R^n,R^n)$ and

$$-b_0 \le \eta_0(t) - t \le 0, \qquad -b \le \eta(t,x) - x \le b.$$

Then $\psi_\star(t,x,,z_{(t,x)}) = \tilde{\psi}(\,t,x,z(\eta_0(t),\eta(t,x)\,)$. We assume that there are L_0, $L_1 \in R_+$ such that

$$\|\eta(t,x) - \eta(t,\bar{x})\| \le L_0\|x-\bar{x}\|,$$
$$\|\tilde{\psi}(t,x,y) - \tilde{\psi}(t,\bar{x},\bar{y})\| \le L_1\,[\|x-\bar{x}\| + \|y-\bar{y}\|]$$

where $(t,x),(,\bar{x}) \in [0,a]\times R^n$, $y,\bar{y} \in R^n$. Then function $\psi_\star$ given by (8) satisfies condition (3) and does not satisfy (7).

Note that we have assumed the Lipschitz condition for $\varrho(t,\,\cdot\,)$, and this condition is local with respect to the functional variable.

3. Existence and uniqueness of Carathéodory solutions. First we formulate integral functional equations corresponding to problem (1), (2). Suppose that $\varphi \in J[\omega_0,p]$, $c \in [0,a]$, $z \in K_{\varphi.c}[\omega,q]$ and

$$(\,g_1[z](\,\cdot\,,t,x),\dots,g_k[z](\,\cdot\,,t,x)\,) = g[z](\,\cdot\,,t,x)$$

is the set of bicharacteristics of system (1) corresponding to z. Write

$$\Delta_i[z](\tau,t,x) = \psi(\tau,g_i[z](\tau,t,x),z_{(\tau,g_i[z](\tau,t,x))}), \quad 1 \le i \le k,$$

and

$$f^*(\tau, g[z](\tau,t,x), z_{\Delta[z](\tau,t,x)}) \\ = \big(f_1(\tau, g_1[z](\tau,t,x), z_{\Delta_1[z](\tau,t,x)}), \dots , \\ f_k(\tau, g_k[z](\tau,t,x), z_{\Delta_k[z](\tau,t,x)})\big)$$

and

$$\varphi^\star(0, g[z](0,t,x)) = (\, \varphi_1(0, g_1[z](0,t,x)), \dots , \varphi_k(0, g_k[z](0,t,x))\,)\,.$$

Let us define the operator F for all $z \in K_{\varphi.c}[\omega, q]$ by the formula

$$F[z](t,x) = \varphi^\star(0, g[z](0,t,x)) + \int_0^t f^*(\tau, g[z](\tau,t,x), z_{\Delta[z](\tau,t,x)})\, d\tau,$$

for $(t,x) \in [0,c] \times R^n$, and

$$F[z](t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0.$$

Assumption $H[f]$. Suppose that 1) the function $f(\,\cdot\,, x, w) : [0,a] \to R^k$ is measurable for $(x,w) \in R^n \times C(B, R^k)$ and $f(t, \cdot\,) : R^n \times C(B, R^k) \to R^k$ is continuous for almost all $t \in [0,a]$;

2) there exist γ_0, $\gamma \in \Theta$ such that

$$\|f(t,x,w)\| \le \gamma_0(t,\kappa)$$

for $(x,w) \in R^n \times C(B, R^k; \kappa)$ almost everywhere on $[0,a]$ and

$$\|f(t,x,w) - f(t,\bar{x},\bar{w})\| \le \gamma(t,\kappa)\ [\, \|x-\bar{x}\| + \|w - \bar{w}\|_0\,]$$

for $(x,w), (\bar{x},\bar{w}) \in R^n \times C(B, R^k; \kappa)$ almost everywhere on $[0,a]$.

Assumption $H[\omega, q]$. Suppose that the constants $(q_0, q_1) = q$, $c \in (0,a]$, and the function $\omega : [-b_0, c] \to R_+$ satisfy the conditions

$$q_0 \ge p_0 + \int_0^c \gamma_0(s, q_0)\, ds,$$
$$q_1 \ge \left[p_1 + \delta(q) \int_0^c \gamma(s, q_0)\, ds \right] \Lambda(c, 0)$$

and

$$\omega(t) \geq \alpha_0(t, q_0)\left[p_1 + \delta(q)\int_0^c \gamma(s, q_0)\,ds\right]\Lambda(c, 0) + \gamma_0(t, q_0)$$

for almost all $t \in [0, c]$.

Remark 3.1. If $q_0 > p_0$ and $q_1 > p_1$, then there is a $c \in (0, a]$ such that (q_0, q_1) satisfies Assumption H $[\omega, q]$.

Lemma 3.2. *If* $\varphi \in J[\omega_0, p]$ *and Assumptions* H $[\varrho]$, $[\psi]$, H $[f]$ *and* H $[\omega, q]$ *are satisfied, then*

$$F : K_{\varphi.c}[\omega, q] \to K_{\varphi.c}[\omega, q].$$

Proof. For $z \in K_{\varphi.c}[\omega, q]$, by Assumptions $H[f]$ and $H[\omega, q]$, we have

$$\|F[z](t, x)\| \leq p_0 + \int_0^c \gamma_0(s, q_0)\,ds \leq q_0,$$

on $[0, c] \times R^n$, and

$$\begin{aligned}\|F[z](t, x) - F[z](\bar{t}, \bar{x})\| \leq{}& \|\varphi^\star(0, g[z](0, t, x)) - \varphi^\star(0, g[z](0, \bar{t}, \bar{x}))\| \\ &+ \int_0^t \| f^*(s, g[z](s, t, x), z_{\Delta[z](s,t,x)}) \\ &- f^*(s, g[z](s, \bar{t}, \bar{x}), z_{\Delta[z](s,\bar{t},\bar{x})})\|\,ds \\ &+ \left|\int_t^{\bar{t}} \|f^\star s, g[z](s, \bar{t}, \bar{x}), z_{\Delta[z](s,\bar{t},\bar{x})}\,)\|\,ds\right| \\ \leq{}& p_1 \max_{1\leq i\leq k} \| g_i[z](0, t, x) - g_i[z](0, \bar{t}, \bar{x})\,\| \\ &+ \delta(q)\int_0^t \gamma(s, |q|) \max_{1\leq i\leq k} \|g_i[z](0, t, x) \\ &- g_i[z](0, \bar{t}, \bar{x})\|\,ds + \left|\int_t^{\bar{t}} \gamma_0(s, q_0)\,ds\right|\end{aligned}$$

for $(t,x),(\bar t,\bar x)\in[0,c]\times R^n$. Hence, by Lemma 1 we get

$$\begin{aligned}\| F[z](t,x)-F[z](\bar t,\bar x)\| &\\ \le \left[\|x-\bar x\|+\int_t^{\bar t}\alpha_0(s,q_0)\,ds\right]&\left[p_1+\delta(q)\int_0^c\gamma(s,q_0)\,ds\right]\Lambda(c,0)\\ +\left|\int_t^{\bar t}\gamma_0(s,q_0)\,ds\right|&\end{aligned}$$

and consequently

$$\|F[z](t,x)-F[z](\bar t,\bar x)\|\le\left|\int_t^{\bar t}\omega(s)\,ds\right|+q_1\,\|x-\bar x\|$$

on $[0,c]\times R^n$. Therefore $Fz\in K_{\varphi.c}[\omega,q]$, which completes the proof of Lemma 3.2.

Theorem 3.3. *Suppose that* $\varphi\in J[\omega_0,p]$ *and Assumptions* H $[\varrho]$, $[\psi]$, H $[f]$ *and* H $[\omega,q]$ *are satisfied. Then there exists exactly one solution* $u\in K_{\varphi.c}[\omega,q]$ *of problem* (1), (2).

If $\bar\varphi\in J[\omega_0,p]$ *and* $\bar u\in K_{\bar\varphi.c}[\omega,q]$ *is the solution of system* (1) *with the initial condition* $z(t,x)=\bar\varphi(t,x)$ *on* E_0, *then there is* $\eta\in C([0,c],R_+)$ *such that*

$$\|u-\bar u\|_t\le\eta(t)\,\|\varphi-\bar\varphi\|_0,\quad t\in[0,c]. \tag{9}$$

Proof. Lemma 3.2 shows that the operator F maps $K_{\varphi.c}[\omega,q]$ into itself. Put

$$\begin{aligned}\Gamma(t)&=(1+q_1\beta(|q|))\ \Lambda(c,0)\left[p_1+\delta(q)\int_0^t\gamma(s,q_0)\,ds\right],\\ \bar q&=(1+q_1\beta(|q|))\ \max\{\,1,\,\Gamma(c)\},\end{aligned}$$

and

$$\lambda(t)=\bar q\,(\gamma(t,q_0)+\alpha(r,q_0))\,,\quad t\in[0,c].$$

For $z,\ \bar z\in K_{\varphi.c}[\omega,q]$ we put

$$[|z-\bar z|]=\sup\left\{\|z-\bar z\|_t\ \exp\left[-2\int_0^t\lambda(s)\,ds\right]:t\in[0,c]\right\}.$$

It follows from Assumption H $[f]$ that

$$\begin{aligned}
&\|F[z](t,x) - F[\bar{z}](t,x)\| \\
&\qquad \le \|\varphi^\star(0, g[z](0,t,x)) - \varphi^\star(0, g[\bar{z}](0,t,x))\| \\
&\qquad\quad + \int_0^t \| f^*(s, g[z](s,t,x), z_{\Delta[z](s,t,x)}) \\
&\qquad\quad - f^*(s, g[\bar{z}](s,t,x), z_{\Delta[\bar{z}](s,t,x)}) \| \, ds \\
&\qquad \le p_1 \max_{1\le i\le k} \|g_i[z](0,t,x) - g_i[\bar{z}](0,t,x)\| \\
&\qquad\quad + (1 + q_1\beta(|q|)) \int_0^t \gamma(s, q_0) \, \|z - \bar{z}\|_s \, ds \\
&\qquad\quad + \delta(q) \int_0 \gamma(s, q_0) \max_{1\le i\le k} \|g_i[z](s,t,x) - g_i[\bar{z}(s,t,x)\| \, ds,
\end{aligned}$$

and consequently

$$\begin{aligned}
\|F[z](t,x) - F[\bar{z}](t,x)\| &\le \int_0^t \lambda(s) \, \|z - \bar{z}\|_s \, ds \\
&\le [|z - \bar{z}|] \int_0^t \lambda(s) \, \exp\left[2 \int_0^s \lambda(\tau) \, d\tau\right] \, ds \\
&\le \frac{1}{2} \, [|z - \bar{z}|] \, \exp\left[2 \int_0^t \lambda(s) \, ds\right], \\
&\qquad (t,x) \in [0,c] \times R^n.
\end{aligned}$$

The result is

$$[| \, F[z] - F[\bar{z}] \, |] \le \frac{1}{2} \, [| \, z - \bar{z} \, |]$$

for z, $\bar{z} \in K_{\varphi.c}[\omega, q]$. By the Banach fixed point theorem there exists a unique solution $u \in K_{\varphi.c}[\omega, q]$ of the equation $z = F[z]$. Now, we prove that u is a solution of system (1). We have

$$u_i(t,x) = \varphi(0, g_i[u](0,t,x)) + \int_0^t f_i(s, g[u](s,t,x), z_{\Delta_i[u](s,t,x)}) \, ds, \tag{10}$$

where $1 \le i \le k$, $(t,x) \in [0,c] \times R^n$. For a given $1 \le i \le k$, $(t,x) \in [0,c] \times R^n$, let us put $\eta^{(i)} = g_i[u](0,t,x)$. Hence, we obtain

$$g_i[u](\tau,t,x) = g_i[u](t,0,\eta^{(i)}) \quad \text{for } \tau \in [0,c] \text{ and } x = g_i[u](t,0,\eta^{(i)}).$$

From (10) it follows that

$$u_i(t, g_i[u](t, 0, \eta^{(i)})) \\ = \varphi_i(0, \eta^{(i)}) + \int_0^t f_i(s, g_i[u](s, 0, \eta^{(i)}), u_{\Delta_i[u](s,0,\eta^{(i)})})\, ds, \quad 1 \le i \le k.$$

By differentiating the last expressions with respect to t and by putting again $\eta^{(i)} = g_i[u](0, t, x)$, we obtain that u satisfies system (11) for almost all $(t, x) \in [0, c] \times R^n$.

If $u = F[u]$ and $\bar{u} = F[\bar{u}]$, then we have the integral inequality

$$\|u - \bar{u}\|_t \le \|\varphi - \bar{\varphi}\|_0 + \int_0^t \lambda(s)\, \|u - \bar{u}\|_s\, ds, \quad t \in [0, c].$$

Using the Gronwall inequality, we obtain (9) with

$$\eta(t) = \exp\left[\int_0^t \lambda(s)\, ds\right], \quad t \in [0, c].$$

This completes the proof. □

4. Some noteworthy particular case. Now we list examples of systems which can be derived from (11) by specializing the operators ϱ and f. Assume that

$$\tilde{\varrho} : [0, a] \times R^n \times R^k \longrightarrow M_{k\times n}, \quad \tilde{\varrho} = [\tilde{\varrho}_{ij}]_{i=1,\dots,n,\, j=1,\dots,n}\,,$$
$$\tilde{f} : [0, a] \times R^n \times R^k \longrightarrow R^k, \quad \tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_k),$$

and

$$\psi_0 : [0, a] \longrightarrow R,\ \psi_* : \Omega \longrightarrow R^n$$

are given functions. Consider the operators

$$\varrho : \Omega \longrightarrow M_{k\times n} \quad \text{and} \quad f : \Omega \longrightarrow R^k$$

defined by

$$\varrho(t, x, w) = \tilde{\varrho}(t, x, w(0, 0)),$$
$$f(t, x, w) = \tilde{f}(t, x, w(0, 0)), \ \ (t, x, w) \in \Omega.$$

Then

$$\varrho(t,x,z_{\psi(t,x,z_{(t,x)})}) = \tilde{\varrho}(\,t,x,z(\psi(t,x,z_{(t,x)})\,)$$

and

$$f(t,x,z_{\psi(t,x,z_{(t,x)})}) = \tilde{f}(\,t,x,z(\psi(t,x,z_{(t,x)})\,),$$

and system (1) reduces to the system of differential equations with delay dependent on the unknown function

$$\begin{aligned}(11)\quad \partial_t z_i(t,x) + \sum_{j=1}^{n} \tilde{\varrho}_{ij}(\,t,x,z(\psi(t,x,z_{(t,x)}))\,)\,\partial_{x_j} z_i(t,x)& \\ = \tilde{f}(\,t,x,z(\psi(t,x,z_{(t,x)}))\,),& \quad i=1,\dots,k.\end{aligned}$$

Assumption H $[\tilde{\varrho},\tilde{f}]$. Suppose that 1) the functions $\tilde{\varrho}(\,\cdot\,,x,\zeta) : [0,a] \to M_{k\times n}$ and $\tilde{f}(\,\cdot\,,x,\zeta) : [0,a] \to R^k$ are measurable for $(x,\zeta) \in R^n \times R^k$ and $\tilde{\varrho}(t,\,\cdot\,) : R^n \times R^k \to M_{k\times n}$, $\tilde{f}(t,\,\cdot\,) : R^n \times R^k \to R^k$ are continuous for almost all $t \in [0,a]$;

2) there exist α_0, α, γ_0, $\gamma \in \Theta$ such that

$$\|\tilde{\varrho}(t,x,\zeta)\| \le \alpha_0(t,\kappa), \quad \|\tilde{f}(t,x,\zeta)\| \le \gamma_0(t,\kappa)$$

for $(x,\zeta) \in R^n \times R^k$, $\|\zeta\| \le \kappa$, almost everywhere on $[0,a]$ and

$$\begin{aligned}\|\tilde{\varrho}(t,x,\zeta) - \tilde{\varrho}(t,\bar{x},\bar{\zeta})\| &\le \alpha(t,\kappa)\,[\|x-\bar{x}\| + \|\zeta-\bar{\zeta}\|]\\ \|\tilde{f}(t,x,\zeta) - \tilde{f}(t,\bar{x},\bar{\zeta})\| &\le \gamma(t,\kappa)\,[\|x-\bar{x}\| + \|\zeta-\bar{\zeta}\|]\end{aligned}$$

for (x,ζ), $(\bar{x},\bar{\zeta}) \in R^n \times R^k$, $\|\zeta\|$, $\|\bar{\zeta}\| \le \kappa$, almost everywhere on $[0,a]$.

We formulate a theorem on the solvability of problem (11), (2).

Theorem 4.1. *Suppose that $\varphi \in J[\omega_0,p]$ and Assumption* H $[\tilde{\varrho},\tilde{f}]$, H $[\psi]$, H $[\omega,q]$ *are satisfied. Then there exists exactly one solution $u \in K_{\varphi.c}[\omega,q]$ of problem* (11), (2). *If $\bar{\varphi} \in J[\omega_0,p]$ and $\bar{u} \in K_{\varphi.c}[\omega,q]$ is the solution of system* (11) *with the initial condition $z(t,x) = \bar{\varphi}(t,x)$ for $(t,x) \in E_0$, then there is $\eta \in C([0,c],R_+)$ such that estimate* (10) *holds.*

The theorem follows as an immediate consequence of Theorem 3.3.

Consider the functions $\varrho : \Omega \to M_{k\times n}$ and $f : \Omega \to R^k$ given by

$$\varrho(t,x,w) = \tilde{\varrho}(t,x,\int_B w(\tau,y)\,d\tau\,dy),$$
$$f(t,x,w) = \tilde{f}(t,x,\int_B w(\tau,y)\,d\tau\,dy).$$

Write

$$D[t,x,z] = \big\{\,(\tau,y)\in R^{1+n}:\ \ \psi_0(t)-b_0 \le \tau \le \psi_0(t),\\ \psi_\star(t,x,z_{(t,x)}) - b \le y \le \psi_\star(t,x,z_{(t,x)}) + b\,\big\}\,.$$

Then (1) reduces to the differential integral system

$$(12)\quad \partial_t z_i(t,x) + \sum_{j=1}^n \tilde{\varrho}_{ij}\bigg(t,x,\int_{D[t,x,z]} z(\tau,y)\,d\tau\,dy\bigg)\,\partial_{x_j} z_i(t,x) \\ = \tilde{f}\bigg(t,x,\int_{D[t,x,z]} z(\tau,y)\,d\tau\,dy\bigg),\quad i=1,\dots,k.$$

It is easy to formulate sufficient conditions for the existence, uniqueness and continuous dependence of the Carathéodory solutions to problem (12), (A2) again as an application of Theorem 3.3.

Note that results of the papers [**14, 18, 20, 21**] and the monograph [**16**, Chapter 4] are not applicable to systems (11) and (12).

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