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A NEW PROOF OF LIEBMANN CLASSICAL RIGIDITY THEOREM FOR SURFACES IN SPACE FORMS

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ABSTRACT. In this paper we provide a new direct proof of the Liebmann classical rigidity theorem for surfaces in space forms, showing that the only compact surfaces with constant Gaussian curvature which are immersed into the Euclidean space $\mathbf{E}^3$, into the hyperbolic space $\mathbf{H}^3$, or into an open hemisphere $\mathbf{S}^3_+$ are the totally umbilical round spheres. Our proof is an application of the Gauss-Bonnet theorem along with a formula involving the Gaussian curvatures of the first and second fundamental forms of the surface, which is interesting *per se*.

1. Introduction. In 1897 Hadamard [**6**] proved that an ovaloid, that is, a compact connected surface with positive Gaussian curvature, in the three-dimensional Euclidean space $\mathbf{E}^3$ is a topological sphere. In view of this result, it was natural to look for conditions which allowed one to conclude that such a surface was necessarily a totally umbilical round sphere. In 1899 Liebmann [**11**] obtained his celebrated rigidity result, which states that every compact connected surface in $\mathbf{E}^3$ with constant Gaussian curvature is necessarily a totally umbilical round sphere.

The most famous proof of Liebmann theorem was given by Hilbert, just a short time after Liebmann's original proof, using local computations [**7**, Appendix 5], see also [**4**], as well as [**3**, Theorem 1, p. 317] for

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an accessible reference). On the other hand, Minkowski formulas provide one with another nice proof of Liebmann result, now from a global approach, see for instance [**8**, Theorem 6.2.10, p. 137], [**10**, Theorem 5.3] or [**13**, Corollary 6.8, p. 218].

Afterwards, there have been different generalizations of Liebmann classical rigidity theorem from several points of view for surfaces, and more generally hypersurfaces, into the Euclidean space [**9, 15, 16, 18, 19**], or into the hyperbolic space or an open hemisphere [**12**]. Nevertheless, it would be desirable a direct simple proof of the result using the same techniques as in [**9, 18, 19**], which could be easily understood for beginning researchers. Our main objective in this paper is to provide with such a proof. Specifically, we will give a proof of the following version of Liebmann classical rigidity result for the case of surfaces in space forms.

Theorem 1. *The only compact surfaces which are immersed into the Euclidean space* $\mathbf{E}^3$*, into the hyperbolic space* $\mathbf{H}^3$*, or into an open hemisphere* $\mathbf{S}^3_+$ *with constant Gaussian curvature are the totally umbilical round spheres.*

Our proof follows from a nice reasoning as a consequence of a formula involving the Gaussian curvatures of the first and second fundamental forms of the surface (10), jointly with the Gauss-Bonnet theorem. Although the formula (10) was already known for the case of convex hypersurfaces into Euclidean space [**18**], we prove it from a modern setting according to the development of the current differential geometry. Actually, the formula (10), and its n-dimensional version involving the scalar curvature (9), is obtained in Section 2 from a detailed study of the relationship between the Riemannian curvature tensors and the Ricci tensors of the first and second fundamental forms of the hypersurface.

In our opinion, formulas (9) and (10) are interesting per se, and they also have interest in affine differential geometry. In that context, the metric defined by the second fundamental form II is the so-called relative metric induced from the Euclidean normalization; the equiaffine normalization leads to the form $G^{-1/(n+2)}\mathrm{II}$ [**17**], where G stands for the Gauss-Kronecker curvature of the hypersurface. Therefore, a careful study of the geometric properties of the metric defined by II is of interest also in affine differential geometry, see [**1**], and it should be useful to have a modern proof of formulas (9) and (10). For that reason, a second interest of this paper is to provide with such a proof of formulas (9) and (10), writing and proving them in a more intrinsically way. Our proof here is a nice application of connection theory using uniquely elementary properties of Riemannian connections on manifolds.

2. Riemannian curvature of the second fundamental form.

Our objective in this section is to compute the curvature of the second fundamental form of a surface, in the case where the second fundamental form defines a non-degenerate pseudo-Riemannian metric on the surface. Although we are particularly interested in the case of surfaces immersed into a Riemannian three-space form, it is not much extra work to consider the more general case of hypersurfaces of a Riemannian space form with non-degenerate second fundamental form. That computation was first made by Schneider [**18**] for the case of convex hypersurfaces in Euclidean space, using old fashion index gymnastics. Although more general, our proof here follows essentially similar ideas to Schneider's proof, but it is written using modern and conceptual terminology and classical results about surfaces expressed in terms of connections.

In order to set up the notation to be used later on, let us denote by $\overline{M}(c)$ the standard model of an $(n+1)$-dimensional Riemannian space form of constant curvature c, $c = 0, 1, -1$. That is, $\overline{M}(c)$ denotes the Euclidean space $\mathbf{E}^{n+1}$ when $c = 0$, the sphere $\mathbf{S}^{n+1}$ when $c = 1$, and the hyperbolic space $\mathbf{H}^{n+1}$ when $c = -1$. Let us consider M an orientable hypersurface immersed into $\overline{M}(c)$, and let N be a globally defined unit normal field on M. As usual, we agree to denote by $\langle , \rangle$ both the constant curvature Riemannian metric on $\overline{M}(c)$ and the Riemannian metric induced on M. In what follows, we will assume that the second fundamental form II of the hypersurface is non-degenerate, so that

$$\mathrm{II}(X, Y) = \langle AX, Y\rangle, \quad \text{for every tangent vector field } X, Y \in \mathfrak{X}(M),$$

defines a pseudo-Riemannian metric on M, where A denotes the shape operator of M associated to the chosen normal field N. Observe that the condition that II is non-degenerate is equivalent to the fact that $G = \det A \neq 0$, and it means that the principal curvatures of the hypersurface do not vanish on M.

Let us consider on M both the Levi-Civita connection ∇ of the Riemannian metric $\langle , \rangle$ and the Levi-Civita connection ∇^{II} of the pseudo-Riemannian metric II. Our aim is to study the relationship between their Riemannian curvatures. To do that, let us denote by

$$T(X,Y) = \nabla^{\mathrm{II}}_X Y - \nabla_X Y, \quad X, Y \in \mathfrak{X}(M),$$

the difference tensor of the Levi-Civita connections, which is a symmetric tensor on M since ∇ and ∇^{II} are both torsion-free. Using the well-known Koszul formula for ∇^{II} [**14**, Theorem 3.11, p. 61], it follows that

$$\begin{aligned}\mathrm{II}(\nabla^{\mathrm{II}}_X Y, Z) = \langle A(\nabla^{\mathrm{II}}_X Y), Z\rangle = \langle A(\nabla_X Y), Z\rangle + \frac{1}{2}\langle(\nabla A)(Y,X), Z\rangle \\ + \frac{1}{2}\langle(\nabla A)(Z,Y), X\rangle - \frac{1}{2}\langle(\nabla A)(X,Z), Y\rangle,\end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$, where in our notation $(\nabla A)(X,Y) = (\nabla_Y A)X$. Since $\overline{M}(c)$ has constant curvature, Codazzi equation implies that ∇A is symmetric [**19**, Corollary 4.34, p. 115], and therefore $\langle(\nabla A)(X,Y), Z\rangle$ is symmetric in all three variables, so that the expression above simplifies to

$$\langle A(\nabla^{\mathrm{II}}_X Y), Z\rangle = \langle A(\nabla_X Y), Z\rangle + \frac{1}{2}\langle(\nabla A)(X,Y), Z\rangle,$$

or equivalently

$$T(X,Y) = \frac{1}{2} A^{-1}((\nabla A)(X,Y)), \tag{1}$$

for every tangent vector field $X, Y \in \mathfrak{X}(M)$.

Now, let us denote by R and R^{II} the Riemannian curvature tensors of $\langle , \rangle$ and II, respectively. By a direct computation we obtain that

$$R^{\mathrm{II}}(X,Y)Z = R(X,Y)Z + Q_1(X,Y)Z + Q_2(X,Y)Z, \tag{2}$$

where

$$\begin{aligned}Q_1(X,Y)Z &= (\nabla^{\mathrm{II}}_Y T)(X,Z) - (\nabla^{\mathrm{II}}_X T)(Y,Z),\\ Q_2(X,Y)Z &= T(X,T(Y,Z)) - T(Y,T(X,Z)).\end{aligned}$$

Here we are following O'Neill's choice of the Riemannian curvature tensor, that is, $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. Contracting in (2) we obtain

$$\mathrm{Ric}^{\mathrm{II}}(X,Y) = \mathrm{Ric}\,(X,Y) + \widehat{Q}_1(X,Y) + \widehat{Q}_2(X,Y),$$

where Ric and $\mathrm{Ric}^{\mathrm{II}}$ denote, respectively, the Ricci tensor of $\langle , \rangle$ and II, and

$$\widehat{Q}_1(X,Y) = \mathrm{tr}\,\big(Z \mapsto Q_1(X,Z)Y\big),$$
$$\widehat{Q}_2(X,Y) = \mathrm{tr}\,\big(Z \mapsto Q_2(X,Z)Y\big).$$

Taking now traces with respect to the (pseudo-Riemannian) metric II it follows that

$$S_{\mathrm{II}} = \mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}) + \mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) + \mathrm{tr}_{\mathrm{II}}(\widehat{Q}_2), \tag{3}$$

where $S_{\mathrm{II}} = \mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}^{\mathrm{II}})$ stands for the scalar curvature of II, and the trace $\mathrm{tr}_{\mathrm{II}}$ of a 2-covariant tensor L is, as usual, the trace of the (1,1)-tensor $L^\sharp$ defined by

$$\mathrm{II}\big(L^\sharp(X), Y\big) = L(X,Y), \quad X, Y \in \mathfrak{X}(M).$$

Now, let us compute independently each term in (3). We will do this by computing in a local $\langle , \rangle$-orthonormal frame on M that diagonalizes A. It is worth pointing out that such a frame does not always exist; problems occur when the multiplicity of the principal curvatures changes (also the principal curvatures are not necessarily everywhere differentiable). For that reason, we will work on the subset M_0 of M consisting of points at which the number of distinct principal curvatures is locally constant. Let us recall that M_0 is an open dense subset of M, and in every connected component of M_0, the principal curvatures form mutually distinct smooth principal curvature functions and, for such a principal curvature κ, the assignment $p \mapsto V_{\kappa(p)}(p)$ defines a smooth distribution, where $V_{\kappa(p)}(p) \subset T_pM$ denotes the eigenspace associated to $\kappa(p)$, see, for instance, [**2**, Paragraph 16.10]. Therefore, for every $p \in M_0$ there exists a local $\langle , \rangle$-orthonormal frame defined on a neighborhood of p that diagonalizes A, that is, $\{e_1, \dots, e_n\}$ such that $Ae_i = \kappa_i e_i$, with each $\kappa_i \neq 0$ smooth. Therefore, $E_i = (1/\sqrt{|\kappa_i|})e_i$, for $1 \leq i \leq n$, defines a local II-orthonormal frame, with signature $\varepsilon_i = \mathrm{II}(E_i, E_i) = \operatorname{sign} \kappa_i$. Using this frame, we can compute each term in (3).

For the first term in (3), let us recall that the Gauss equation of the hypersurface [**19**, Corollary 4.20, p. 107] implies that its Ricci curvature can be expressed in terms of the constant curvature c of $\overline{M}(c)$ and the shape operator A by

$$\mathrm{Ric}(X,Y) = c(n-1)\langle X,Y\rangle + nH\langle AX,Y\rangle - \langle AX,AY\rangle, \quad X,Y \in \mathfrak{X}(M),$$

where as usual $H = (1/n)\mathrm{tr}\,(A)$ stands for the mean curvature of the hypersurface. It follows from here that

$$\begin{aligned} \mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}) &= \sum_{i=1}^{n} \varepsilon_i \mathrm{Ric}(E_i,E_i) = \sum_{i=1}^{n} \frac{1}{\kappa_i}\,\mathrm{Ric}(e_i,e_i) \\ &= c(n-1)\bigg(\sum_{i=1}^{n}\frac{1}{\kappa_i}\bigg) + n^2H - \sum_{i=1}^{n}\kappa_i. \end{aligned}$$

That is,

$$\mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}) = (n-1)\bigg(nH + \frac{c\sigma_{n-1}}{G}\bigg), \tag{4}$$

where $G = \det A \neq 0$ is the Gauss-Kronecker curvature of M and

$$\sigma_{n-1} = \sum_{i_1<\cdots<i_{n-1}} \kappa_{i_1}\cdots\kappa_{i_{n-1}}$$

is the $(n-1)$th elementary symmetric function of the principal curvatures.

On the other hand, we claim that the second term in (3) vanishes,

$$\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) = 0. \tag{5}$$

In fact, it follows from (1) that

$$\mathrm{II}\big(T(X,Y),Z\big) = \frac{1}{2}\,\langle(\nabla A)(X,Y),Z\rangle$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Consequently by the Codazzi equation $\mathrm{II}\big(T(X,Y),Z\big)$ is symmetric in all three variables, and therefore

$$\mathrm{II}\big((\nabla^{\mathrm{II}}_U T)(X,Y),Z\big), \quad X,Y,Z,U \in \mathfrak{X}(M),$$

is symmetric in X, Y, Z. Using this fact, we can easily compute the second term in (3), obtaining (5),

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1)\\
&= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}(Q_1(E_i,E_j)E_i,E_j)\\
&= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}((\nabla^{\mathrm{II}}_{E_j}T)(E_i,E_i),E_j) - \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}((\nabla^{\mathrm{II}}_{E_i}T)(E_j,E_i),E_j)\\
&= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}((\nabla^{\mathrm{II}}_{E_i}T)(E_j,E_j),E_i) - \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}((\nabla^{\mathrm{II}}_{E_i}T)(E_j,E_i),E_j)\\
&= 0.
\end{aligned}$$

Finally, let us compute the third term in (3). By the symmetries of $\mathrm{II}(T(X,Y),Z)$, we have

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_2)\\
&= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}(Q_2(E_i,E_j)E_i,E_j)\\
&= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}(T(E_i,T(E_j,E_i)),E_j) - \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}(T(E_j,T(E_i,E_i),E_j)\\
&= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}(T(E_i,E_j),T(E_i,E_j)) - \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \mathrm{II}(T(E_i,E_i),T(E_j,E_j))\\
&= \mathrm{II}(T,T) - \mathrm{II}\big(\mathrm{tr}_{\mathrm{II}}(T),\mathrm{tr}_{\mathrm{II}}(T)\big),
\end{aligned}$$

where $\mathrm{tr}_{\mathrm{II}}(T)$ is the vector field obtained from the II-contraction of T.

Remark 2. Observe that II is in general a pseudo-Riemannian metric. For that reason, we prefer to write $\mathrm{II}(T,T)$ and $\mathrm{II}\big(\mathrm{tr}_{\mathrm{II}}(T),\mathrm{tr}_{\mathrm{II}}(T)\big)$ instead of $||T||^2_{\mathrm{II}}$ and $||\mathrm{tr}_{\mathrm{II}}T||^2_{\mathrm{II}}$, respectively. We will reserve that terminology for the case where II is Riemannian.

It only remains to compute $\mathrm{tr}_{\mathrm{II}}T$. Observe that

$$\mathrm{tr}_{\mathrm{II}}(T)=\sum_{i=1}^{n}\varepsilon_i T(E_i,E_i)=\frac{1}{2}\,A^{-1}\bigg(\sum_{i=1}^{n}\frac{1}{\kappa_i}\,(\nabla A)(e_i,e_i)\bigg). \tag{6}$$

On the other hand, $\log|G|=\sum_i\log|\kappa_i|$ is a smooth function on M and, for every $X\in\mathfrak{X}(M)$, we have

$$X(\log|G|)=\sum_i\frac{1}{\kappa_i}\,X(\kappa_i)=\sum_i\frac{1}{\kappa_i}\,\langle(\nabla A)(e_i,e_i),X\rangle,$$

where we have used again Codazzi equation. This implies that

$$\frac{1}{G}\,\mathrm{II}(X,\nabla^{\mathrm{II}}G)=\frac{1}{G}\,\langle X,A(\nabla^{\mathrm{II}}G)\rangle=\sum_i\frac{1}{\kappa_i}\,\langle(\nabla A)(e_i,e_i),X\rangle,$$

for every $X\in\mathfrak{X}(M)$, which by (6) means that

$$\mathrm{tr}_{\mathrm{II}}(T)=\frac{1}{2G}\,\nabla^{\mathrm{II}}G.$$

Consequently, we have

$$\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_2)=\mathrm{II}(T,T)-\frac{1}{4G^2}\,\mathrm{II}(\nabla^{\mathrm{II}}G,\nabla^{\mathrm{II}}G). \tag{7}$$

Summing up, using (4), (5) and (7) in formula (3) we conclude that the scalar curvature of II is given by

$$S_{\mathrm{II}}=(n-1)\left(nH+\frac{c\sigma_{n-1}}{G}\right)+\mathrm{II}(T,T)-\frac{1}{4G^2}\,\mathrm{II}(\nabla^{\mathrm{II}}G,\nabla^{\mathrm{II}}G). \tag{8}$$

Recall that a hypersurface in a Riemannian space form $\overline{M}(c)$ is said to be convex if its second fundamental form is everywhere positive (or negative) definite. In particular, every convex hypersurface in $\overline{M}(c)$

is orientable and, by choosing the appropriate orientation, II defines a Riemannian metric on M. As an application of (8), we can state the following result.

Proposition 2. *Let M be a convex hypersurface immersed into $\overline{M}(c)$. Then the scalar curvature S_{II} of the Riemannian metric defined by its second fundamental form* II *is given by*

$$S_{\mathrm{II}} = (n-1)\left(nH + \frac{c\sigma_{n-1}}{G}\right) + ||T||^2_{\mathrm{II}} - \frac{1}{4G^2}\,||\nabla^{\mathrm{II}} G||^2_{\mathrm{II}}, \tag{9}$$

where H and G stands for the mean and the Gauss-Kronecker curvatures of M, respectively, σ_{n-1} is the $(n-1)$th elementary symmetric function of its principal curvatures, $||T||^2_{\mathrm{II}}$ is the square II*-length of the difference tensor $T = \nabla^{\mathrm{II}} - \nabla$, and $\nabla^{\mathrm{II}} G$ denotes the* II*-gradient of G.*

In [**18**] Schneider gave a proof of formula (9) for the case of Euclidean hypersurfaces, $c = 0$. Schneider's proof follows the ideas of Eisenhart [**5**], see also [**5**, Exercise I.18, p. 33], and makes use of local computations and index gymnastics. One of the main interests of this paper is to give a modern proof of formula (9), writing and proving it in a more intrinsically way. Our proof here is a nice application of connection theory using uniquely elementary properties of Riemannian connections on manifolds.

3. Proof of Theorem 1. In this section we will derive Liebmann classical rigidity theorem for surfaces in space forms as a nice application of our formula (9) and Gauss-Bonnet theorem. First of all, observe that in the two-dimensional case, $S_{\mathrm{II}} = 2K_{\mathrm{II}}$, where K_{II} is the Gaussian curvature of II, $\sigma_{n-1} = \sigma_1 = 2H$ and $G = K - c$, where K is the Gaussian curvature of the surface. Therefore, our formula (9) simplifies to

$$K_{\mathrm{II}} = \frac{HK}{K-c} + \frac{1}{2}\,||T||^2_{\mathrm{II}} - \frac{1}{8(K-c)^2}\,||\nabla^{\mathrm{II}} K||^2_{\mathrm{II}}. \tag{10}$$

On the other hand, let us observe that every compact oriented surface M immersed into the Euclidean space $\mathbf{E}^3$, into the hyperbolic space $\mathbf{H}^3$, or into an open hemisphere $\mathbf{S}^3_+$ admits at least one elliptic point $p_0 \in M$ where

$$K(p_0) > 0 \quad \text{and} \quad K(p_0) > c.$$

When $c = 0$, such an elliptic point is precisely the point where the square of the distance to the origin attains its maximum. Certainly, let $\psi : M \to \mathbf{E}^3$ be an immersion of a compact oriented surface into the Euclidean space, and let $u = \langle \psi, \psi \rangle$ be the square of the distance to the origin. Then, the gradient and the Hessian of u are respectively given by

$$\nabla u = 2\psi^\top \quad \text{and} \quad \nabla^2 u(X,Y) = 2(\langle X,Y\rangle + \langle \psi, N\rangle\langle AX,Y\rangle),$$

where $\psi^\top = \psi - \langle \psi, N\rangle N$ denotes the tangent component of the position vector field. In particular,

$$u = |\psi^\top|^2 + \langle \psi, N\rangle^2 = \frac{1}{4}\,|\nabla u|^2 + \langle \psi, N\rangle^2.$$

Therefore, at a point $p_0 \in M$ where u attains its maximum, it follows from $\nabla u(p_0) = 0$ that $u(p_0) = \langle \psi, N\rangle^2(p_0) > 0$, so that we may assume, by changing the orientation of M if necessary, that $\langle \psi, N\rangle(p_0) = -\sqrt{u(p_0)} < 0$. Moreover, $\nabla^2 u(v,v) \leq 0$ for every tangent vector $v \in T_{p_0}M$. Let $\{e_1, e_2\}$ be the basis of principal directions at p_0, and let us denote its corresponding principal curvatures by $\kappa_1(p_0)$ and $\kappa_2(p_0)$. Then, $\nabla^2 u(e_i, e_i) \leq 0$ implies that

$$\kappa_i(p_0) \geq \frac{1}{\sqrt{u(p_0)}} > 0, \quad i = 1, 2,$$

and $K(p_0) = \kappa_1(p_0)\kappa_2(p_0) \geq 1/u(p_0) > 0$.

Analogously, in the case where $c = -1$, such an elliptic point is the point where the hyperbolic distance to a fixed arbitrary point attains its maximum. In order to see it, let us consider the Minkowskian model of the hyperbolic space. Let $\mathbf{E}^4_1$ be the four-dimensional Minkowski space endowed with canonical coordinates (x_0, x_1, x_2, x_3) and the Lorentzian metric given by

$$\langle , \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

The three-dimensional hyperbolic space $\mathbf{H}^3$ is the simply connected Riemannian manifold with sectional curvature -1, which is realized as the hyperboloid

$$\mathbf{H}^3 = \{x \in \mathbf{E}_1^4 : \langle x, x\rangle = -1,\ x_0 > 0\}$$

with Riemannian metric induced from $\mathbf{E}_1^4$. In this model, the hyperbolic distance between two points $x, y \in \mathbf{H}^3$ is given by

$$d(x, y) = \arg\cosh(-\langle x, y\rangle).$$

Consider $\psi : M \to \mathbf{H}^3$ an immersion of a compact oriented surface into the hyperbolic space, fix an arbitrary point $a \in \mathbf{H}^3$, and let $u = \langle a, \psi\rangle$. Then, the hyperbolic distance $d(a, \psi) = \arg\cosh(-u)$ attains its maximum precisely at a point $p_0 \in M$ where the function u attains its minimum, $u(p_0) < -1$. Observe now that the gradient and the Hessian of u are respectively given by

$$\nabla u = a^\top \quad \text{and} \quad \nabla^2 u(X, Y) = u\langle X, Y\rangle + \langle a, N\rangle\langle AX, Y\rangle,$$

where $a = a^\top + \langle a, N\rangle N - u\psi$, $a^\top$ denoting the tangent component of a. In particular,

$$-1 = |\nabla a|^2 + \langle a, N\rangle^2 - u^2.$$

Therefore, at a point $p_0 \in M$ where u attains its minimum, it follows from $\nabla u(p_0) = 0$ that $\langle a, N\rangle^2(p_0) = u(p_0)^2 - 1 > 0$, so that we may assume, by changing the orientation of M if necessary, that $\langle a, N\rangle(p_0) = \sqrt{u(p_0)^2 - 1} > 0$. Moreover, $\nabla^2 u(v, v) \geq 0$ for every tangent vector $v \in T_{p_0}M$. Let $\{e_1, e_2\}$ be the basis of principal directions at p_0, and let us denote its corresponding principal curvatures by $\kappa_1(p_0)$ and $\kappa_2(p_0)$. Then, $\nabla^2 u(e_i, e_i) = \langle a, N\rangle(p_0)\kappa_i(p_0) + u(p_0) \geq 0$ implies that

$$\kappa_i(p_0) \geq \frac{-u(p_0)}{\langle a, N\rangle(p_0)} = \frac{\sqrt{1 + \langle a, N\rangle^2(p_0)}}{\langle a, N\rangle(p_0)} > 1, \quad i = 1, 2.$$

In this case, the Gaussian curvature of the surface is given by $K = -1 + \kappa_1\kappa_2$, so that $K(p_0) > 0$ and $K(p_0) > -1$.

Finally, in the case where $c = 1$, such an elliptic point is the point where the spherical distance to the center of an open hemisphere $\mathbf{S}^3_+$

attains its maximum. Before showing it, let us remark that the result is not true if we do not assume that the surface is contained in an open hemisphere. Actually, the family of flat tori $\mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2) \subset \mathbf{S}^3$, with $r_1^2 + r_2^2 = 1$ are examples of flat compact surfaces without elliptic points. The proof of the result for surfaces into an open hemisphere $\mathbf{S}^3_+$ is similar to that of the hyperbolic space. Consider $\mathbf{S}^3_+$ an immersion of a compact oriented surface into an open hemisphere $\mathbf{S}^3_+$ centered at a point $a \in \mathbf{S}^3_+$. In this case, the spherical distance to the center is given by $d(a, \psi) = \arccos(u)$, where $u = \langle a, \psi \rangle$ (here, $\langle , \rangle$ stands now for the Euclidean metric in $\mathbf{E}^4$). Therefore, the function $d(a, \psi)$ attains its maximum precisely at a point $p_0 \in M$ where the function u attains its minimum, $0 < u(p_0) < 1$. A similar reasoning as in the hyperbolic space leads to

$$\kappa_i(p_0) \geq \frac{u(p_0)}{\langle a, N \rangle (p_0)} = \frac{\sqrt{1 - \langle a, N \rangle^2 (p_0)}}{\langle a, N \rangle (p_0)}, \quad i = 1, 2.$$

In this case, the Gaussian curvature of the surface is given by $K = 1 + \kappa_1 \kappa_2$, so that we get $K(p_0) > 0$ and $K(p_0) > 1$.

Now we are ready to prove Theorem 1. Since K is constant, it must be $K > 0$ and $K - c > 0$. Besides, equation (10) reduces to

$$K_{\mathrm{II}} = \frac{HK}{K - c} + \frac{1}{2} \, ||T||^2_{\mathrm{II}} \geq \frac{HK}{K - c}. \tag{11}$$

As

$$H \geq \sqrt{K - c} \tag{12}$$

by the inequality between arithmetic and geometric means applied to the principal curvatures, we obtain

$$K_{\mathrm{II}} \geq \frac{K}{\sqrt{K - c}} \tag{13}$$

on M. On the other hand, if we denote by $\mathrm{d}A$ and $\mathrm{d}A_{\mathrm{II}}$ the area elements on M with respect to $\langle , \rangle$ and II, respectively, it can be easily seen that $\mathrm{d}A_{\mathrm{II}} = \sqrt{K - c}\, \mathrm{d}A$. Then, using the Gauss-Bonnet theorem we obtain from (13)

$$\int_M K \mathrm{d}A = \int_M K_{\mathrm{II}} \mathrm{d}A_{\mathrm{II}} \geq \int_M \frac{K}{\sqrt{K - c}} \, \mathrm{d}A_{\mathrm{II}} = \int_M K \mathrm{d}A,$$

so that equality holds in (12), and the proof finishes because equality (12) holds if and only if M is a totally umbilical round sphere (note that M is assumed to be connected).

REFERENCES

1. M. Becker and W. Kühnel, *Hypersurfaces with constant inner curvature of the second fundamental form, and the non-rigidity of the sphere*, Math. Z. **223** (1996), 693–708.

2. A.L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.

3. M.P. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall, New Jersey, 1976.

4. S.S. Chern, *Some new characterizations of the Euclidean sphere*, Duke Math. J. **12** (1945), 270–290.

5. L.P. Eisenhart, *Riemannian geometry*, 6th ed., Princeton Univ. Press, Princeton, 1966.

6. J. Hadamard, *Sur certaines propriétés des trajectoires en dynamique*, J. Math. Pures Appl. **3** (1897), 331–387.

7. D. Hilbert, *Grundlagen der geometrie*, 3rd ed., Leipzig, 1909.

8. W. Klingenberg, *A course in differential geometry*, Springer Verlag, New York, 1978.

9. D. Koutroufiotis, *Two characteristic properties of the sphere*, Proc. Amer. Math. Soc. **44** (1974), 176–178.

10. K. Leichtweiss, *Convexity and differential geometry*, in *Handbook of convex geometry* (P.M. Gruber and J.M. Wills, eds.), Elsevier, Amsterdam, 1993.

11. H. Liebmann, *Eine neue eigenschaft der kugel*, Nachr. Kgl. Ges. Wiss. Göttingen, Math.-Phys. Klasse (1899), 44–55.

12. S. Montiel and A. Ros, *Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures*, in *Differential geometry*, Pitman Monographs Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 279–296.

13. S. Montiel and A. Ros, *Curvas y Superficies*, Proyecto Sur, Granada, 1997.

14. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

15. A. Ros, *Compact hypersurfaces with constant scalar curvature and a congruence theorem*, J. Differential Geom. **27** (1988), 215–220.

16. ———, *Compact hypersurfaces with constant higher order mean curvatures*, Rev. Mat. Iberomericana **3** (1987), 447–453.

17. R. Schneider, *Zur affinen Differentialgeometrie im Grossen*, I, Math. Z. **101** (1967), 375–406.

18. ———, *Closed convex hypersurfaces with second fundamental form of constant curvature*, Proc. Amer. Math. Soc. **35** (1972), 230–233.

19. U. Simon, *Characterizations of the sphere by the curvature of the second fundamental form*, Proc. Amer. Math. Soc. **55** (1976), 382–384.

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