ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

OSCILLATION RESULTS FOR LINEAR MATRIX HAMILTONIAN SYSTEMS

YUAN GONG SUN AND FANWEI MENG

ABSTRACT. In this paper we present new oscillation criteria in terms of the coefficient functions for the matrix linear Hamiltonian systems $X' = A(t)X + B(t)Y$, $Y' = C(t)X - A^*(t)Y$, which are not contained in our recent paper [**15**], and improve the main results in [**15**] to some extent.

1. Introduction. Consider the linear Hamiltonian system

$$(1.1)\qquad \begin{cases} X' = A(t)X + B(t)Y \\ Y' = C(t)X - A^*(t)Y, \end{cases} \qquad t \geq t_0$$

where $X(t)$, $Y(t)$, $A(t)$, $B(t)$, $C(t)$ are $n \times n$ real continuous matrix functions such that $B(t)$ and $C(t)$ are symmetric and $B(t)$ is positive definite, i.e., $B(t) > 0$ for $t \geq t_0$. By M^* we mean the transpose of the matrix M.

For any two solutions $X_1(t)$, $Y_1(t)$ and $X_2(t)$, $Y_2(t)$ of (1.1) the *Wronskian* $X_1^*(t)\,Y_2(t) - Y_1^*(t)X_2(t)$ is a constant matrix. In particular, for any solution $X(t)$, $Y(t)$ of (1.1), $X^*(t)Y(t) - Y^*(t)X(t)$ is a constant matrix. We now recall for the sake of convenience of reference the following definitions from the earlier literature.

Definition 1.1. A solution $X(t)$, $Y(t)$ of (1.1) is said to be nontrivial if $\det X(t) \neq 0$ for at least one $t \in [t_0, \infty)$.

Definition 1.2. A nontrivial solution $X(t)$, $Y(t)$ of (1.1) is said to be prepared if, for every $t \in [t_0, \infty)$,

$$(1.2)\qquad X^*(t)Y(t) - Y^*(t)X(t) = 0.$$

2000 AMS *Mathematics Subject Classification.* Primary 34A30, 34C10.
Key words and phrases. Oscillation, linear, matrix Hamiltonian system.
Received by the editors on June 13, 2003, and in revised form on October 25, 2003.

Definition 1.3. System (1.1) is said to be oscillatory if one nontrivial prepared solution $X(t)$, $Y(t)$ of (1.1) has the property that $\det X(t)$ vanishes on $[T, \infty)$ for sufficiently large $T \geq t_0$.

We also need for stating our results the following definition of a positive linear functional on the space of $n \times n$ matrices.

Definition 1.4. Let $\Re$ be the linear space of $n \times n$ matrices with real entries, $\wp \subset \Re$ be the subspace of $n \times n$ symmetric matrices, and g be a linear functional on $\Re$. The functional g is said to be *positive* if $g(M) > 0$ whenever $M \in \wp$ and $M > 0$.

In the case when $A(t) \equiv 0$, $B(t) > 0$, (1.1) reduces to the second order self-adjoint matrix differential system

$$(P(t)X')' + Q(t)X = 0 \tag{1.3}$$

with $P(t) = B^{-1}(t)$, $Q(t) = -C(t)$. The oscillation and non-oscillation of (1.3) have been extensively studied by many authors [**1**–**9, 11, 12, 16**–**18**]. A discrete version of (1.3) is studied in [**19**]. The oscillation of (1.1) has been studied by Sowjanya Kumari and Umamaheswaram [**10**], Sun [**20**], Meng and Sun [**15**], Meng and Mingarelli [**14**] and Meng [**13**].

We recall the following concept from [**3**]. For any subset E of the real line R, $\mu(E)$ denotes the Lebesgue measure of E. If $f\colon [t_0, \infty) \to R$ is continuous and if l, m satisfy $-\infty \leq l$, $m \leq \infty$, then $\lim \text{approxinf}_{t\to\infty} f(t) = l$ if and only if $\mu\{t \in [t_0, \infty) : f(t) \leq l_1\} < +\infty$ for all $l_1 < l$ and $\mu\{t \in [t_0, \infty) : f(t) \leq l_2\} = +\infty$ for all $l_2 > l$. Similarly, $\lim \text{approxsup}_{t\to\infty} f(t) = m$ if and only if $\mu\{t \in [t_0, \infty) : f(t) \geq m_1\} = +\infty$ for all $m_1 < m$ and $\mu\{t \in [t_0, \infty) : f(t) \geq m_2\} < +\infty$ for all $m_2 > m$. We define $\lim \text{approx}_{t\to\infty} f(t) = \lambda$ in case

$$\lim_{t\to\infty} \text{approxsup}\, f(t) = \lim_{t\to\infty} \text{approxinf}\, f(t) = \lambda.$$

In general,

$$\begin{aligned}\liminf_{t\to\infty} F(t) \leq \lim_{t\to\infty} \text{approxinf}\, F(t) &\leq \lim_{t\to\infty} \text{approxsup}_{t\to\infty} F(t) \\ &\leq \limsup_{t\to\infty} F(t).\end{aligned}$$

The motivation for the present work has come chiefly from our recent paper [**15**]. One of our results is stated as follows:

Theorem 1. *Assume there exists a smooth and real-valued function* $f(t)$ *on* $[t_0, \infty)$ *such that* $a^{-1}(t)(\Phi^{-1}B\Phi^{*-1})(t) \geq I$ *(an* $n \times n$ *identity matrix) for* $t \geq t_0$*, where* $a(t) = \exp(-2\int_{t_0}^t f(s)\,ds)$*, and*

$$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^T \left(\operatorname{tr}\int_{t_0}^t C_1(s)\,ds\right) dt > -\infty.$$

If one of the conditions

$$(A_1) \quad \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^T \left(\operatorname{tr}\int_{t_0}^t C_1(s)\,ds\right) dt = +\infty,$$

$$(A_2) \quad \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^T \left[\operatorname{tr}\int_{t_0}^t C_1(s)\,ds\right]^2 dt = +\infty,$$

$$(A_3) \quad \lim_{T\to\infty} \operatorname{approxsup}\left[\operatorname{tr}\int_{t_0}^T C_1(s)\,ds\right] = +\infty,$$

$$(A_4) \quad \lim_{T\to\infty} \operatorname{approxinf}\left[\operatorname{tr}\int_{t_0}^T C_1(s)\,ds\right] = -\infty,$$

holds, where

$$C_1(t) = -a(t)\Phi^*(t)[C(t) + f(t)(B^{-1}A + A^*B^{-1})(t) + (fB^{-1})'(t) - (f^2(t)B^{-1})(t)]\Phi(t),$$

and $\Phi(t)$ *is a fundamental matrix of the linear equation* $v' = A(t)v$*. Then* (1.1) *is oscillatory.*

In this paper, the following result has been established:

Theorem 2.1. *Assume there exist a smooth and real-valued function* $f(t)$ *on* $[t_0, \infty)$ *and a positive linear functional* g *on* $\Re$ *such that* $a(t)\,g[B^{-1}(t)] \leq m$ *(*$m > 0$ *is a constant), where* $a(t) = \exp(-2\int_{t_0}^t f(s)\,ds)$*, and*

$$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^T g\left[D(t) + \int_{t_0}^t E(s)\,ds\right] dt > -\infty.$$

If one of the conditions

$$(B_1) \quad \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} g\left[D(t)+\int_{t_0}^{t} E(s)\,ds\right]dt = +\infty,$$

$$(B_2) \quad \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} g^2\left[D(t)+\int_{t_0}^{t} E(s)\,ds\right]dt = +\infty,$$

$$(B_3) \quad \lim_{t\to\infty} \operatorname{approxsup} g\left[D(t)+\int_{t_0}^{t} E(s)\,ds\right] = +\infty,$$

$$(B_4) \quad \lim_{t\to\infty} \operatorname{approxinf} g\left[D(t)+\int_{t_0}^{t} E(s)\,ds\right] = -\infty,$$

holds, where $D(t) = -a(t)B^{-1}(t)A(t) - \int_{t_0}^{t} a(s)A^*(s)B^{-1}(s)A(s)\,ds$ *and*

$$E(t) = -a(t)[C(t)+f(t)(B^{-1}A+A^*B^{-1})(t)+(fB^{-1})'(t)-(f^2B^{-1})(t)],$$

then (1.1) *is oscillatory.*

Compared with Theorem 1, Theorem 2.1 has the following advantages. First, Theorem 2.1 removes the fundamental matrix of the linear equation $v' = A(t)v$. At present, we do not have a general method to find a fundamental matrix of the equation $v' = A(t)v$. Therefore, Theorem 2.1 can be conveniently applied to (1.1). Second, in some cases the assumption $a(t)\,g[B^{-1}(t)] \le m$ is weaker than the assumption $a^{-1}(t)(\Phi^{-1}B\Phi^{*-1})(t) \ge I$ for $t \ge t_0$. For example, for the case when $a(t) \equiv 1$ and $\Phi(t) \equiv I$, let $g[M] = m_{11}$, where $M = (m_{ij})$ is a matrix, and $B(t) = \operatorname{diag}(t, 1/t)$ for $t \ge 1$, then we have that $g[B^{-1}(t)] = 1/t \le 1$. However, $B(t) \ge I$ does not hold for $t \ge 1$. Finally, with an appropriate choice of the positive linear functional g such as $g[M] = m_{ii}$ for $i = 1, 2, \dots, n$, $g[M] = \operatorname{tr} M$, and $g[M] = c^*Mc$ where c is an arbitrary but fixed vector in R^n, we may give many possibilities for oscillation criteria of (1.1).

2. Main results. Let $f(t)$ be a smooth and real-valued function on $[t_0, \infty)$, and let

$$a(t) = \exp\left(-2\int_{t_0}^{t} f(s)\,ds\right).$$

If a prepared solution $X(t)$, $Y(t)$ of (1.1) is nonoscillatory, then $X(t)$ is nonsingular for all sufficiently large t, without loss of generality say $t \geq t_0$. Let

$$W(t) = a(t)[Y(t)X^{-1}(t) + f(t)B^{-1}(t)], \quad t \geq t_0. \tag{2.1}$$

It is easy to see that $W(t)$ is symmetric on $[t_0, \infty)$. From (1.1) we have

$$W'(t) = -E(t) - A^*(t)W(t) - W(t)A(t) - a^{-1}(t)W(t)B(t)W(t), \tag{2.2}$$

where

$$E(t) = -a(t)[C(t) + f(t)(B^{-1}A + A^*B^{-1})(t) + (fB^{-1})'(t) - (f^2B^{-1})(t)], \tag{2.3}$$

where fB^{-1} is differentiable on $[t_0, \infty)$. Integrating both sides of (2.2) from t_0 to t we obtain

$$\begin{aligned} W(t) = W(t_0) &- \int_{t_0}^{t} E(s)\,ds \\ &- \int_{t_0}^{t} [A^*(s)W(s) + W(s)A(s) + a^{-1}(s)W(s)B(s)W(s)]\,ds \end{aligned}$$

Now the substitution $P(t) = W(t) + a(t)B^{-1}(t)A(t)$ in the above equation gives us

$$P(t) = W(t_0) - D(t) - \int_{t_0}^{t} E(s)\,ds - \int_{t_0}^{t} a^{-1}(s)P^*(s)B(s)P(s)\,ds, \tag{2.4}$$

where

$$D(t) = -a(t)B^{-1}(t)A(t) - \int_{t_0}^{t} a(s)A^*(s)B^{-1}(s)A(s)\,ds. \tag{2.5}$$

In the sequel, we use the following lemmas.

Lemma 2.1 [**16**]. *If g is a positive linear functional on $\Re$, then for all $P,\ Q \in \Re$, $|g[P^*Q]|^2 \leq g[P^*P]\,g[Q^*Q]$.*

Lemma 2.2. *If g is a positive linear functional on $\Re$ then for all $P \in \Re$ and $B \in \wp$ with $B > 0$, $g[B^{-1}]\, g[P^*BP] \geq g^2[P]$.*

Proof. By Lemma 2.1, we have

$$\begin{aligned} g\,[B^{-1}]\, g\,[P^*BP] &= g\,[B^{-1/2*}B^{-1/2}]\, g\,[(B^{1/2}P)^*(B^{1/2}P)] \\ &\geq g^2[B^{-1/2}B^{1/2}P] = g^2[P]. \end{aligned}$$

Hence, Lemma 2.2 is true. □

Lemma 2.3. *Assume that* (1.1) *is nonoscillatory on* $[a, \infty)$. *If there exists a positive linear functional g on $\Re$ such that $a(t)\, g[B^{-1}(t)] \leq m$ ($m > 0$ is a constant), then there is a $t_0 \geq a$ such that*

$$\lim_{T\to\infty} \int_t^T a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds < +\infty, \quad \text{for} \quad t \geq t_0 \tag{2.6}$$

if and only if

$$\liminf_{T\to\infty} \frac{1}{T} \int_{t_0}^T g\left[D(t) + \int_{t_0}^t E(s)\, ds\right] dt > -\infty, \tag{2.7}$$

where $a(t)$, $D(t)$ and $E(t)$ are the same as above.

Proof. Applying the positive linear functional g to both sides of (2.4), we have

$$\begin{aligned} g\,[P(t)] = g\,[W(t_0)] - g\left[D(t) + \int_{t_0}^t E(s)\, ds\right] \\ - \int_{t_0}^t a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds. \end{aligned} \tag{2.8}$$

From (2.6) and (2.8) we obtain

$$g\,[P(t)] - M(t) = -g\left[D(t) + \int_{t_0}^t E(s)\, ds\right] + L, \tag{2.9}$$

where

$$L = g\,[W(t_0)] - \int_{t_0}^\infty a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds$$

and

$$M(t) = \int_t^\infty a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds.$$

By Lemma 2.2 and the assumption of Lemma 2.3, we have

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T g^2[P(s)]\, ds \\ &\le \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T a(s)\, g\,[B^{-1}(s)]a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds \\ &\le m \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds = 0.\end{aligned}$$

That is,

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T g^2[P(s)]\, ds = 0. \tag{2.10}$$

On the other hand, by (2.6), we observe that, for every $\varepsilon > 0$, it is possible to find a $t_1 > t_0$ such that, for $t \ge t_1$, $M(t) < \varepsilon$. Hence,

$$\frac{1}{T}\int_{t_0}^T M^2(t)\, dt = \frac{1}{T}\int_{t_0}^{t_1} M^2(t)\, dt + \frac{1}{T}\int_{t_1}^T M^2(t)\, dt \le \varepsilon^2.$$

Since ϵ is arbitrary, then

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T M^2(t)\, dt = 0. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T \{g\,[P(t)] - M(t)\}^2\ dt \\ &\le 2 \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T \{g^2[P(t)] + M^2(t)\}\, dt = 0.\end{aligned}$$

Therefore, from (2.9), it follows that

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T \left\{L - g\left[D(t) + \int_{t_0}^t E(s)\, ds\right]\right\}^2 dt = 0. \tag{2.12}$$

By the Cauchy-Schwartz inequality and (2.12), we can easily obtain that

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \left\{L - g\left[D(t) + \int_{t_0}^{t} E(s)\,ds\right]\right\} dt = 0,$$

i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} g\left[D(t) + \int_{t_0}^{t} E(s)\,ds\right] dt = L > -\infty,$$

so that (2.7) holds.

Conversely, suppose that (2.7) holds. From (2.7) and (2.8), we have

$$\text{(2.13)}\quad \lim_{T\to\infty} \sup\left\{\frac{1}{T}\int_{t_0}^{T} g\,[P(s)]\,ds + \frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\,dt\right\} < +\infty.$$

Since $g[P^*(t)B(t)P(t)] \geq 0$ for $t \geq t_0$, it follows that $\lim_{t\to\infty}\int_{t_0}^{t} a^{-1}(s) \times g[P^*(s)B(s)\ P(s)]\,ds$ exists, finite or infinite. Suppose that

$$\lim_{t\to\infty} \int_{t_0}^{t} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds = +\infty.$$

Hence,

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\,dt = +\infty.$$

Then (2.13) yields

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} g\,[P(s)]\,ds = -\infty.$$

So for large T we have, again using (2.13),

$$\text{(2.14)}\quad \frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\,dt \leq -\frac{2}{T}\int_{t_0}^{T} g\,[P(s)]\,ds.$$

Now by the Cauchy-Schwartz inequality and Lemma 2.2, we have

$$\begin{aligned}\left|\frac{1}{T}\int_{t_0}^{T} g\,[P(s)]\,ds\right| &\le \left\{\frac{1}{T}\int_{t_0}^{T} g^2[P(s)]\,ds\right\}^{1/2} \times \left[\frac{T-t_0}{T}\right]^{1/2} \\ &\le \left\{\frac{m}{T}\int_{t_0}^{T} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\right\}^{1/2},\end{aligned}$$

so that (2.14) gives

$$\begin{aligned}\left\{\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\,dt\right\}^2 \\ \le \frac{4m}{T}\int_{t_0}^{T} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\end{aligned} \tag{2.15}$$

for large T, say, for $T \ge T_1$. Setting

$$H(T) = \int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\,dt > 0,$$

we obtain

$$H'(T) = \int_{t_0}^{T} a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds.$$

Thus, (2.15) yields $H^2(T) \le 4mTH'(T)$ for $T \ge T_1$. Integrating this inequality from T_1 to T and noting that $H(T) > 0$ for $T \ge T_1$, we get

$$\frac{1}{4m}\,[\log T - \log T_1] \le \frac{1}{H(T)}.$$

A contradiction is obtained as $T \to \infty$. Thus $\lim_{t\to\infty}\int_{t_0}^{t} a^{-1}(s) \times g[P^*(s)B(s)P(s)]\,ds$ exists as a finite limit. Consequently, we have (2.6). Thus Lemma 2.3 is proved. □

Now, let us give the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose that (1.1) is not oscillatory. Then there exists a prepared solution $X(t), Y(t)$ of (1.1) such that $X(t)$ is nonsingular. Without loss of generality, we assume that $\det X(t) \neq 0$ for $t \geq t_0$. Denote $W(t)$ by(2.1); then we have that (2.8) holds. By (2.7) and Lemma 2.3, it follows that (2.6) holds.

Suppose that (B_1) holds. From (2.8) we obtain

$$g\,[W(t_0)] - g\,[P(t)] \geq g\biggl[D(t) + \int_{t_0}^{t} E(s)\,ds\biggr], \quad t \geq t_0.$$

Thus, for $t \geq t_0$

$$\frac{1}{T}\int_{t_0}^{T} -g\,[P(t)]\,dt + \frac{T-t_0}{T}\,g\,[W(t_0)] \geq \frac{1}{T}\int_{t_0}^{T} g\biggl[D(t) + \int_{t_0}^{t} E(s)\,ds\biggr]\,dt.$$

By the assumption (B_1) and the above inequality, we have that there exists a sequence $\{T_n\}$ such that $T_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{T_n}\int_{t_0}^{T_n} -g\,[P(t)]\,dt = +\infty. \tag{2.16}$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\left|\frac{1}{T_n}\int_{t_0}^{T_n} g\,[P(t)]\,dt\right| &\leq \left\{\frac{1}{T_n}\int_{t_0}^{T_n} g^2[P(t)]\,dt\right\}^{1/2} \times \left[\frac{T_n - t_0}{T_n}\right]^{1/2} \\ &\leq \left\{\frac{m}{T_n}\int_{t_0}^{T_n} a^{-1}(t)\,g\,[P^*(t)B(t)P(t)]\,dt\right\}^{1/2}.\end{aligned}$$

Using (2.16), we obtain

$$\lim_{n\to\infty} \frac{1}{T_n}\int_{t_0}^{T_n} a^{-1}(t)\,g\,[P^*(t)B(t)P(t)]\,dt = +\infty,$$

which in turn implies that

$$\lim_{n\to\infty} \int_{t_0}^{T_n} a^{-1}(t)\,g\,[P^*(t)B(t)P(t)]\,dt = +\infty.$$

On the other hand, by (2.6) we have

$$\lim_{n\to\infty} \int_{t_0}^{T_n} a^{-1}(t)\,g\,[P^*(t)B(t)P(t)]\,dt < +\infty.$$

This contradiction completes the proof of the part under the assumption (B_1) of the theorem.

Let (B_2) be true. From (2.9) we have

$$g^2\left[D(t)+\int_{t_0}^{t}E(s)\,ds\right]=\{-g\,[P(t)]+M(t)+L\}^2$$
$$\leq 4\left\{g^2[P(t)]+M^2(t)\right\}+2L^2.$$

Thus,

$$\frac{1}{T}\int_{t_0}^{T}g^2\left[D(t)+\int_{t_0}^{t}E(s)\,ds\right]dt \leq \frac{4}{T}\int_{t_0}^{T}g^2[P(t)]\,dt+\frac{4}{T}\int_{t_0}^{T}M^2(t)\,dt+2L^2\,\frac{T-t_0}{T}. \tag{2.17}$$

Noting that

$$\frac{1}{T}\int_{t_0}^{T}g^2[P(t)]\,dt\leq\frac{m}{T}\int_{t_0}^{T}a^{-1}(t)\,g\,[P^*(t)B(t)P(t)]\,dt.$$

From (2.6) we have

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}g^2[P(t)]\,dt=0$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}M^2(t)\,dt =\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(\int_{t}^{\infty}a^{-1}(s)\,g\,[P^*(s)B(s)P(s)]\,ds\right)^2dt=0.$$

Therefore, we obtain from (2.17)

$$\lim_{T\to\infty}\sup\frac{1}{T}\int_{t_0}^{T}g^2\left[D(t)+\int_{t_0}^{t}E(s)\,ds\right]dt<+\infty$$

which contradicts the assumption (B_2). This contradiction completes the proof of the part under the assumption (B_2) of the theorem.

Let us assume that (B_3) holds; then, for any real l, we have

$$\mu\biggl\{t \in [t_0, \infty) : g\biggl[D(t) + \int_{t_0}^{t} E(s)\, ds\biggr] \geq l\biggr\} = +\infty. \tag{2.18}$$

We may write (2.8) in the form

$$\begin{aligned} -g\,[P(t)] = -g\,[W(t_0)] &+ g\biggl[D(t) + \int_{t_0}^{t} E(s)\, ds\biggr] \\ &+ \int_{t_0}^{t} a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds. \end{aligned}$$

Since (2.6) holds, we have for any real k,

$$\begin{aligned} &\{t \in [t_0, \infty) : -g\,[P(t)] \geq k\} \\ &\quad = \biggl\{t \in [t_0, \infty) : g\biggl[D(t) + \int_{t_0}^{t} E(s)\, ds\biggr] \\ &\qquad \geq g\,[W(t_0)] + k - \int_{t_0}^{\infty} a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds\biggr\}. \end{aligned}$$

Consequently, from (2.18) it follows that, for any real k,

$$\mu\{t \in [t_0, \infty) : -g\,[P(t)] \geq k\} = +\infty.$$

In particular, $\mu(E_k) = +\infty$, where $E_k = \{t \in [t_0, \infty) : -g\,[P(t)] \geq k > 0\}$. Thus,

$$\int_{E_k} g^2[P(t)]\, dt \geq k^2 \mu(E_k) = +\infty.$$

On the other hand,

$$\begin{aligned} \int_{E_k} g^2[P(t)]\, dt &\leq m \int_{E_k} a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds \\ &\leq m \int_{t_0}^{\infty} a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds < +\infty \end{aligned} \tag{2.19}$$

due to (2.6). It is a contradiction. Hence, the proof of the part under the assumption (B_3) of the theorem is complete.

Suppose that (B_4) holds. Since (2.6) holds, then for every $\epsilon > 0$, there exists a $T_0 > t_0$ such that $t \geq T_0$ implies that

$$M(t) = \int_t^\infty a^{-1}(s)\, g\left[P^*(s)B(s)P(s)\right] ds < \epsilon.$$

The assumption (B_4) yields for every real l

$$\mu\left\{t \in [t_0, \infty) : g\left[D(t) + \int_{t_0}^t E(s)\, ds\right] \leq l\right\} = +\infty. \tag{2.20}$$

For any real k, we have in view of (2.9)

$$\begin{aligned}
&\{t \in [T_0, \infty) : -g\left[P(t)\right] \leq k\}\\
&\quad = \left\{t \in [T_0, \infty) : g\left[D(t) + \int_{T_0}^t E(s)\, ds\right] \leq k + L + M(t)\right\}\\
&\quad = \left\{t \in [T_0, \infty) : g\left[D(t) + \int_{t_0}^t E(s)\, ds\right] \leq k + L + \epsilon + \int_{t_0}^{T_0} g\left[E(s)\right] ds\right\}.
\end{aligned}$$

Thus, for every real k, (2.20) yields that

$$\mu\{t \in [t_0, \infty) : -g\left[P(t)\right] \leq k\} = +\infty.$$

Set $E_k = \{t \in [t_0, \infty) : -g\left[P(t)\right] \leq k < 0\}$, then $\mu\left(E_k\right) = +\infty$ and

$$\int_{E_k} g^2[P(t)]\, dt \geq k^2 \mu\left(E_k\right) = +\infty.$$

However, from (2.19) we have $\int_{E_k} g^2[P(t)]\, dt < +\infty$. This contradiction completes the proof of the part under the assumption (B_4) of the theorem.

This completes the proof of Theorem 2.1. □

The following theorem complements Theorem 2.1.

Theorem 2.2. *If there exists a positive linear functional g on $\Re$ such that $a(t)\, g[B^{-1}(t)] \leq M$ ($M > 0$ is a constant),*

$$\liminf_{T \to \infty} \frac{1}{T} \int_{t_0}^T g\left[D(t) + \int_{t_0}^t E(s)\, ds\right] dt = -\infty,$$

and

$$\lim_{t\to\infty} \text{approxsup}\, g\left[D(t) + \int_{t_0}^{t} E(s)\, ds\right] = m > -\infty, \tag{2.21}$$

then (1.1) *is oscillatory, where* $a(t)$, $D(t)$ *and* $E(t)$ *are the same as above.*

Proof. Suppose that (1.1) is not oscillatory. Then there exists a prepared solution $X(t), Y(t)$ of (1.1) such that $X(t)$ is nonsingular. Without loss of generality, we assume that $\det X(t) \neq 0$ for $t \geq t_0$. Denote $W(t)$ by (2.1), then we have (2.8) holds. From (2.21) and Lemma 2.3, it follows that

$$\lim_{T\to\infty} \int_{t_0}^{T} a^{-1}(s)\, g\left[P^*(s)B(s)P(s)\right] ds = +\infty. \tag{2.22}$$

For any $\epsilon > 0$, the given condition yields

$$\mu\left\{t \in [t_0,\infty) : g\left[D(t) + \int_{t_0}^{t} E(s)\, ds\right] \geq m - \epsilon\right\} = +\infty.$$

Thus, using (2.8) we have

$$\mu\left\{t \in [t_0,\infty) : \int_{t_0}^{t} a^{-1}(s)\, g\left[P^*(s)B(s)P(s)\right] ds \leq -g\left[P(t)\right] + g\left[W(t_0)\right] - m + \epsilon\right\} = +\infty.$$

Consequently, for large t,

$$\int_{t_0}^{t} a^{-1}(s)\, g\left[P^*(s)B(s)P(s)\right] ds \leq -g\left[P(t)\right] + g\left[W(t_0)\right] - m + \epsilon.$$

Hence, in view of (2.22), we have $\lim_{t\to\infty} -g[P(t)] = +\infty$. We may choose $T_0 > t_0$ such that $-g[P(t)] > g[W(t_0)] - m + \epsilon$ for $t \geq T_0$. Then we have

$$\mu\left\{t \in [T_0,\infty) : \int_{t_0}^{t} a^{-1}(s)\, g\left[P^*(s)B(s)P(s)\right] ds \leq -2g\left[P(t)\right]\right\} = +\infty.$$

Let

$$E = \left\{ t \in [T_0, \infty) : \int_{t_0}^{t} a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds \le -2g\,[P(t)] \right\}.$$

Hence $\mu(E) = +\infty$. Setting, for $t \ge T_0$,

$$H(t) = \int_{t_0}^{t} a^{-1}(s)\, g\,[P^*(s)B(s)P(s)]\, ds > 0,$$

we get $H'(t) = a^{-1}(t)\, g[P^*(t)B(t)P(t)] \ge 0$. For $t \in E$,

$$H^2(t) \le 4\, g^2[P(t)] \le 4\, M a^{-1}(t)\, g\,[P^*(t)B(t)P(t)] = 4\, M H'(t).$$

Integrating over E and noting that $H(t) > 0$ for $t \ge T_0$, we obtain

$$\frac{1}{4M}\, \mu(E) \le \int_E \frac{H'(t)}{H(t)}\, dt \le \lim_{T \to \infty} \int_{T_0}^{T} \frac{H'(t)}{H(t)}\, dt < \frac{1}{H(T_0)} < +\infty,$$

which is a contradiction, since $\mu(E) = +\infty$. This completes the proof of Theorem 2.2. □

In order to illustrate our theorems, we consider the following example.

Example. Consider the Hamiltonian system

$$\text{(2.23)} \qquad \begin{cases} X' = A(t)X + B(t)Y \\ Y' = C(t)X - A^*(t)Y, \end{cases} \qquad t \ge t_0,$$

where

$$A(t) = \begin{bmatrix} a_1(t) & a_3(t) \\ a_2(t) & a_4(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} b_1(t) & 0 \\ 0 & b_2(t) \end{bmatrix},$$
$$C(t) = \begin{bmatrix} -c_1(t) & c_2(t) \\ c_2(t) & c_3(t) \end{bmatrix},$$

$a_i(t)$, $b_j(t) > 0$, $c_k(t)$ are continuous functions on $[t_0, \infty)$ for $i = 1, 2, 3, 4$, $j = 1, 2$ and $k = 1, 2, 3$, and there exists a constant $m > 0$

such that $b_1^{-1}(t) \le m$ for $t \ge t_0$. If we let $\rho(t) \equiv 0$, $g[M] = m_{11}$, where $M = (m_{ij})$ is a 2×2 matrix, then we have

$$\begin{aligned} g\,[D(t)] &= -g\,[B^{-1}(t)A(t)] - \int_{t_0}^{t} g\,[A^*(s)B^{-1}(s)A(s)]\,ds \\ &= -a_1(t)\,b_1^{-1}(t) - \int_{t_0}^{t} [a_1^2(s)\,b_1^{-1}(s) + a_2^2(s)\,b_2^{-1}(s)]\,ds, \end{aligned}$$

and

$$g\left[\int_{t_0}^{t} E(s)\,ds\right] = \int_{t_0}^{t} g\,[E(s)]\,ds = -\int_{t_0}^{t} g\,[C(s)]\,ds = \int_{t_0}^{t} c_1(s)\,ds.$$

Set

$$U(t) = \int_{t_0}^{t} [c_1(s) - a_1^2(s)\,b_1^{-1}(s) - a_2^2(s)\,b_2^{-1}(s)]\,ds - a_1(t)\,b_1^{-1}(t).$$

Now, let us consider the following two cases.

Case 1. If

$$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} U(t)\,dt > -\infty,$$

and one of the following conditions holds

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} U(t)\,dt = +\infty, \\ &\limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} U^2(t)\,dt = +\infty, \\ &\lim_{t\to\infty} \operatorname{approxsup} U(t) = +\infty, \\ &\lim_{t\to\infty} \operatorname{approxinf} U(t) = -\infty, \end{aligned}$$

then (2.23) is oscillatory by Theorem 2.1.

Case 2. If

$$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} U(t)\,dt = -\infty,$$

and

$$\lim_{t\to\infty} \operatorname{approxsup} U(t) > -\infty,$$

then (2.23) is oscillatory by Theorem 2.2. However, it is difficult to apply Theorems 1 and 2 in our recent paper [**15**] to (2.23), since the continuous functions $a_3(t)$, $a_4(t)$, $b_2(t) > 0$, $c_2(t)$, $c_3(t)$ and $c_4(t)$ are arbitrary and the fundamental matrix of the linear equation $v' = A(t)v$ is not very easy to obtain.

Remark. With an appropriate choice of the positive linear functional g such as $g[M] = m_{ii}$ for $i = 1, 2, \dots, n$, $g[M] = \operatorname{tr} M$, and $g[M] = c^* M c$ where c is an arbitrary but fixed vector in R^n, we may derive many possibilities for oscillation criteria of (1.1) from Theorems 2.1 and 2.2. Because of the limited space, we omit them here.

Acknowledgments. The authors thank the referee for correcting errors in the original manuscript and for his helpful suggestions.

REFERENCES

1. G.J. Butler and L.H. Erbe, *Oscillation results for second order differential systems*, SIAM J. Math. Anal. **17** (1986), 19–29.

2. ———, *Oscillation results for self-adjoint differential systems*, J. Math. Anal. Appl. **115** (1986), 470–481.

3. G.J. Butler, L.H. Erbe and A.B. Mingarelli, *Riccati techniques and variational principles in oscillation theory for linear systems*, Trans. Amer. Math. Soc. **303** (1987), 263–282.

4. R. Byers, B.J. Harris and M.K. Kwong, *Weighted means and oscillation conditions for second order matrix differential equations*, J. Differential Equations **61** (1986), 164–177.

5. W.A. Coppel, *Disconjugacy*, Lecture Notes in Math., vol. 220, Springer-Verlag Berlin, 1971.

6. L.H. Erbe, Q. Kong and S. Ruan, *Kamenev type theorems for second order matrix differential systems*, Proc. Amer. Math. Soc. **117** (1993), 957–962.

7. P. Hartman, *Self-adjoint, non-oscillatory systems of ordinary second order linear differential equations*, Duke Math. J. **24** (1957), 25–36.

8. D.B. Hinton and R.T. Lewis, *Oscillation theory for generalized second order differential equations*, Rocky Mountain J. Math. **10** (1980), 751–766.

9. W. Kratz, *Quadratic functionals in variational analysis and control theory*, Akademie Verlag, Berlin, 1995.

10. I. Sowjanya Kumari and S. Umamaheswaram, *Oscillation criteria for linear matrix Hamiltonian systems*, J. Differential Equations **165** (2000), 174–198.

11. M.K. Kwong and H.G. Kaper, *Oscillation of two-dimensional linear second order differential systems*, J. Differential Equations **56** (1985), 195–205.

12. M.K. Kwong, H.G. Kaper, K. Akiyama and A.B. Mingarelli, *Oscillation of linear second-order differential systems*, Proc. Amer. Math. Soc. **91** (1984), 85–91.

13. F. Meng, *Oscillation results for linear Hamiltonian systems*, Appl. Math. Comput. **131** (2002), 357–372.

14. F. Meng and A.B. Mingarelli, *Oscillation of linear Hamiltonian systems*, Proc. Amer. Math. Soc. **131** (2003), 897–904.

15. F. Meng and Y.G. Sun, *Oscillation of linear Hamiltonian systems*, Comput. Math. Appl. **44** (2002), 1467–1477.

16. F. Meng, J. Wang and Z. Zhen, *A note on Kamenev type theorems for second order matrix differential systems*, Proc. Amer. Math. Soc. **126** (1998), 391–395.

17. A.B. Mingarelli, *On a conjecture for oscillation of second order ordinary differential systems*, Proc. Amer. Math. Soc. **82** (1981), 593–598.

18. N. Parhi and P. Praharaj, *Oscillation of linear second order matrix differential equations*, J. Math. Anal. Appl. **221** (1998), 287–305.

19. A. Peterson and J. Ridenhour, *Oscillation of second order linear matrix difference equations*, J. Differential Equations **89** (1991), 69–88.

20. Y.G. Sun, *New oscillation criteria for linear matrix Hamiltonian systems*, J. Math. Anal. Appl. **279** (2003), 651–658.

Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China
E-mail address: yugsun@263.net

Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China