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THE DIVERGENCE-FREE JACOBIAN CONJECTURE IN DIMENSION TWO

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ABSTRACT. A special case, called the divergence-free case, of the Jacobian conjecture in dimension two is proved.

1. Introduction. This note outlines an argument for a special case of the Jacobian conjecture in dimension two: Suppose $F : C^2 \to C^2$ is a polynomial so that

$$F(0) = 0, \quad F'(0) = I, \quad \det(F'(z)) = 1, \quad z \in C^2. \tag{1}$$

where I is the identity transformation on C^2.

Write

$$F(x,y) = \begin{pmatrix} r(x,y) + x \\ s(x,y) + y \end{pmatrix}, \quad (x,y) \in C^2$$

where r, s have no nonzero constant or linear terms and observe that

$$\det F' = \{r, s\} + \nabla \cdot \begin{pmatrix} r \\ s \end{pmatrix} + 1$$

so that (1) gives

$$\{r, s\} + \nabla \cdot \begin{pmatrix} r \\ s \end{pmatrix} = 0 \tag{2}$$

with

$$\{r, s\} = r_1 s_2 - r_2 s_1, \quad \nabla \cdot \begin{pmatrix} r \\ s \end{pmatrix} = r_1 + s_2,$$

the Poisson bracket and divergence respectively of the vector field (r, s), subscripts in these instances indicating partial derivatives in first and second arguments.

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The main purpose of this note is to prove the following

Theorem 1. *Suppose in addition to the above that*

$$\{r, s\} = 0, \quad \nabla \cdot \begin{pmatrix} r \\ s \end{pmatrix} = 0. \tag{3}$$

Then F is bijective, i.e., the Jacobian conjecture holds in this case.

We hasten to point out that (3) does not follow (no matter what this writer may have suspected for some time) automatically. The following example communicated by Hyman Bass showed the author's expectation to be false.

Example 1. Suppose that each of a and b is a positive integer greater than one and

$$F(x, y) = \begin{pmatrix} (x^a + y)^b + x \\ x^a + y \end{pmatrix}, \quad (x, y) \in C^2.$$

It is easy to check that (1) holds but that (3) does not. Nevertheless the following indicates instances in which (3) holds.

Corollary 1. *Suppose that all terms of F of degree higher than one are even. Then* (3) *holds.*

Corollary 2. *Suppose that $m > n$ are positive integers and $(n-1)^2 > m-1$. If F is such that all of its terms of degree higher than one have their degree in $[n, m]$, then* (3) *holds.*

In both cases there is no term in $\{s, r\}$ which has a degree in common with a term of $\nabla \cdot \binom{r}{s}$. Hence these two quantities that sum to zero must each be zero. Therefore (2) implies (3) in these two cases.

References on the Jacobian conjecture are [**1, 4, 5**]. An argument for Theorem 1 is based on the following.

Theorem 2. *In order for* (3) *to hold, it is necessary and sufficient that there be a linear transformation* $L : C^2 \to C$ *and a polynomial* $Q : C \to C^2$ *so that*

$$F(z) = z + Q(Lz) \quad \text{and} \quad L(Q(Lz)) = 0, \quad z \in C^2. \tag{4}$$

Before a proof of Theorem 2, three lemmas are given which indicate that if (4) holds then Theorem 1 follows.

Lemma 1. *If* (4) *holds,* $M \subset C^2$ *and* $F(M)$ *is bounded, then* M *is bounded, i.e.,* F *is proper.*

Proof [Lemma 1]. Suppose M is a subset of C^2 and $F(M)$ is bounded. Since $L(F(M))$ is then bounded, (4) yields that $L(M)$ is also bounded. But then $Q(L(M))$ is bounded and so then is M. □

Lemma 2. *Under* (4)*, suppose* $v, w \in C^2$*. There is a unique function* $u : [0, \infty) \to C^2$ *such that*

$$u(0) = w, \quad u'(t) = -(F'(u(t)))^{-1}(F(u(t)) - v), \quad t \geq 0. \tag{5}$$

Moreover,

$$q = \lim_{t \to \infty} u(t) \quad \text{exists and} \quad F(q) = v. \tag{6}$$

Proof [Lemma 2]. Suppose that each of $w, v \in C^2$ and the equation in (5) holds with solution u for some maximal interval $[0, c)$, $c > 0$. Then

$$(F(u) - v)' = -(F(u) - v) \quad \text{on} \quad [0, c)$$

and so

$$F(u(t)) - v = e^{-t}(F(w) - v), \quad t \in [0, c). \tag{7}$$

Assume that c is finite. Then $F(u([0, c)))$ is bounded and therefore, using Lemma 1, $u([0, c))$ is also bounded. Consequently, due to

(5), $u'([0,c)))$ is bounded too. But this last conclusion gives that $p = \lim_{t\to c-} u(t)$ exists. Therefore u may be extended by continuity to $[0,c]$ and consequently the solution u can be further extended beyond $[0,c]$, contradicting the maximality of $[0,c)$. Thus $c = \infty$. It follows that there is a solution u to (5) exactly as stated there. Hence, using (5), $|u'|$ decreases to zero exponentially and so

$$q = \lim_{t\to\infty} u(t)$$

exists. That $F(q) = v$ follows from (7). $\square$

For $v \in C^2$, define $G_v : C^2 \to C^2$ so that $G_v(w) = \lim_{t\to\infty} u(t)$ where u satisfies (5). Given $v \in C^2$, one says that w is in the domain of attraction of q relative to (5), i.e., $(G_v)^{-1}(q)$ is this domain of attraction.

Lemma 3. *Under* (4), *F has an inverse defined on all of C^2.*

Proof [Lemma 3]. Suppose that $v \in C^2$. From Lemmas 1 and 2, it follows that every member of C^2 is in the domain of attraction of some z so that of $F(z) = v$. Denote by S_v the preimage of v under F. The collection S_v has no limit point since such a limit point would be a place at which F' is singular. Hence, from general principles of ordinary differential equations, a domain of attraction of an element of S_v is an open set. Now C^2 is not the union of mutually separated open sets. Hence, the domain of attraction of an element of S_v is all of C^2 and in fact S_v contains just one point. Thus, there cannot be two elements $q, \bar{q}$ so that $F(q) = F(\bar{q})$ since two such elements would be distinct members of S_v. $\square$

Lemmas 1, 2 and 3 imply that under (3) F is a bijection and hence Theorem 1 follows from Theorem 2 since these lemmas follow from Theorem 2. See also [**4**] in connection with this lemma. It remains to prove Theorem 2.

Proof [Theorem 2]. Under the hypotheses of Theorem 2, $\{r, s\} = 0$. Now if r, s are both zero, the conclusion surely holds. Accordingly, suppose that one of r and s is not zero, say r. Note that r is not constant. Denote by (α, β) a point of C^2 at which at least one of the partial derivatives r_1, r_2 is not zero. A classical result on functional dependence, cf. [**2**] for matters of differentiability and cf. [**3**, p. 426] for functional dependence, gives that there is an $\varepsilon > 0$ and an analytic function h with domain the open ball B_ε (radius ε, center (α, β)) so that not both of r_1, r_2 are zero at any point of B_ε and

$$s(x, y) = h(r(x, y)), \quad (x, y) \in B_\varepsilon. \tag{8}$$

Note that then

$$s_2(x, y) = h'(r(x, y))\, r_2(x, y), \quad (x, y) \in B_\varepsilon. \tag{9}$$

Equation (9) together with (3) yields that

$$r_1(x, y) + h'(r(x, y))\, r_2(x, y) = 0, \quad (x, y) \in B_\varepsilon. \tag{10}$$

For r satisfying the above and $(\gamma, \delta) \in B_\varepsilon$ denote by $\binom{u}{v}$ functions with maximal domain in C so that

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} r_2 \\ -r_1 \end{pmatrix}(u, v), \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Note that

$$r(u, v)' = 0$$

and consequently, $r(u, v)$ and $h(r(u, v))$ are constant. Denote the common value of $h'(r(u, v))$ by c.

Using (10),

$$\begin{pmatrix} u \\ v \end{pmatrix}' = r_2(u, v) \begin{pmatrix} 1 \\ c \end{pmatrix}.$$

This implies that directions of members of the range of $\binom{u}{v}'$ are constant and hence the range of $\binom{u}{v}$ lies on the (complex) line

$$W_{\gamma, \delta} = \left\{ s \begin{pmatrix} 1 \\ c \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix} : s \in C \right\},$$

the line of slope c through $\binom{\gamma}{\delta}$. Hence r is constant on the intersection of this line and B_ε, and so by analyticity, r is constant on all of $W_{\gamma,\delta}$.

It will be seen that each member of the set of lines

$$\{W_{\gamma,\delta} : (\gamma,\delta) \in B_\varepsilon\} \tag{11}$$

has slope c. If two of these lines had different slopes, they would cross; then every member of (11) would cross at least one of these two and hence r would be constant on all of B_ε, and hence all of C^2, a contradiction. Thus the members of (11) are parallel, all with slope c. Put another way, r satisfies on B_ε the partial differential equation

$$r_1 + c\,r_2 = 0. \tag{12}$$

Hence there is a function f from a subset of C to C so that

$$r(x,y) = f(cx - y), \quad (x,y) \in B_\varepsilon. \tag{13}$$

The function f clearly is a polynomial. Hence relation (13) extends by analyticity to all of C^2. Moreover (8), with h now known to be linear and homogeneous (actually the action of h is just multiplication by c), must extend to all of C^2 and consequently the relationship (8) extends to all of C^2. The two extensions noted above give that

$$\begin{pmatrix} r(x,y) \\ s(x,y) \end{pmatrix} = \begin{pmatrix} f(cx-y) \\ cf(cx-y) \end{pmatrix}, \quad (x,y) \in C^2.$$

Defining $L : C^2 \to C$ and $Q : C \to C^2$ by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = cx - y, \quad (x,y) \in C^2 \quad \text{and} \quad Q(w) = \begin{pmatrix} f(w) \\ cf(w) \end{pmatrix}, \quad w \in C,$$

one has

$$F(z) = z + Q(Lz), \quad z \in C^2.$$

Since

$$LQ(Lz) = 0, \quad z \in C^2,$$

it is shown that (3) implies (4).

Now it is to be shown that (4) implies (3). Choose Q, L so that (4) holds. Choose $a, b \in C$ such that

$$L\begin{pmatrix} x \\ y \end{pmatrix} = ax + by, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in C^2.$$

Denote by each of g, h a polynomial from C to C so that

$$Q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(ax+by) \\ h(ax+by) \end{pmatrix}, \quad (x,y) \in C^2.$$

If $a = 0 = b$, then $r = 0 = s$ and the conclusion holds, so suppose that at least one of a, b is not zero. From (4) it follows that

$$ag(ax+by) + bh(ax+by) = 0, \quad (x,y) \in C^2,$$

and so

$$ag + bh = 0$$

since with proper choice for x, y, $ax + by$ may be any member of C. Thus

$$\left(\nabla \cdot \begin{pmatrix} r \\ s \end{pmatrix}\right)(x,y) = (ag+bh)'(ax+by) = 0, \quad (x,y) \in C^2.$$

Thus (3) holds and the argument is finished. $\square$

Proof. It has already been noted that Theorem 1 follows from Theorem 2.

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