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MULTIPLE POSITIVE SOLUTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. We extend ODE results of Henderson and Thompson, see [**10**], to a large class of boundary value problems for both ODEs and PDEs. Our method of proof combines upper and lower solutions with degree theory.

1. Introduction. In [**10**] the following theorem is proved:

Theorem 1. *Let* $0 < a < b < (c/2)$*, and suppose that* $f : [0, \infty) \to [0, \infty)$ *is a continuous function satisfying*

(i) $f(t) < 8a$*, for* $0 \le t \le a$*,*

(ii) $f(t) \ge 16b$*, for* $b \le t \le 2b$*, and*

(iii) $f(t) \le 8c$*, for* $0 \le t \le c$*.*

Then the boundary value problem

$$\begin{aligned} u'' + f(u) = 0 \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0, \end{aligned} \tag{1}$$

has at least three symmetric nonnegative solutions y_1*,* y_2*, and* y_3 *satisfying* $||y_1||_\infty < a$*,* $b < \min_{[(1/4),(3/4)]} y_2$*,* $||y_3||_\infty > a$*, and* $\min_{[(1/4),(3/4)]} y_3 < b$*.*

(It has been observed by Henderson and Thompson, and others, that it suffices to impose condition (ii) on the interval $[b, 3b/2]$.)

It is instructive to consider problem (1) with

$$f(t) = \begin{cases} k & : 0 \le t \le 1, \\ K & : t > 1. \end{cases}.$$

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We assume that k and K are positive constants. Solutions to this problem can be constructed by assembling quadratic splines that satisfy $-u'' = k$ when $u \leq 1$ and $-u'' = K$ when $u > 1$. It follows that there is one solution satisfying $\max_{[0,1]} u(x) \leq 1$ if and only if $k \leq 8$, and there are two solutions satisfying $\max_{[0,1]} u(x) > 1$ if and only if $K^2 - 16K + 8k > 0$. Both inequalities must be satisfied in order to have three solutions. (Allowing for the possibility that k might be any constant in $(0, 8]$ leads to the condition $K \geq 16$.) Problem (1) serves as an idealized example for Theorem 1 with $a = b = 1$. Condition (iii) is automatically satisfied for large enough c because f is bounded. Of course, the continuity hypothesis is not satisfied, but it is not difficult to reach similar conclusions using continuous approximations of the given f which are bounded above by 8 on $[0, 1]$, and below by 16 for $t \geq b = 1 + \varepsilon$. It follows that the choice of constants, 8 and 16, in Theorem 1 is sharp. For another discussion demonstrating that these conditions are sharp, see [**12**].

The proof in [**10**] is an application of the Leggett-Williams fixed point theorem, see [**16**]. Using similar methods Henderson and Thompson have generalized Theorem 1 to nth order conjugate boundary value problems, see [**11**]. In [**12**] these authors demonstrate that Theorem 1 can be proved via a combination of degree theory and upper and lower solutions, which is the approach that we use in this paper. Other generalizations have been proved by Baxley, et al. In [**4**] elementary shooting techniques are used to prove that an appropriately generalized sequence of inequalities leads to any desired odd number of positive solutions. Moreover, [**4**] includes an interesting theorem showing that if f is also assumed to be superlinear, then an additional solution can be found. In [**3**] similar problems are considered for a class of generalized Sturm-Liouville differential operators of order $2n$. In this case the Krasnoselskii fixed point theorem, see [**15**], is the primary tool used in the proofs. In all of the work mentioned above the necessary estimates for proof rely on ODE techniques that do not readily generalize to the PDE case.

Theorem 1 is complementary to the many papers that investigate boundary value problems of the form

$$\begin{aligned} \Delta u + \lambda f(u) = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2}$$

whose solutions, $(\lambda, u(\lambda))$, describe S-shaped bifurcation curves. For a sampling of the literature, see [**2, 5, 8, 13, 14, 17, 19, 20**]. A common prototype for $f(t)$ is the function $e^{at/(a+t)}$, which is motivated by problems in combustion theory. An important example of such results can be found in the paper of Brown, Ibrahim, and Shivaji, [**5**], where they consider a class of functions with properties similar to the combustion example above, and prove that if the graph of f is sufficiently convex for small t, then there is a solution curve, $(\lambda, u(\lambda))$, that is essentially S-shaped. In particular, there is an interval $(\underline{\lambda}, \overline{\lambda})$, such that problem (2) has at least three solutions for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$. For the ODE case the authors use a quadrature technique to derive very explicit results, and for the PDE case the authors use an upper and lower solution approach combined with degree theory that is quite similar to what we will use in later sections. This result has been improved and extended in several ways, but effectively demonstrates how the literature on S-shaped bifurcation curves relies on a different set of hypotheses than those in Theorem 1. For example, [**5**] imposes the following smoothness conditions: $f \in C^2[0, r)$, $f(t)$ is bounded, $f(t)$ satisfies a Lipschitz lower bound, i.e., there is an $l > 0$ such that $f(t) - f(s) \geq -l(t-s)$ for all $t > s$, and $f(t)$ is increasing on some interval $[0, c]$. It would be interesting to explore the overlap of these complementary theorems to see what insight each has to offer the other.

In this paper we generalize Theorem 1, and the main result in [**4**], to the boundary value problem

$$\begin{aligned} \Delta u + f(u) = 0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{3}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary. It will be clear that our results generalize to a much broader class of elliptic operators and boundary conditions. Any criteria that provide a strong maximum principle and a degree theoretical structure will allow similar theorems and proofs.

Our method of proof is to establish the existence of upper and lower solutions and then apply Leray-Schauder degree theory. In particular we will confront the situation where we have lower solutions, $\{\underline{u}_1, \underline{u}_2\}$, and upper solutions, $\{\overline{u}_1, \overline{u}_2\}$, satisfying the ordering $\underline{u}_1 \leq \underline{u}_2 \leq \overline{u}_2$ and $\underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_2$. The ordering of $\underline{u}_2$ and $\overline{u}_1$ is generally not known. These methods have a rich history going back to the work of Amann, see [**1**], with important generalizations provided by Shivaji, [**18**], and others. One technical difference between our situation and those in [**1**] and [**18**] is that our function f does not necessarily satisfy a Lipschitz lower bound. As a consequence we do not rely on monotone iteration schemes to solve (3). However, the degree theoretical structure is still present, as in the recent work of [**7**], and our proof provides appropriate modifications to demonstrate that the theory applies to problem (3).

We call particular attention to the construction of lower solutions, which depends upon an analysis of problem (5). This problem provides an important special case for understanding Theorem 1 and its generalizations, and for determining whether or not the multiple solutions criteria are sharp. It seems clear that a better understanding of this problem will lead to a better understanding of multiple solutions for problem (3). In particular, it might be of interest to explore the role that the shape of Ω plays in determining the multiplicity of solutions.

2. Constructing upper and lower solutions. We say that $\overline{u} \in W^{2,p}(\Omega)$, $p > N$, is an upper solution for problem (3) if

$$\begin{aligned} \Delta u + f(u) \leq 0 \quad &\text{in} \quad \Omega, \\ u \geq 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

A lower solution satisfies the opposite inequalities. We say that $\overline{u}$ is a strict upper solution if any solution, u, with $u \leq \overline{u}$ in $\overline{\Omega}$ must satisfy $u < \overline{u}$ in Ω and $\partial u/\partial\nu > \partial\overline{u}/\partial\nu$ at points of $\partial\Omega$ such that $u = \overline{u}$, where ν represents the unit outward normal vector. For convenience in all that follows we will denote this last type of comparison as $u \prec \overline{u}$. We will typically establish that an upper or lower solution is strict via an application of the Strong Maximum Principle (SMP), see [**6**].

2.1 *Upper solutions.* Let ϕ be the unique positive solution of

$$\begin{aligned} \Delta\phi + 1 = 0 \quad &\text{in} \quad \Omega, \\ \phi = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{4}$$

and let $m := \max_{\overline{\Omega}} \phi(x)$.

Lemma 1. *If $a > 0$ and if $f : [0, \infty) \to [0, \infty)$ is a continuous function such that $0 \leq f(t) < a/m$ for all $t \in [0, a]$, then $(a/m)\phi$ is a strict upper solution for problem* (3).

Proof. Observe that $0 \le (a/m)\phi \le a$, so $f((a/m)\phi) < a/m$. Thus $0 = \Delta((a/m)\phi) + (a/m) > \Delta((a/m)\phi) + f((a/m)\phi)$, so $(a/m)\phi$ is an upper solution. If u is any solution of (3) satisfying $u \le (a/m)\phi$, then $u \le a$ and we have $0 \le f(u) < (a/m)$, so $-\Delta u = f(u) < (a/m) = -\Delta((a/m)\phi)$, so $u \prec (a/m)\phi$ by the SMP. Hence, $(a/m)\phi$ is a strict upper solution. □

Remark 1. In order to prove the lemma above it suffices to assume that $0 \le f(t) \le a/m$ for all $t \in [0, a]$ with strict inequality $f(t) < a/m$ at some point in the interval. Of course, by continuity, this would imply strict inequality on a nontrivial open subset of the interval.

2.2 *Lower solutions.* Consider the boundary value problem

$$\begin{aligned} \Delta u + Kh(u) = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{5}$$

where

$$h(t) := \begin{cases} 1 & \text{for } t \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

Before continuing we check one fact.

Lemma 2. *Suppose that* ψ_K *is a nontrivial solution of* (5) *and that* $M_K := \max_{\overline{\Omega}} \psi_K$. *Then* $M_K > 1$.

Proof. If $M_K < 1$, then $\psi_K < 1$ in Ω, so $-\Delta\psi_K = Kh(\psi_K) \equiv 0$, and ψ_K must be the trivial solution, a contradiction. Hence $M_K \ge 1$. Consider the open set $\Omega' := \{x \in \Omega : 0 < \psi_K(x) < 1\}$. By assumption $\partial\Omega' \setminus \partial\Omega = \{x \in \Omega : \psi_K(x) = 1\}$ is a nontrivial closed subset of Ω. It is clearly possible to find a sphere $B_\varepsilon(x_0) \subset \Omega'$ such that $\partial B_\varepsilon(x_0) \bigcap \{x \in \Omega : \psi_K(x) = 1\}$ is nontrivial. Let x' be an element of this intersection. Since the interior sphere condition is satisfied at this point, and since ψ_K is a harmonic function in Ω' that reaches a maximum at the boundary point x', it must be that $(\partial\psi_K/\partial\nu)(x') > 0$, where ν is the unit outward normal to $\partial B_\varepsilon(x_0)$ at x'. It follows that $\psi_K > 1$ at a point near x', and thus $M_K > 1$. The proof is done. □

Lemma 3. *Suppose that* ψ_K *is a nontrivial solution of* (5) *and that* $M_K := \max_{\overline{\Omega}} \psi_K$. *If* $b > 0$ *and if* $f : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $f(t) > Kb$ *for all* $t \in [b, M_K b]$, *then* $b\psi_K$ *is a strict lower solution for* (3).

Proof. If $x \in \Omega$ such that $\psi_K(x) \geq 1$, then $b \leq b\psi_K(x) \leq M_K b$, so $f(b\psi_K(x)) > Kb = bKh(\psi_K(x))$. If $x \in \Omega$ such that $\psi_K(x) < 1$, then $f(b\psi_K(x)) \geq 0 = Kbh(\psi_K(x))$. Therefore $0 = \Delta(b\psi_K) + bKh(\psi_K) \leq \Delta(b\psi_K) + f(b\psi_K)$ in Ω, so $b\psi_K$ is a lower solution.

Suppose that u is any solution of (3) satisfying $u \geq b\psi_K$. Notice that u must attain all values between 0 and $M_K b$. On $\{x : 0 < u(x) < b\}$, we must have $0 < \psi_K < 1$, so $f(u) \geq 0 = bKh(\psi_K)$. Moreover, since $f(u(x)) \geq Kb > 0$ when $u(x) = b$, we can appeal to continuity to show that the open set $\{x : 0 < u(x) < b, f(u(x)) > 0 = bKh(\psi_K)\}$ is nontrivial. Since $f(t) > Kb$ for all $t \in [b, M_K b]$, there must be an $\varepsilon > 0$ such that $f(t) \geq Kb$ for all $t \in [b, M_K b + \varepsilon]$. It follows that $f(u) \geq Kbh(\psi_K)$ on $\{x : b \leq u(x) \leq M_K b + \varepsilon\}$. Thus we have $f(u) \geq Kbh(\psi_K)$ on $\Omega' := \{x : 0 < u(x) < M_K b + \varepsilon\}$ with strict inequality on a nontrivial open subset. We also have $u = b\psi_K = 0$ on $\partial\Omega$ and $u = bM_K + \varepsilon > b\psi_K$ on $\partial\Omega' \setminus \partial\Omega$. Hence, by the SMP, $u > b\psi_K$ in $\{x : 0 < u(x) < M_K b + \varepsilon\}$, with $(\partial u/\partial\nu) < (\partial(b\psi_K)/\partial\nu)$ on $\partial\Omega$. Of course $u > b\psi_K$ on $\{x : u \geq bM_K + \varepsilon\}$. We have proved that $b\psi_K$ is a strict lower solution. □

Remark 2. For the ODE case, or for the PDE case over special domains such as a sphere, it is not hard to verify that $b\psi_K$ is a strict lower solution even if we assume only that $f(t) \geq bK$ on $[b, M_K b]$, as in [**10**]. We demonstrate such an argument in greater detail in the proof of Theorem 3.

The difficulty with the PDE case over general domains is that the inequality $f(u) \geq bKh(\psi_K)$ may no longer be satisfied for $u > bM_K$, so the application of the maximum principle is not as clear. Of course $u > bM_K$ implies $u > b\psi_K$, but the possibility remains that there could be a point, or points, where $u(x) = bM_K = b\psi_K(x)$. Such points would have to be quite special, for example they would have to be maxima for $b\psi_K$. Since $u \geq b\psi_K$, they would also have to be critical points for u. Since the maximum principle prevents interior local minima for u we see that these points have to be on the boundary of $\{x : 0 < u(x) < bM_K\}$. However, they cannot satisfy an interior sphere condition from within this set or a contradiction would arise because we already know that the two gradients have to be equal. Thus they are not maxima for u, but rather some type of saddle points. (This observation indicates why such points do not arise in the ODE case or in the spherically symmetric PDE case.) We suspect that the right technical argument will show that no such points exist in general, but we have not discovered such an argument.

Since h is nondecreasing, problem (5) cooperates well with the method of upper and lower solutions and monotone iteration. However, this approach is most often applied to boundary value problems with continuous forcing terms, so a few preliminary remarks are in order. Suppose that we can find upper and lower solutions, $\overline{u}$ and $\underline{u}$, respectively, such that $\underline{u} \leq \overline{u}$. We set up a standard iteration scheme using $u_0 = \overline{u}$ and

$$\begin{aligned} \Delta u_{n+1} + Kh(u_n) = 0 \quad \text{in} \quad \Omega, \\ u_{n+1} = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

It follows from the usual maximum principle arguments that $\{u_n\}$ is a monotonically decreasing sequence which is bounded below by $\underline{u}$. Hence the sequence converges pointwise to some ψ_K. Moreover, by standard regularity and compactness arguments we know that, without loss of generality, this convergence is in $C^1(\overline{\Omega})$. Since $t_n \searrow t$ implies $h(t_n) \searrow h(t)$, we see that $h(u_n(x)) \searrow h(\psi_K(x))$ for all x, and it follows from dominated convergence that this convergence is actually L^p for any $1 \leq p < \infty$. It follows that ψ_K is a solution of (5). If we had initiated the monotone iteration from the lower solution, then it would not be as clear that the limiting function is a solution of (5). The reason consists in the fact that the function h is not left continuous at $t = 1$.

Consider any $K' \geq K$. It is straightforward to check, as in the previous section, that $\overline{u} = K'\phi$ is a strict upper solution for (5). It is also clear that $\underline{u} \equiv 0$ is a lower solution. Moreover, if ψ_K is any solution, then we have $\Delta(K'\phi) - \Delta(\psi_K) = -K' + Kh(\psi_K) \leq 0$, which implies that $\psi_K < K'\phi$. Hence using monotone iteration starting from

$\overline{u}$ we can find a maximal solution of (5). All that remains is to argue that this maximal solution is nontrivial for some choices of K, and that there is an optimal choice of K.

Lemma 4. *There is a $K^* \geq 1/m$ such that* (5) *has a maximal nontrivial solution for every $K \geq K^*$, and* (5) *has only the trivial solution for $K < K^*$.*

Proof. First we argue that (5) has only trivial solutions for small K. Suppose that $0 < K < 1/m$, and let ψ_K be a nonnegative solution of (5). By the comparison above we see that $0 \leq \psi_K < K\phi \leq (1/m)\phi \leq 1$ in Ω. Thus $h(\psi_K) \equiv 0$ and $\Delta(\psi_K) \equiv 0$ in Ω. Hence $\psi_K \equiv 0$.

Now we show that (5) has a positive solution for some choice of K. Let Ω' be an open set such that $\Omega' \subset\subset \Omega$, let $\chi_{\Omega'}$ be the characteristic function for this set, and let ψ be the solution of

$$\begin{aligned} \Delta\psi + \chi_{\Omega'} = 0 \quad &\text{in} \quad \Omega, \\ \psi = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

By the maximum principle it is clear that ψ is strictly positive in Ω. Also, $u = K\psi$ is the solution of

$$\begin{aligned} \Delta u + K\chi_{\Omega'} = 0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

Choose K large enough so that $K\psi > 1$ on Ω'. Then $0 = \Delta(K\psi) + K\chi_{\Omega'} \leq \Delta(K\psi) + Kh(K\psi)$, so $K\psi$ is a nontrivial lower solution for (5). We see that $K\phi$ is an upper solution such that $K\phi > K\psi$ in Ω. Thus the problem has a nontrivial solution for all K large enough.

Now suppose that (5) has a maximal nontrivial solution, ψ_{K_1}, for some $K_1 > 0$, and consider (5) for $K = K_2 > K_1$. As before, $K_2\phi$ is an upper solution. Also, $0 = \Delta(\psi_{K_1}) + K_1 h(\psi_{K_1}) \leq \Delta(\psi_{K_1}) + K_2 h(\psi_{K_1})$, so ψ_{K_1} is a lower solution. Another simple comparison argument shows that $\psi_{K_1} \leq K_2\phi$, so the lower and upper solutions are well-ordered. Hence (5) has a maximal positive solution satisfying $\psi_{K_1} \leq \psi_{K_2}$.

Finally, we let $K^* := \inf\{K > 0 : (5) \text{ has a positive solution}\}$, let K_n be a decreasing sequence with limit K^*, and let ψ_{K_n} represent the corresponding maximal solutions of (5) for $K = K_n$. By previous comparisons this is a decreasing sequence and thus converges pointwise to some ψ_{K^*}. As we argued before stating this lemma, $h(\psi_{K_n}(x)) \searrow h(\psi_{K^*}(x))$ for all x. It follows that ψ_{K^*} is a solution of (5) for $K = K^*$. $\square$

3. Main theorem.

Theorem 2. *Let $\{a_j\}_{j=1}^{n+1}$ and $\{b_j\}_{j=1}^{n+1}$ be sequences of nonnegative numbers such that $M_{K^*}b_j < a_j$ for $j = 1, \ldots, n+1$, and $a_j < b_{j+1}$ for $j = 1, \ldots, n$. Assume that $f : [0, \infty) \to [0, \infty)$ is a continuous function such that the following inequalities are satisfied for all j:*

(i) *$f(t) < a_j/m$, for $0 \le t \le a_j$, and*

(ii) *$f(t) > K^* b_j$, for $b_j \le t \le M_{K^*} b_j$.*

Then the boundary value problem

$$\begin{aligned} \Delta u + f(u) = 0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

has at least $2n + 1$ nonnegative solutions.

Proof. The sequence of $n+1$ functions $\{(a_j/m)\phi\}_{j=1}^{n+1}$ is a set of strict upper solutions, by Lemma 1. Also, $\{b_j \psi_{K^*}\}_{j=1}^{n+1}$ is a sequence of strict lower solutions, by Lemma 3, with the possible exception of $b_1 \psi_{K^*}$. If $f(0) = 0$, then $u \equiv 0$ is a solution of problem (3), so $b_1 = 0$, and thus the first lower solution is not strict. This does not harm the following arguments in a significant way, but for consistency in the statements that follow we can extend f so that $f(x) \equiv 0$ for $x \le 0$, and then use a negative constant function as the first element in the list of strict lower solutions.

Consider a pair of well-ordered upper and lower solutions from the lists above. Call them $\underline{u}$ and $\overline{u}$, and note that we are assuming $\underline{u} \le \overline{u}$. Define

$$\overline{f}(x,t) := \begin{cases} f(\underline{u}(x)) & \text{if } t < \underline{u}(x) \\ f(u(x)) & \text{if } \underline{u}(x) \le t \le \overline{u}(x) \\ f(\overline{u}(x)) & \text{if } \overline{u}(x) < t. \end{cases}$$

Consider the modified boundary value problem

$$\begin{aligned} \Delta u + \overline{f}(x,u) = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{6}$$

Note that the substitution operator $N : C_0^1(\overline{\Omega}) \to C(\overline{\Omega}) : N(u(x)) := \overline{f}(x,u(x))$ is bounded and continuous, and that $(-\Delta)^{-1} : C(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is compact. Thus solutions of (6) correspond to a fixed point problem for a compact operator, i.e., $u = Mu$ for $u \in C_0^1(\overline{\Omega})$ and $M := (-\Delta)^{-1}N$.

If u is a solution of (6), then we can show that $\underline{u} \prec u \prec \overline{u}$. Since $\underline{u}$ and $\overline{u}$ are strict, it suffices to show that $\underline{u} \le u \le \overline{u}$. Suppose $\Omega' = \{x \in \Omega : \underline{u}(x) > u(x)\}$ is nonempty. On this set we have $\Delta\underline{u} - \Delta u = \Delta\underline{u} + \overline{f}(x,u(x)) = \Delta\underline{u} + f(\underline{u}(x)) \ge 0$, so $\underline{u} - u$ has no interior maximum in Ω'. But $\underline{u} - u = 0$ on $\partial\Omega'$, so $\underline{u} \le u$ in Ω', a contradiction. Similarly, we can prove that $u \le \overline{u}$.

Since $\overline{f}$ is bounded it is clear that the solutions of (6) satisfy an a priori bound $||u||_1 < R$, where $||\cdot||_1$ is the standard norm on $C^1(\overline{\Omega})$. If we let $(\underline{u},\overline{u}) := \{u \in C^1(\overline{\Omega}) : ||u||_1 < R \text{ and } \underline{u} \prec u \prec \overline{u}\}$, then $(\underline{u},\overline{u})$ is a bounded open set in $C^1(\overline{\Omega})$ that contains all solutions of problem (6), so we can apply a standard computation of the Leray-Schauder degree to get $deg(I - M, (\underline{u},\overline{u}), 0) = 1$. (See [**1**], for example.)

Now consider a pair of lower solutions, $\{\underline{u}_1,\underline{u}_2\}$, and a pair of upper solutions, $\{\overline{u}_1,\overline{u}_2\}$, taken from the lists above. Suppose that they satisfy $\underline{u}_1 \le \underline{u}_2 \le \overline{u}_2$ and $\underline{u}_1 \le \overline{u}_1 \le \overline{u}_2$. We can consider the modified problem as described above with $\underline{u} = \underline{u}_1$ and $\overline{u} = \overline{u}_2$, and through similar arguments compute

$$\begin{aligned} \deg(I - M, (\underline{u}_1,\overline{u}_1), 0) &= \deg(I - M, (\underline{u}_2,\overline{u}_2), 0) \\ &= \deg(I - M, (\underline{u}_1,\overline{u}_2), 0) = 1. \end{aligned}$$

In particular we can conclude that there are two solutions lying in $(\underline{u}_1,\overline{u}_1)$ and $(\underline{u}_2,\overline{u}_2)$, respectively. To find a third solution define the set $\mathcal{S} := \{u \in (\underline{u}_1,\overline{u}_2) :$ there exists $x_0, x_1 \in \Omega$ such that $u(x_0) > \overline{u}_1(x_0)$ and $u(x_1) < \underline{u}_2(x_1)\}$. If there were a solution, u, on the boundary of $\mathcal{S}$, then it would follow that either $u \le \overline{u}_1$ in Ω, or $u \ge \underline{u}_2$ in Ω. But these are strict upper and lower solutions, so this would imply either $u \prec \overline{u}_1$ or $u \succ \underline{u}_2$, which contradicts the fact that u is on the boundary of $\mathcal{S}$. Thus there are no solutions on $\partial\mathcal{S}$ and we can apply the excision property of degree to get

$$\begin{aligned}\deg(I - M, (\underline{u}_1, \overline{u}_2), 0) = \deg(I - M, (\underline{u}_1, \overline{u}_1), 0)\\ + \deg(I - M, (\underline{u}_2, \overline{u}_2), 0) + \deg(I - M, \mathcal{S}, 0),\end{aligned}$$

which implies that $\deg(I - M, \mathcal{S}, 0) = -1$. Hence there is a solution in $\mathcal{S}$, which is clearly distinct from the solutions in $(\underline{u}_1, \overline{u}_1)$ and $(\underline{u}_2, \overline{u}_2)$.

Now consider the pair of lower solutions $\{b_j\psi_{K^*}, b_{j+1}\psi_{K^*}\}$ and the corresponding pair of upper solutions $\{(a_j/m)\phi, (a_{j+1})/m\phi\}$. It is implicit in our hypotheses that $b_jK^* \leq (a_j/m)$ for all j, so a straightforward comparison shows that $b_j\psi_{K^*} \leq (a_j/m)\phi$ for all j. Hence, $b_j\psi_{K^*} \leq b_{j+1}\psi_{K^*} \leq (a_{j+1}/m)\phi$ and $b_j\psi_{K^*} \leq (a_j/m)\phi \leq (a_{j+1}/m)\phi$. (It is not clear, in general, how $(a_j/m)\phi$ compares to $b_{j+1}\psi_{K^*}$.) Therefore, we can apply the argument above to conclude that there are three solutions in $(b_j\psi_{K^*}, (a_{j+1}/m)\phi)$.

The remainder of the proof follows easily once we observe that the overlapping sets $(b_j\psi_{K^*}, (a_{j+1}/m)\phi)$ and $(b_{j+1}\psi_{K^*}, (a_{j+2}/m)\phi)$ each contain three solutions as described above, but can only share at most one of the solutions, i.e., the solution in $(b_{j+1}\psi_{K^*}, (a_{j+1}/m)\phi)$. $\square$

Remark 3. The assumption $(a_j/mK^*) > b_j$, which is implicit in the hypotheses of our theorem as mentioned in the proof above, corresponds to the inequality $b < (c/2)$ in Theorem 1.

Remark 4. Condition (i) can be relaxed to require a strict inequality only for the case $j = 1$. For $j > 1$ we can still argue that the upper solutions are strict because previous inequalities prevent $f \equiv (a_j/m)$.

4. Examples.

4.1 $N = 1$, $\Omega = (0, 1)$**.** For this case we consider the problem (1) exactly as in [**10**]. It is an elementary exercise to compute the relevant constants. First consider

$$\begin{aligned}u'' + 1 = 0 \quad \text{in} \quad (0, 1),\\ u(0) = u(1) = 0,\end{aligned}$$

which has the solution $u = (1/2)x(1-x)$ with maximum $m = 1/8$. Next consider

$$u'' + Kh(u) = 0 \quad \text{in} \quad (0,1),$$
$$u(0) = u(1) = 0,$$

whose solutions are symmetric about 1/2 and can be constructed by using line segments to connect the boundaries of the interval to a parabolic *cap* in the center of the interval. An elementary computation shows that in $[0,(1/2)]$ we have

$$u(x) = \begin{cases} (1/\rho)x & \text{in } [0,\rho), \\ -(K/2)(x-(1/2))^2 + 1 + (K/2)(\rho-(1/2))^2 & \text{in } [\rho,(1/2)] \end{cases}$$

where $\rho = 1/4 \pm 1/4\sqrt{1-(16/K)}$. u has a maximum $M_K := 1 + (K/2)(\rho-(1/2))^2$. By substituting the expression for ρ into this formula for M_K we can draw an explicit bifurcation diagram describing the lower solutions for our problem. Moreover, it follows that the minimal choice for K is $K^* = 16$ with corresponding $M_{K^*} = 3/2$.

4.2 $N \geq 2$, $\Omega = B_1(0)$**.** For this case we consider the problem

$$\Delta u + f(u) = 0 \quad \text{in} \quad B_1(0),$$
$$u = 0 \quad \text{on} \quad \partial B_1(0),$$

where $B_1(0)$ is the unit ball in $\mathbf{R}^N$. For simplicity we will work out some details for the case $N = 2$, and then state conclusions for $N \geq 3$ without checking details. First consider

$$\Delta u + 1 = 0 \quad \text{in} \quad B_1(0),$$
$$u = 0 \quad \text{on} \quad \partial B_1(0),$$

whose solution is $u(x,y) = 1/4-(1/4)(x^2+y^2)$ with maximum $m = 1/4$. Second look at

$$\Delta u + Kh(u) = 0 \quad \text{in} \quad B_1(0),$$
$$u = 0 \quad \text{on} \quad \partial B_1(0),$$

whose solutions can be constructed by connecting a harmonic function in an annulus, whose outer boundary is $\partial B_1(0)$, to a paraboloid *cap*. More specifically, we get

$$u(r) = \begin{cases} \ln(r)/\ln(\rho) & \text{in } (\rho, 1], \\ 1 + (K/4)\rho^2 - (K/4)r^2 & \text{in } [0, \rho]. \end{cases}$$

In order to have a smooth connection at the common boundary of the annulus and the interior circle, we require $K = (-2/\rho^2 \ln(\rho))$. It follows that $K^* = 4e$ and $M_{K^*} = 2$. Also notice that for $K > K^*$ we get exactly two solutions, one large and one small.

Similar arguments for $N \geq 3$ show that $m = 1/2N$, $K^* = 2^{2/(N-2)} N^{N/(N-2)}$ and $M_{K^*} = 1 + (N/2)$.

5. Generalization. As was mentioned previously, it is clear that Theorem 2 generalizes to a large class of boundary value problems where Δ can be replaced by any homogeneous quasilinear operator satisfying a strong maximum principle as well as sufficient compactness properties so that degree theory is applicable. In this section we consider the particular case of the p-Laplacian.

As a first example we generalize Theorem 1 to a class of boundary value problems for the one-dimensional p-Laplacian.

Theorem 3. *Let* $0 < a < b < (c/M)$, $1 < p < \infty$, $(1/p) + (1/q) = 1$, $k = q2^q$, $K = (p-1)2^p q^p$, *and* $M = 1 + (1/p)$. *If* $f : [0, \infty) \to [0, \infty)$ *is a continuous function satisfying*

(i) $f(t) < (ka)^{p-1}$, *for* $0 \leq t \leq a$,

(ii) $f(t) \geq (Kb)^{p-1}$, *for* $b \leq t \leq Mb$, *and*

(iii) $f(t) \leq (kc)^{p-1}$, *for* $0 \leq t \leq c$,

then the boundary value problem

$$(|u'|^{p-2}u')' + f(u) = 0 \quad \text{in} \quad (0,1),$$
$$u(0) = u(1) = 0,$$

has at least three symmetric nonnegative solutions y_1, y_2, *and* y_3 *satisfying* $||y_1||_\infty < a$, $b < \min_{[(1/2q),1-(1/2q)]} y_2$, $||y_3||_\infty > a$ *and* $\min_{[(1/2q),1-(1/2q)]} y_3 < b$.

Proof. It is clear that all solutions must be symmetric about 1/2. The result will follow from an explicit computation of upper and lower solutions.

To find upper solutions we consider

$$\begin{gathered}(|\phi'|^{p-2}\phi')' + 1 = 0 \quad \text{in} \quad (0,1),\\ \phi(0) = \phi(1) = 0,\end{gathered}$$

and compute $\phi(x) = -(1/q)|(1/2) - x|^q + (1/q)(1/2)^q$, which has a maximum value of $m = (1/q)(1/2)^q$. To rescale this equation we multiply through by a constant γ^{p-1}, such as $(ka)^{p-1}$, to get

$$\begin{gathered}(|\gamma\phi'|^{p-2}\gamma\phi')' + \gamma^{p-1} = 0 \quad \text{in} \quad (0,1),\\ \gamma\phi(0) = \gamma\phi(1) = 0,\end{gathered}$$

and it follows, as in Lemma 1 and the subsequent remark, that $ak\phi = (a/m)\phi$ and $ck\phi = (c/m)\phi$ are strict upper solutions. We refer to these as $\overline{u}_1$ and $\overline{u}_2$, respectively.

To find lower solutions, consider the appropriate modification of (5), i.e.,

$$\begin{gathered}(|\psi'|^{p-2}\psi')' + Kh(\psi) = 0 \quad \text{in} \quad (0,1),\\ \psi(0) = \psi(1) = 0,\end{gathered}$$

whose solution can be constructed on $[0,(1/2)]$ using a line segment of slope $s > 2$ on the interval $[0,(1/s)]$ attached to a *cap* satisfying $(|\psi'|^{p-2}\psi')' = -K$ on $[(1/s),(1/2)]$. The solution on $[(1/2),1]$ can then be obtained by reflection. Solving the *cap* ODE subject to the conditions $\psi'(1/2) = 0$ and $\psi(1/s) = 1$ leads to

$$\psi' = K^{q-1}\left(\frac{1}{2} - x\right)^{q-1},$$

and

$$\psi = -\frac{K^{q-1}}{q}\left(\frac{1}{2} - x\right)^q + 1 + \frac{K^{q-1}}{q}\left(\frac{1}{2} - \frac{1}{s}\right)^q.$$

In order to have a smooth connection at $x = 1/s$, we require that the equation $s^{p-1} = K((1/2) - (1/s))$ be satisfied. An elementary analysis of this equation shows that if $K^* = (p-1)2^p q^p$, then there is exactly one solution $s = 2q$. For $K > K^*$ we find two solutions for s, and for $K < K^*$ we find no solutions for s. Substituting these values for K and s yields

$$\psi(x) := \begin{cases} 2qx & \text{if } 0 \leq x \leq 1/(2q) \\ -p^{q-1}2^q((1/2)-x)^q + 1 + (1/p) & \text{if } 1/(2q) < x \leq 1/2. \end{cases}$$

Notice that the maximum of this function is $M = 1+(1/p)$. We denote the lower solutions as $\underline{u}_1 \equiv 0$ and $\underline{u}_2 = b\psi$. The argument in Lemma 3 must be modified somewhat to show that $\underline{u}_2$ is strict. By symmetry we restrict our arguments to the interval $[0,(1/2)]$. As in Lemma 3 we can show that any solution, u, satisfying $u \geq \underline{u}_2$ must satisfy $u > \underline{u}_2$ in $\{x : 0 < u(x) < bM\}$, and the derivatives of the two functions must be distinct if the functions are equal at a boundary point of this open set. In particular, $u'(0) < b\psi'(0)$. It is elementary that any solution will be concave down and will reach its maximum at $x = 1/2$, so it must be that $\{x : 0 < u(x) < bM\} = (0,d)$ for some $d \leq 1/2$. If $d = 1/2$, then $u(1/2) = Mb = b\psi(1/2)$ and $\{x : 0 < u(x) < bM\} = (0,(1/2))$. However the fact $u'(1/2) = \underline{u}_2'(1/2) = 0$ contradicts the SMP. If $d < 1/2$, then we can conclude that $u > Mb$ in $(d,(1/2)]$. Since $\underline{u}_2$ only reaches its maximum of Mb at $x = 1/2$, we would see that $u > b\psi$ in $(0,(1/2)]$ with $u'(0) < b\psi'(0)$, and so $b\psi$ is a strict lower solution. Notice that we have $\underline{u}_1 \leq \underline{u}_2 \leq \overline{u}_2$ and $\underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_2$, where $\underline{u}_2$ and $\overline{u}_1$ are strict. Thus the standard degree argument applies to find three solutions.

y_1 represents the solution between $\underline{u}_1$ and $\overline{u}_1$, so y_1 must lie strictly below $\overline{u}_1$ and so $||y_1||_\infty < a$. y_2 represents the solution between $\underline{u}_2$ and $\overline{u}_2$, so y_2 lies strictly above $\underline{u}_2$ and so $b < \min_{[(1/2q),1-(1/2q)]} y_2$. y_3 represents a solution in the set $\mathcal{S} := \{u \in (\underline{u}_1, \overline{u}_2) : \exists x_0$ and x_1 such that $u(x_0) < \underline{u}_2$ and $u(x_1) > \overline{u}_1\}$. If $y_3(x) \leq a$ for all x, then $y_3 \prec \overline{u}_1$, which contradicts $y_3 \in \mathcal{S}$. Thus $||y_3||_\infty > a$. If $y_3 \geq b$ in $[(1/2q), 1-(1/2q)]$, then there is a $\beta \leq 1/2$ such that $b \leq y_3(x) \leq Mb$ for x in $[(1/2q),\beta]$ and $y_3(x) \geq Mb \geq b\psi(x)$ for x in $[\beta,(1/2)]$. (Recall that solutions are symmetric about $x = 1/2$, so it is only necessary to do estimates on $[0,(1/2)]$.) It follows that $-(|y_3'|^{p-2}y_3')' = f(y_3) \geq K^{p-1}b^{p-1}h(\psi) = -(|b\psi'|^{p-2}b\psi')'$ in $[0,\beta]$, and $y_3 \geq b\psi$ at the boundaries of $[0, \beta]$. Thus $y_3 \geq b\psi$ in $[0, \beta]$. Hence, $y_3 \geq b\psi$ on all of $[0, 1]$, and so $y_3 \succ \phi$, which contradicts $y_3 \in \mathcal{S}$. Therefore, $\min_{[(1/2q),1-(1/2q)]} y_3 < b$. □

Finally, we state the generalization of Theorem 2 to the case where Δ is replaced by the p-Laplacian $-\Delta_p$. The proofs are identical with one small exception, which is already mentioned in the proof of Theorem 3, and that is that Lemmas 1 and 3 must be rewritten to allow for rescaling by a constant such as $(a_j/m)^{p-1}$.

Theorem 4. *Let $\{a_j\}_{j=1}^{n+1}$ and $\{b_j\}_{j=1}^{n+1}$ be sequences of nonnegative numbers such that $M_{K^*} b_j < a_j$ for $j = 1, \dots, n+1$, and $a_j < b_{j+1}$ for $j = 1, \dots, n$. Assume that $f : [0, \infty) \to [0, \infty)$ is a continuous function such that the following inequalities are satisfied for all j:*

(i) $f(t) < (a_j/m)^{p-1}$, *for* $0 \leq t \leq a_j$, *and*

(ii) $f(t) > (K^* b_j)^{p-1}$, *for* $b_j \leq t \leq M_{K^*} b_j$.

Then the boundary value problem

$$\begin{aligned} \Delta_p u + f(u) = 0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}$$

has at least $2n + 1$ nonnegative solutions.

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