ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

WEAKLY KRULL AND RELATED DOMAINS OF THE FORM $D+M$, $A+XB[X]$ AND $A+X^2B[X]$

DAVID F. ANDERSON, GYU WHAN CHANG AND JEANAM PARK

ABSTRACT. Let $T = K + M$ and $R = D + M$ be integral domains, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K. We show that R is a weakly Krull domain, respectively, WFD, AWFD, GWFD, if and only if $\operatorname{ht} M = 1$, D is a field, and T is a weakly Krull domain, respectively, WFD, AWFD, GWFD. Let $A \subsetneq B$ be an extension of integral domains, $R = A + XB[X]$, and $D = A + X^2B[X]$. We also show that R is a weakly Krull domain if and only if D is a weakly Krull domain, if and only if $B_{A-\{0\}}$ is a field, $qf(A) \cap B = A$, and $B[X]$ is a weakly Krull domain; that R is a WFD, respectively AWFD, if and only if $qf(A) \cap B = A$, $B[X]$ is a WFD, respectively AWFD, and for each $0 \neq b \in B$, there is a unit u of B such that $ub \in A$ (respectively, an integer $n = n(b) \geq 1$ and a unit u of B such that $ub^n \in A$); and that if $\operatorname{char} B \neq 0$, then R is an AWFD if and only if D is an AWFD.

1. Introduction. In this paper, we determine when three different pullback constructions yield weakly Krull domains, weakly factorial domains, or almost weakly factorial domains. The first two pullbacks we consider are the well known $D + M$ and $A + XB[X]$ constructions, and the third concerns domains of the form $A + X^2B[X]$.

Let R be an integral domain. Then R is called a *weakly Krull domain* if $R = \cap_{P \in X^1(R)} R_P$, where $X^1(R)$ is the set of height-one prime ideals of R, and R has finite character. Examples of weakly Krull domains include Krull domains, one-dimensional Noetherian domains, and one-dimensional semi-quasi-local domains. It is well known that if R is weakly Krull, then $t\text{-}\dim R = 1$ [**9**, Lemma 2.1], that is, every prime t-ideal of R has height-one. A nonzero element a of R is said to be *primary* if aR is a primary ideal of R. As in [**8**], we will call R a *weakly factorial domain* (WFD) if each nonzero nonunit of R is a product of

2000 AMS *Mathematics Subject Classification.* Primary 13A15, 13B25, 13G05.
The second author's work was supported by the University of Incheon Research Fund, 2003.
The third author's work was supported by Korea Research Foundation Grant (KRF-2001-DP0024).
Received by the editors on April 15, 2003.

primary elements. In [**11**, Theorem], it was shown that R is a WFD if and only if R is a weakly Krull domain and $Cl_t(R) = 0$, where $Cl_t(R)$ is the t-class group of R. This is the analog of the following well known characterization of UFDs: R is a UFD if and only if R is a Krull domain and $Cl\,(R) = 0$, where $Cl\,(R)$ is the divisor class group of R.

As in [**9**], an integral domain R is called an *almost weakly factorial domain* (AWFD) if for each nonzero nonunit $x \in R$, there is an integer $n = n(x) \geq 1$ such that x^n is a product of primary elements. A Krull domain R is called an *almost factorial domain* if $Cl\,(R)$ is torsion. As the weakly Krull domain analog of this, R is an AWFD if and only if R is a weakly Krull and $Cl_t(R)$ is torsion [**9**, Theorem 3.4]. It is easy to show that a Krull domain R is an almost factorial domain if and only if every nonzero prime ideal of R contains a primary element [**5**, Proposition 3.1]. But it is not known whether R is an AWFD if and only if every nonzero prime ideal of R contains a primary element. As in [**5**], we will call R a *generalized weakly factorial domain* (GWFD) if every nonzero prime ideal of R contains a primary element. It is known that WFD $\Rightarrow$ AWFD $\Rightarrow$ GWFD $\Rightarrow$ weakly Krull domain $\Rightarrow$ t-$\dim R = 1$.

Let $T = K + M$ be an integral domain, where K is a field and M is a nonzero maximal ideal of T. If D is a proper subring of K, then $R = D+M$ is a proper subring of T. This construction has been studied extensively and has proved very useful for constructing examples, see [**10, 13, 15, 22**]. In Section 2, we show that R is a weakly Krull domain, respectively WFD, AWFD, GWFD, if and only if $\operatorname{ht} M = 1$, D is a field, and T is a weakly Krull domain, respectively WFD, AWFD, GWFD.

Let $A \subsetneq B$ be an extension of integral domains. Then $R = A+XB[X]$ and $D = A + X^2B[X]$ are proper subrings of $B[X]$ and $B[X^2, X^3]$, respectively. The $A + XB[X]$ construction has also proved useful for constructing examples, see [**2, 22, 24**]. In Section 3, we prove that R is a weakly Krull domain if and only if $B_{A-\{0\}}$ is a field, $qf(A) \cap B = A$, and $B[X]$ is a weakly Krull domain; and that R is a WFD, respectively AWFD, if and only if $qf(A) \cap B = A$, $B[X]$ is a WFD, respectively AWFD, and for each $0 \neq b \in B$, there is a $u \in U(B)$ such that $ub \in A$, respectively an integer $n = n(b) \geq 1$ and $u \in U(B)$ such that $ub^n \in A$. Now let $\{X_\alpha\}$ be a nonempty set of indeterminates over A, $N_v = \{f \in A[\{X_\alpha\}] | (A_f)_v = A\}$, $B = A[\{X_\alpha\}]_{N_v}$, and

$R = A + XB[X]$. As corollaries, we have that R is a WFD if and only if A is a weakly factorial GCD-domain and that if A is integrally closed, then R is an AWFD if and only if A is an almost weakly factorial AGCD-domain.

In the final section, we show that $D = A + X^2B[X]$ is a weakly Krull domain if and only if $B_{A-\{0\}}$ is a field, $qf(A) \cap B = A$, and $B[X]$ is a weakly Krull domain. Unlike the $R = A + XB[X]$ case mentioned above, $D = A + X^2B[X]$ is never a WFD and is not an AWFD if $\operatorname{char} B = 0$. However, we do show that if $\operatorname{char} B \neq 0$, then D is an AWFD if and only if $qf(A) \cap B = A$, $B[X]$ is an AWFD, and for each $0 \neq b \in B$, there is an integer $n = n(b) \geq 1$ and $u \in U(B)$ such that $ub^n \in A$.

All rings R considered in this paper are commutative integral domains with quotient field $qf(R)$, $U(R)$ is the group of units of R, and $X^1(R)$ denotes the set of height-one prime ideals of R. As usual, for $f = a_nX^n + \cdots + a_0 \in qf(R)[X]$, the *content of* f is the fractional ideal $A_f = (a_0, \ldots, a_n)$. Recall that, for nonzero fractional ideals I and J of R, $(I : J) = \{x \in qf(R) | xJ \subseteq I\}$, $I^{-1} = (R : I) = \{x \in qf(R) | xI \subseteq R\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{(a_1, \ldots, a_n)_v | 0 \neq (a_1, \ldots, a_n) \subseteq I\}$. We say that I is a *divisorial* (or v-)*ideal*, respectively t*-ideal*, if $I_v = I$, respectively $I_t = I$. Note that any divisorial ideal is also a t-ideal. It is well known that every proper integral t-ideal is contained in some (necessarily prime) t-ideal maximal among proper integral t-ideals, that every prime ideal minimal over a t-ideal is a t-ideal, and that for nonzero fractional ideals I, J of R, $(IJ)_t = (IJ_t)_t = (I_tJ_t)_t$ and $(I_t)_t = I_t$.

A fractional ideal I of R is said to be t*-invertible* if $(II^{-1})_t = R$. If a fractional ideal I is t-invertible, then $I_t = J_t$ for some finitely generated fractional ideal J, and hence I_t is a divisorial ideal. The set of t-invertible fractional t-ideals of R forms an abelian group under the t-product $I * J = (IJ)_t$. The t*-class group* of R is $Cl_t(R)$—the group of t-invertible fractional t-ideals of R modulo its subgroup of principal fractional ideals. For R a Krull domain, $Cl_t(R) = Cl\,(R)$; while for R a Prüfer domain or one-dimensional integral domain, $Cl_t(R) = C(R) = Pic(R)$, the ideal class group, or Picard group, of R. The reader is referred to [**16**, Sections 32 and 34], [**17**] and [**24**] for the t-operation; to [**10, 13, 15, 22**] for $D + M$ constructions; to [**2, 22, 25**] for $A + XB[X]$ constructions; to [**1, 7, 10, 14**] for the t-class group; and to [**16, 20**] for standard definitions and notations.

2. The subring $R = D + M$ of $T = K + M$. Throughout this section, $T = K+M$ and $R = D+M$ are integral domains, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K. In this section, we prove that R is a weakly Krull domain, respectively WFD, AWFD, GWFD, if and only if $\operatorname{ht} M = 1$, D is a field, and T is a weakly Krull domain, respectively WFD, AWFD, GWFD.

The following results are well known. For easy reference, we collect them in two lemmas.

Lemma 2.1. *Let S be a multiplicative subset of an integral domain R and I a nonzero fractional ideal of R. Then*

(1) *$(IR_S)_t = (I_tR_S)_t$, and if I is t-invertible, then $(IR_S)_t = I_tR_S$.*

(2) *If R is a weakly Krull domain, respectively* WFD, AWFD, GWFD, *then R_S is a weakly Krull domain, respectively* WFD, AWFD, GWFD.

Proof. (1) is in [**19**, Lemma 3.4] and [**14**, Lemma 2.9]. For (2), see [**7**, Proposition 4.7 and Theorem 4.8], and use the fact that if aR is a primary ideal, then so is aR_S. □

Lemma 2.2. *Let $T = K + M$ and $R = D + M$, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K.*

(1) *The height of M as a prime ideal of R is equal to its height as a prime ideal of T.*

(2) *M is a divisorial ideal, and hence a t-ideal, of R.*

Proof. These are [**10**, Proposition 2.1 (2) and (3)]. □

Theorem 2.3. *Let $T = K+M$ and $R = D+M$, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K. Then R is a weakly Krull domain if and only if $\operatorname{ht} M = 1$, D is a field, and T is a weakly Krull domain.*

Proof. $\Rightarrow$. Assume that R is a weakly Krull domain. Note that M is a t-ideal of R by Lemma 2.2 (2), and hence $\operatorname{ht} M = 1$ because t-$\dim R = 1$. Assume that D is not a field, and let aD be a proper nonzero ideal of D. Then $aD + M = a(D + M)$ is a proper nonzero principal ideal of R. Let Q be a prime ideal of R minimal over $aD+M$; then Q is a prime t-ideal of R and $M \subsetneq Q$. Since M is a t-ideal by Lemma 2.2 (2), $t\text{-}\dim R \geq 2$, which is contrary to the fact that $t\text{-}\dim R = 1$. Thus D is a field. (Here is another proof: It is necessary that $\operatorname{ht} M = 1$. Otherwise, the intersections of R with respect to the height-one prime ideals would be T. Similarly, it is necessary that D is a field. Otherwise the same intersection would be $qf(D) + M$.)

Next, we show that T is weakly Krull. Let Q be a prime ideal of T such that $Q \neq M$, and let $P = Q \cap R$. Then P is a prime ideal of R such that $P \neq M$ and $R_P = T_Q$ [**10**, Proposition 2.1 (2)]. Hence $T_Q = R_P$ is weakly Krull by Lemma 2.1 (2). Moreover, since $\operatorname{ht} M = 1$, $T = \cap_{Q \in X^1(T)} T_Q$. Since T is an overring of R and R is weakly Krull, the intersection $T = \cap_{Q \in X^1(T)} T_Q$ also has finite character, and hence T is a weakly Krull domain.

$\Leftarrow$. Assume that $\operatorname{ht} M = 1$, D is a field and T is weakly Krull. Let P be a prime ideal of R such that $P \neq M$. Then $T \subseteq R_P$ because for any $k \in K$, $k = (km)/m \in R_P$ for $m \in M - P$. Thus $R_P = T_{PR_P \cap T}$. Since T is weakly Krull and $\operatorname{ht} M = 1$, $R = \cap_{P \in X^1(R)} R_P$. Moreover, since T contains R and for each $P \in X^1(R) - \{M\}$, $R_P = T_Q$ for some $Q \in X^1(T)$, the intersection has finite character. Thus R is a weakly Krull domain. □

The case where T is quasilocal in our next theorem was observed in [**5**, Remark 2.5 (3)].

Theorem 2.4. *Let $T = K + M$ and $R = D + M$, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K. Then R is a* GWFD *if and only if* $\operatorname{ht} M = 1$*, D is a field, and T is a* GWFD*.*

Proof. $\Rightarrow$. Assume that R is a GWFD. Since a GWFD is weakly Krull [**5**, Corollary 2.3], $\operatorname{ht} M = 1$, D is a field, and T is weakly Krull by Theorem 2.3. Thus it suffices to show that each height-one prime ideal of T is the radical of a principal ideal [**5**, Theorem 2.2]. Let Q be a height-one prime ideal of T and let $P = Q \cap R$.

Case 1. $P = M$. Since R is a GWFD and M is a height-one prime ideal of R, $M = \sqrt{xR}$ for some $x \in R$, and hence $M = \sqrt{xT}$.

Case 2. $P \neq M$. Then $T_Q = R_P$ and $\operatorname{ht} P = 1$. Thus $P = \sqrt{aR}$ for some $a \in R$. Assume that Q' is a prime ideal of T containing a. Then $Q' \cap R$ is a prime ideal of R containing a, and hence $P \subseteq Q' \cap R$. Thus $Q \subseteq Q'$, which implies that $Q = \sqrt{aT}$.

$\Leftarrow$. Assume that D is a field, $\operatorname{ht} M = 1$, and T is a GWFD. Since R is weakly Krull by Theorem 2.3, it suffices to show that every height-one prime ideal of R is the radical of a principal ideal [**5**, Theorem 2.2]. Let P be a height-one prime ideal of R.

Case 1. $P = M$. Since M is a height-one prime ideal of T, $M = \sqrt{xT}$ for some $x \in M \subsetneq R$, and hence $M = \sqrt{xR}$.

Case 2. $P \neq M$. Then $R_P = T_{PR_P \cap T}$; so $PR_P \cap T$ is a height-one prime ideal of T, and thus $PR_P \cap T = \sqrt{xT}$ for some $x \in T$. Since $P \neq M$, $x \notin M$, and hence $x = a + m = a(1 + (m/a))$ for some nonzero $a \in K$. Note that since a is a unit in T, $xT = (1/a)xT = (1 + (m/a))T$. Also, note that $1 + (m/a) \in R$ and $PR_P \cap T$ is the unique height-one prime ideal of T containing $1 + (m/a)$. Thus P is the unique height-one prime ideal of R containing $1 + (m/a)$, and hence $P = \sqrt{(1 + (m/a))R}$. □

Theorem 2.5. *Let $T = K + M$ and $R = D + M$, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K. Then R is an* AWFD *if and only if* $\operatorname{ht} M = 1$*, D is a field, and T is an* AWFD.

Proof. $\Rightarrow$. Assume that R is an AWFD. By Theorem 2.3 and [**9**, Theorem 3.4], it suffices to show that if J is a t-invertible t-ideal of T, then $(J^n)_t$ is principal for some integer $n \geq 1$, i.e., $Cl_t(T)$ is torsion. Since M is a t-ideal of T (note that $\operatorname{ht} M = 1$) and J is t-invertible, $JJ^{-1} \nsubseteq M$. Thus there is a $u \in J^{-1}$ such that $uJ \nsubseteq M$. Replacing J with uJ, we may assume that $J \nsubseteq M$. Let $x + m \in K + M$. If $x = 0$, then $x + m \in M \subsetneq R$; and if $x \neq 0$, then x is a unit of T and $1 + (m/x) \in D + M = R$. Hence there is a finitely generated ideal I of R such that $J = (IT)_t$.

Since $J \nsubseteq M$ and M is a t-ideal of T, $I \nsubseteq M$, and hence $IR_M = R_M$. For $P \in X^1(R) - \{M\}$, let $Q = PR_P \cap T$. Then $Q \in X^1(T)$ and $R_P = T_Q$. Note that JT_Q is principal [**19**, Corollary 2.7]. Thus $(IR_P)_t = (IT_Q)_t = ((IT)_tT_Q)_t = (IT)_tT_Q = JT_Q$ is principal by Lemma 2.1 (1) (note that Q is a prime t-ideal of T and J is t-invertible). Hence I is t-locally principal. Since I is finitely generated, I is t-invertible [**19**, Corollary 2.7], and thus $(I^n)_t = aR$ for some $a \in R$ and integer $n \geq 1$ since $Cl_t(R)$ is torsion [**9**, Theorem 3.4].

We claim that $(J^n)_t = aT$. Let $Q \in X^1(T) - \{M\}$ and $P = Q \cap R$. Then $T_Q = R_P$ and $(J^n)_tT_Q = (((IT)_t)^n)_tT_Q = (((IT)^n)_tT_Q)_t = ((IT)^nT_Q)_t = ((IT_Q)^n)_t = ((IR_P)^n)_t = (I^nR_P)_t = ((I^n)_tR_P)_t = (aR_P)_t = aR_P = aT_Q$, Lemma 2.1 (1). Also, since $I \nsubseteq M$, $aT \nsubseteq M$, and hence $(J^n)_tT_M = T_M = (aT)T_M$. Thus $(J^n)_t = \cap_{Q \in X^1(T)}(J^n)_tT_Q$ $= \cap_{Q\in X^1(T)}(aT)T_Q = aT$ [**19**, Proposition 2.8].

$\Leftarrow$. Assume that $\operatorname{ht} M = 1$, D is a field, and T is an AWFD. Let I be a t-invertible t-ideal of R. As in the beginning of the above proof, we may assume that $I \nsubseteq M$. Note that for each $Q \in X^1(T) - \{M\}$, $Q \cap R \in X^1(R)$ and $R_{Q\cap R} = T_Q$. Thus $II^{-1} \nsubseteq P$ for all $P \in X^1(R)$ implies that $II^{-1} \nsubseteq Q$ for all $Q \in X^1(T)$. Hence IT is a t-invertible ideal of T. Since T is an AWFD, $(((IT)_t)^n)_t = (I^nT)_t = aT$ for some integer $n \geq 1$ and $a = b + m \in K + M = T$. Since $I \nsubseteq M$, $a \notin M$ and hence $b \neq 0$. Thus $aT = (1 + (m/b))T$ and $1 + (m/b) \in R$; so we may assume that $a \in R$. Note that $(I^n)_t$ is a t-ideal of R, and that for $P \in X^1(R) - \{M\}$ and $Q := PR_P \cap T$, $(I^nR_P)_t = (I^nT_Q)_t = ((I^nT)_tT_Q)_t$, Lemma 2.1 (1). So by [**19**, Proposition 2.8], we have

$$\begin{aligned}
(I^n)_t &= \cap_{P\in X^1(R)}(I^n)_tR_P \\
&= (I^n)_tR_M \cap (\cap\{(I^n)_tR_P | P \in X^1(R) \quad \text{and} \quad P \neq M\}) \\
&= R_M \cap (\cap\{(I^n)_tT_Q | Q \in X^1(T) \quad \text{and} \quad Q \neq M\}) \\
&= aR_M \cap (\cap\{aT_Q | Q \in X^1(T) \quad \text{and} \quad Q \neq M\}) \\
&= \cap_{P\in X^1(R)}aR_P = aR.
\end{aligned}$$

Hence R is an AWFD. $\square$

The proof of Theorem 2.5 yields the following theorem as a special case for $n = 1$. The "$\Leftarrow$" implication in Theorem 2.6 also follows from Theorem 2.3 and [**10**, Proposition 3.11] since R is a WFD if and only if R is weakly Krull and $Cl_t(R) = 0$ [**11**, Theorem].

Theorem 2.6. *Let $T = K + M$ and $R = D + M$, where K is a field, M is a nonzero maximal ideal of T and D is a proper subring of K. Then R is a* WFD *if and only if* $\operatorname{ht} M = 1$, *D is a field and T is a* WFD.

Remark 2.7. Let $T = K + M$ and $R = D + M$, where K is a field, M is a nonzero maximal ideal of T, and D is a proper subring of K.

(a) Then R is Noetherian if and only if T is Noetherian and D is a field with $[K : D] < \infty$ [**15**, Theorem 4]. Thus for $T = \mathbf{C}[X] = \mathbf{C} + X\mathbf{C}[X]$, by Theorem 2.6 we have that $R_1 = \mathbf{R} + X\mathbf{C}[X]$ is a Noetherian WFD, while $R_2 = \mathbf{Q} + X\mathbf{C}[X]$ is a non-Noetherian WFD.

(b) It is well known that $\dim R = \max\{\dim D + \operatorname{ht} M, \dim T\}$. Thus $\dim R = \dim T$ when D is a subfield of K. We next give an example where R is weakly Krull and $\dim R = 2$ (for more details, see [**18**, Examples 28 and 29, pp. 62–65]). Let K be any field, X and Y indeterminates over K, and $S = K[X, Y]$. Let $T_1 \subset K(X, Y)$ be a DVR with maximal ideal P such that $T_1 = K + P$ with $(X, Y)S \subset P$, and let $T_2 = K[X, Y]_{(X-1,Y)}$. Then $T = T_1 \cap T_2$ is a two-dimensional Noetherian UFD with exactly two maximal ideals, $M = P \cap T$ and $N = (X - 1, Y)_{(X-1,Y)} \cap T$, where $\operatorname{ht} M = 1$ and $\operatorname{ht} N = 2$. Note that $T = K + M = K + N$. Let k be a proper subfield of K. By Theorem 2.3, $R = k + M$ is weakly Krull (in fact, a WFD, but not a UFD), while $A = k + N$ is not weakly Krull. Also, it is possible to have R be Noetherian and/or integrally closed by suitable choice of k. This construction may be easily generalized to obtain a UFD of the form $T = K + M$, where $\dim T$ is any specified positive integer or infinity and $\operatorname{ht} M = 1$. Thus $R = k + M$ will be weakly Krull (a WFD, but not a UFD) with $\dim R = \dim T$ and $\operatorname{ht} M = 1$.

3. The ring $R = A + XB[X]$. Let $A \subsetneq B$ be an extension of integral domains, and let X be an indeterminate over B. Let $R = A + XB[X]$; then R is the subring of $B[X]$ whose constant terms are in A. In this section, we show that R is a weakly Krull domain if and only if $B_{A-\{0\}}$ is a field, $qf(A) \cap B = A$, and $B[X]$ is a weakly Krull domain; that R is a WFD, respectively AWFD, if and only if $qf(A) \cap B = A$, $B[X]$ is a WFD, respectively AWFD, and for each $0 \neq b \in B$, there is a $u \in U(B)$ such that $ub \in A$, respectively an integer $n = n(b) \geq 1$ and $u \in U(B)$

such that $ub^n \in A$. Let $\{X_\alpha\}$ be a nonempty set of indeterminates over A, $N_v = \{f \in A[\{X_\alpha\}] | (A_f)_v = A\}$ and $B = A[\{X_\alpha\}]_{N_v}$. As corollaries, we have that $R = A + XB[X]$ is a WFD if and only if A is a weakly factorial GCD-domain and that if A is integrally closed, then R is an AWFD if and only if A is an almost weakly factorial AGCD-domain.

We first determine when $XB[X]$ is a maximal t-ideal of R.

Lemma 3.1. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. Then $(R : XB[X]) = \{(1/X)f(X) | f(X) \in (A : B) + XB[X]\}$.*

Proof. It is clear that $\{(1/X)f(X) | f(X) \in (A : B) + XB[X]\}$ is contained in $(R : XB[X])$. For the converse, let $u \in (R : XB[X])$. Then $uXB[X] \subseteq R \subsetneq B[X]$. Thus $uX \in B[X]$, and hence $u = (1/X)f(X)$ for some $f(X) \in B[X]$. Thus $f(X)B[X] = (1/X)f(X)XB[X] = uXB[X] \subseteq A + XB[X]$. Hence $f(0)B \subseteq A$; so $f(0) \in (A : B)$. Thus $f(X) \in (A : B) + XB[X]$, which completes the proof. $\square$

Lemma 3.2 [**3**, Lemma 2.2]. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. Then $XB[X]$ is a divisorial prime ideal of R.*

Proof. Clearly $XB[X]$ is a prime ideal of R. Assume that $XB[X] \subsetneq (XB[X])_v$ and let $a + Xb(X) \in (XB[X])_v - XB[X]$. Then $a \neq 0$ and $(a + Xb(X))(R : XB[X]) \subseteq A + XB[X]$.

Case 1. $(A : B) = 0$. Then $(R : XB[X]) = B[X]$ by Lemma 3.1. Thus $(a + Xb(X))B[X] \subseteq A + XB[X]$. In particular, $aB \subseteq A$, and hence $a \in (A : B) = 0$, a contradiction.

Case 2. $(A : B) \neq 0$. Let $0 \neq c \in (A : B)$ and $b'(X) \in B[X]$; then $(1/X)(c + Xb'(X)) \in (R : XB[X])$ by Lemma 3.1. Thus $(a + Xb(X))(1/X)(c + Xb'(X)) \in A + XB[X] \subsetneq B[X]$, and hence $(1/X)ac \in B[X]$, a contradiction.

Thus $XB[X] = (XB[X])_v$. $\square$

Lemma 3.3. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. Then $XB[X]$ is a maximal t-ideal of R if and only if $qf(A) \cap B = A$.*

Proof. ($\Rightarrow$). Assume that $XB[X]$ is a maximal t-ideal of R and $qf(A) \cap B \neq A$. Let $a/b \in (qf(A) \cap B) - A$; then $a/b(bA + XB[X]) \subseteq A + XB[X]$. Thus $R \subsetneq (bA + XB[X])^{-1}$, and hence $XB[X] \subsetneq (bA + XB[X])_v \subsetneq R$, which is contrary to the fact that $XB[X]$ is a maximal t-ideal of R. Thus $qf(A) \cap B = A$.

($\Leftarrow$). Suppose that $qf(A) \cap B = A$. Let $0 \neq a \in A$. For $u \in (a, X)^{-1}$, $ua \in R \subseteq B[X]$, and hence $u \in qf(B)[X]$. Also, since $uX \in R \subseteq B[X]$, $u \in B[X]$. If $u(0) = 0$, then $u \in XB[X] \subseteq R$. If $u(0) \neq 0$, then $u(0)a \in A$, and hence $u(0) \in a^{-1}A \subseteq qf(A)$. Thus $u(0) \in qf(A) \cap B = A$; so $u \in A + XB[X] = R$. Hence $(a, X)^{-1} = R$, and thus $(a, X)_v = R$, which implies that $(aA + XB[X])_t = R$. Therefore $XB[X]$ is a maximal t-ideal of R. $\square$

It is clear that $XB[X]$ is a height-one prime ideal of $B[X]$. But this need not be true as a prime ideal of R. In fact, $\operatorname{ht}_R(XB[X]) = \dim B_{A-\{0\}}[X]$ [**25**, Lemma 2.6], and hence $\operatorname{ht}_R(XB[X]) = 1$ if and only if $B_{A-\{0\}}$ is a field. In particular, $XB[X]$ is a height-one maximal t-ideal of R if and only if $qf(A) \cap B = A$ and $B_{A-\{0\}}$ is a field. Also, recall that $B[X]$ is weakly Krull if and only if B is a weakly Krull UMT-domain [**7**, Proposition 4.11]. (As in [**17**], an integral domain B is said to be a *UMT-domain* if every nonzero prime ideal of $B[X]$ which contracts to zero in B is a maximal t-ideal.)

Theorem 3.4. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. Then the following statements are equivalent.*

(1) *R is a weakly Krull domain.*

(2) *$XB[X]$ is a height-one maximal t-ideal of R and $B[X]$ is a weakly Krull domain.*

(3) *$B_{A-\{0\}}$ is a field, $qf(A) \cap B = A$, and $B[X]$ is a weakly Krull domain.*

Proof. (1) $\Rightarrow$ (2). Let $S = \{X^n\}_{n\geq 0}$; then $R_S = B[X]_S = B[X, X^{-1}]$. Since R is weakly Krull, $R_S = B[X]_S$ is also weakly Krull by Lemma 2.1 (2). Moreover, since $B[X] = B[X]_S \cap B[X]_{XB[X]}$ and $XB[X]$ is a height-one prime ideal of $B[X]$, $B[X]$ is weakly Krull. Also, since R is weakly Krull and $XB[X]$ is a t-ideal of R, $XB[X]$ is a height-one maximal t-ideal of R.

(2) $\Leftrightarrow$ (3). This follows from Lemma 3.3 and [**25**, Lemma 2.6].

(3) $\Rightarrow$ (1). We show that $R = R_S \cap R_{XB[X]}$, where $S = \{X^n\}_{n\geq 0}$. Let $f = h/g \in R_S \cap R_{XB[X]}$, where $f \in B[X]$, $h \in R$, and $g \in R - XB[X]$. If $f(0) = 0$, then $f \in XB[X] \subsetneq R$. If $f(0) \neq 0$, then $f(0) = h(0)/g(0) \in B \cap q(A) = A$. Thus $f(X) \in A + XB[X] = R$, and hence $R = R_S \cap R_{XB[X]}$. Thus R is a weakly Krull domain since $R_S = B[X]_S$ is weakly Krull (Lemma 2.1 (2)) and $R_{XB[X]}$ is one-dimensional quasilocal, and hence weakly Krull, [**25**, Lemma 2.6]. □

Theorem 3.5. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. Then the following statements are equivalent.*

(1) *R is an* AWFD.

(2) *$XB[X]$ is a* (*height-one*) *maximal t-ideal of R, $B[X]$ is an* AWFD, *and for each $0 \neq b \in B$, there is an integer $n = n(b) \geq 1$ and $a \in A$ such that $aB = b^nB$.*

(3) *$qf(A) \cap B = A$, $B[X]$ is an* AWFD *and, for each $0 \neq b \in B$, there is an integer $n = n(b) \geq 1$ and $u \in U(B)$ such that $ub^n \in A$.*

Proof. (1) $\Rightarrow$ (2). Assume that R is an AWFD. Since an AWFD is weakly Krull [**9**, Theorem 3.4], $XB[X]$ is a (height-one) maximal t-ideal of R and $B[X]$ is weakly Krull by Theorem 3.4. Let $S = \{X^n\}_{n\geq 0}$; then $R_S = B[X]_S$, and hence $B[X]_S$ is an AWFD by Lemma 2.1 (2). Moreover, since X is a prime element of $B[X]$, $Cl_t(B[X]) \cong Cl_t(B[X]_S)$ is torsion [**7**, Corollary 4.9]. Thus, since $B[X]$ is weakly Krull, $B[X]$ is an AWFD [**9**, Theorem 3.4].

Let $0 \neq b \in B$. Then $bB \cap A \neq 0$ since $B_{A-\{0\}}$ is a field by Theorem 3.4. For $0 \neq c \in bB \cap A$, let $I = (c, bX)$ be the ideal of R generated by c and bX. We claim that I is t-invertible. Since I is finitely generated, it suffices to show that I is t-locally principal [**19**, Corollary 2.7]. Let P be a maximal t-ideal of R. If $P = XB[X]$, then $IR_P = R_P$ since $c \notin P$. Assume that $P \neq XB[X]$. Then $PR_S \subsetneq R_S$, and hence $IR_P = (IR_S)_{PR_S} = (bR_S)_{PR_S} = bR_P$ (note that $IR_S = IB[X]_S = (c,b)B[X]_S = bB[X]_S = bR_S$). Thus I, and hence I_t, is t-invertible. Since R is an AWFD, there is an integer $n \geq 1$ such that $(I^n)_t = aR$ for some $a \in R$ [**9**, Theorem 3.4]. Also, since $c^n \in (I^n)_t$, $a \in A$. Thus $aB[X]_S = aR_S = (I^n)_t R_S = (I^n R_S)_t = ((IR_S)^n)_t = ((bR_S)^n)_t = (b^n R_S)_t = b^n R_S = b^n B[X]_S$ (the third equality follows from the fact that $(I^n)_t$ is t-invertible, Lemma 2.1 (1). Thus $aB = aB[X]_S \cap B = b^n B[X]_S \cap B = b^n B$.

(2) $\Rightarrow$ (3). Let $0 \neq b \in B$. Then $b^n B = aB$ for some $a \in A$ and integer $n \geq 1$ by the assumption. Thus $ub^n = a \in A$ for some $u \in U(B)$. Also, since $XB[X]$ is a maximal t-ideal of R, $qf(A) \cap B = A$ by Lemma 3.3.

(3) $\Rightarrow$ (1). We first show that: (#) if $gB[X]$ is primary for $0 \neq g \in R$, then so is gR. Note that $B_{A-\{0\}}$ is a field since $bB \cap A \neq$ for all $0 \neq b \in B$. Also, since an AWFD is weakly Krull [**9**, Theorem 3.4], R is weakly Krull (Theorem 3.4), and hence t-dim$R = 1$ [**9**, Lemma 2.1]. Thus to prove that gR is primary, it suffices to show that $\sqrt{gR}$ is a prime ideal [**5**, Lemma 2.1]. If $\sqrt{gB[X]} = XB[X]$, then $\sqrt{gR} = XB[X]$. Assume that $\sqrt{gB[X]} \neq XB[X]$ and let $S = \{X^n\}_{n \geq 0}$. Then $g(0) \neq 0$ and $gR_S \cap R \subseteq gB[X]_S \cap B[X] = gB[X]$ since $gB[X]$ is primary. Hence if $h \in gR_S \cap R$, then $h = gb(X)$ for some $b(X) \in B[X]$. If $b(0) = 0$, then $b(X) \in XB[X] \subsetneq R$, and hence $h \in gR$. If $b(0) \neq 0$, then $b(0) = h(0)/g(0) \in qf(A) \cap B = A$. Thus $b(X) \in R$, and hence $h \in gR$. Thus $gR_S \cap R = gR$. Also, since $\sqrt{gR_S}$ is a prime ideal, $\sqrt{gR} = \sqrt{gR_S} \cap R$ is a prime ideal (note that $gB[X]$, and hence $gB[X]_S = gR_S$ is primary).

Let $0 \neq f \in R$. Since $R \subsetneq B[X]$ and $B[X]$ is an AWFD, there is an integer $n \geq 1$ such that $f^n = X^m f_1 \cdots f_l$ for some integer $m \geq 0$ and primary elements f_i of $B[X]$ with each $f_i(0) \neq 0$. Also, since $f_i(0) \in B$, there is an integer $e_i \geq 1$ and $u_i \in U(B)$ such that $u_i f_i^{e_i}(0) \in A$. Let $e = e_1 \cdots e_l$, $\hat{e_i} = e/e_i$, $u = u_1^{\hat{e_1}} \cdots u_l^{\hat{e_l}}$, and $g_i = (u_i f_i^{e_i})^{\hat{e_i}}$. Then $g_i \in R$ and $uf^{en} = X^{em} g_1 \cdots g_l$. Note that $\sqrt{g_i B[X]} = \sqrt{(u_i f_i^{e_i})^{\hat{e_i}} B[X]} = \sqrt{f_i B[X]}$, t-dim $R = 1$, and $g_i \in R$. Thus $g_i B[X]$ is primary [**5**, Lemma 2.1], and hence $g_i R$ is primary by (#).

Case 1. $m \geq 1$. Then $f^{en} = (u^{-1}X^{em})g_1 \cdots g_l$. Since $u^{-1}X^{em} \in R$ and $(u^{-1}X^{em})B[X]$ is primary, $(u^{-1}X^{em})R$ is primary by (#). Thus f^{en} has a primary factorization in R.

Case 2. $m = 0$. Then $uf^{en}(0) = g_1(0) \cdots g_l(0)$, and hence $u = (g_1(0) \cdots g_l(0))/(f^{en}(0)) \in qf(A) \cap U(B) = U(A) \subseteq U(R)$, where the equality $qf(A) \cap U(B) = U(A)$ follows from the fact that $qf(A) \cap U(B) \subseteq qf(A) \cap B = A$. Thus $f^{en} = u^{-1}g_1 \cdots g_l$, and hence f^{en} has a primary factorization in R.

Therefore R is an AWFD. $\square$

An integral domain R is a *Prüfer v-multiplication domain* (PVMD) if each finite type t-ideal is t-invertible; equivalently, R_P is a valuation domain for each maximal t-ideal P of R [**19**, Theorem 3.2]. An integral domain R is called an *almost GCD-domain* (AGCD-domain) if for all nonzero $a, b \in R$, there is an integer $n = n(a,b) \geq 1$ such that $a^nR \cap b^nR$ is principal; equivalently, $(a^n, b^n)_v$ is principal. It is known that if R is an AGCD-domain, then $Cl_t(R)$ is torsion [**12**, Theorem 3.4]; that if R is integrally closed, then R is an AGCD-domain if and only if R is a PVMD with $Cl_t(R)$ torsion [**23**, Corollary 3.8 and Theorem 3.9]; and that if R is integrally closed, then $R[X]$ is an AWFD if and only if R is an almost weakly factorial AGCD-domain [**5**, Theorem 3.3].

Corollary 3.6. *Let $A \subsetneq B$ be an extension of integral domains such that B is integrally closed, and let $R = A + XB[X]$. If R is an* AWFD, *then A and B are each almost weakly factorial* AGCD-*domains.*

Proof. Assume that R is an AWFD. We first prove that A is an AWFD. Since $A \subsetneq R$, it suffices to show that for $0 \neq a \in A$, if aR is primary, then so is aA. Assume that aR is primary and let $S = \{X^n\}_{n \geq 0}$. Then $aR_S = aB[X]_S$ is primary, and hence $aB = aB[X]_S \cap B$ is primary. Since $qf(A) \cap B = A$ by Theorem 3.5, $aB \cap A = aA$. Thus aA is primary since $\sqrt{aA} = \sqrt{aB} \cap A$.

Next, we show that A is an AGCD-domain. Note that B is an AGCD-domain since $B[X]$ is an AWFD by Theorem 3.5 and B is integrally closed [**5**, Theorem 3.3]. Let $0 \neq a_1, a_2 \in A$. Then there is an integer $n = n(a_1, a_2) \geq 1$ such that $a_1^nB \cap a_2^nB = bB$ for some

$b \in B$ since B is an AGCD-domain. Let $m \geq 1$ be an integer such that $ub^m \in A$ for some $u \in U(B)$, Theorem 3.5. Since $a_1^n B \cap a_2^n B$ is t-invertible, $a_1^{nm} B \cap a_2^{nm} B = ((a_1^n B \cap a_2^n B)^m)_t = (bB)^m = b^m B$, cf. [**12**, Lemma 3.3]. Thus $a_1^{mn} A \cap a_2^{mn} A = (a_1^{mn} B \cap A) \cap (a_2^{mn} B \cap A) = (a_1^{mn} B \cap a_2^{mn} B) \cap A = b^m B \cap A = ub^m B \cap A = ub^m A$, the first and last equalities follow from the fact that $qf(A) \cap B = A$. Thus A is an AGCD-domain. It follows from Theorem 3.5 and [**5**, Theorem 3.3] that B is also an almost weakly factorial AGCD-domain. $\square$

Let A be an integral domain. In [**7**, Corollary 4.13], it was proved that A is a weakly Krull PVMD if and only if $A[X]$ is weakly Krull and $Cl_t(A) = Cl_t(A[X])$ for one indeterminate. The same argument given in the proof of [**7**, Corollary 4.13] shows that A is a weakly Krull PVMD if and only if $A[\{X_\alpha\}]$ is weakly Krull and $Cl_t(A)Cl_t(A[\{X_\alpha\}])$ for any nonempty set $\{X_\alpha\}$ of indeterminates.

Corollary 3.7. *Let A be an integrally closed domain, $\{X_\alpha\}$ a nonempty set of indeterminates over A, $N_v = \{f \in A[\{X_\alpha\}] | (A_f)_v = A\}$, $B = A[\{X_\alpha\}]_{N_v}$, and $R = A + XB[X]$. Then R is an* AWFD *if and only if A is an almost weakly factorial* AGCD-*domain.*

Proof. Assume that R is an AWFD. Since A is integrally closed, $A[\{X_\alpha\}]$, and hence $B = A[\{X_\alpha\}]_{N_v}$ is integrally closed. Thus A is an almost weakly factorial AGCD-domain by Corollary 3.6. For the converse, assume that A is an almost weakly factorial AGCD-domain. Then A is a weakly Krull PVMD with $Cl_t(A)$ torsion [**5**, Theorem 3.3] $\Leftrightarrow A[\{X_\alpha\}]$ is a weakly Krull PVMD with $Cl_t(A[\{X_\alpha\}]) = Cl_t(A)$ torsion $\Rightarrow B = A[\{X_\alpha\}]_{N_v}$ is a weakly Krull PVMD with $Cl_t(B)$ torsion (see Lemma 2.2 (2) for weakly Krull, [**19**, Theorem 3.7] for PVMD and [**7**, Theorem 4.8] for $Cl_t(B)$ torsion) $\Leftrightarrow B[X]$ is an AWFD [**5**, Theorem 3.3].

Let $0 \neq f/g \in B$, where $f \in A[\{X_\alpha\}]$ and $g \in N_v$. Since A is an integrally closed AGCD-domain, there is an integer $n \geq 1$ such that $(A_{f^n})_t = (A_f^n)_t = aA$ for some $a \in A$. Let $f^n = af'$ for $f' \in A[\{X_\alpha\}]$. Then $f' \in N_v$; thus, $f'/g^n \in U(B)$ such that $(f/g)^n g^n/f' = a \in A$.

Let $c/b \in qf(A) \cap U(B)$, where $0 \neq b, c \in A$. Since $c/b \in U(B)$, there are $h, h_1 \in N_v$ such that $ch = bh_1$. Thus $cA = c(A_h)_t =$

$b(A_{h_1})_t = bA$, and hence $c/b \in U(A)$. Thus, $qf(A) \cap U(B) = U(A)$. Let $0 \neq d \in qf(A) \cap B$. The above paragraph shows that there is an integer $n \geq 1$ and $u \in U(B)$ such that $ud^n \in A$. In particular, $u \in d^{-n}A \cap U(B) \subseteq qf(A) \cap U(B) = U(A)$; so $d^n \in A$. Since A is integrally closed, $d \in A$. Thus $qf(A) \cap B = A$. By Theorem 3.5, R is an AWFD. □

We next give the WFD analog of Theorem 3.5. Recall that for an integral domain R, $R[X]$ is a WFD if and only if R is a weakly factorial GCD-domain, and hence R is integrally closed, [**8**, Theorem 17]. The "(3) ⇒ (1)" implication in Theorem 3.8 also follows from Theorem 3.4 and [**4**, Corollary 4.11].

Theorem 3.8. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. Then the following statements are equivalent.*

(1) *R is a* WFD.

(2) *$XB[X]$ is a (height-one) maximal t-ideal of R, $B[X]$ is a* WFD *and, for each $0 \neq b \in B$, there is an $a \in A$ such that $aB = bB$.*

(3) *$qf(A) \cap B = A$, $B[X]$ is a* WFD *and, for each $0 \neq b \in B$, there is a $u \in U(B)$ such that $ub \in A$.*

(4) *$qf(A) \cap B = A$, B is a weakly factorial* GCD*-domain, and for each $0 \neq b \in B$, there is a $u \in U(B)$ such that $ub \in A$.*

(5) *$U(B) \cap qf(A) = U(A)$, B is a weakly factorial* GCD *domain, and for each $0 \neq b \in B$, there is a $u \in U(B)$ such that $ub \in A$.*

Proof. The proofs of (1) ⇒ (2) ⇒ (3) ⇒ (1) are similar to the proof of Theorem 3.5, (3) ⇔ (4) is [**8**, Theorem 17], and (4) ⇒ (5) follows directly from the fact that $qf(A) \cap U(B) \subseteq qf(A) \cap B = A$. For (5) ⇒ (4), let $0 \neq a \in qf(A) \cap B$. Then there is a $u \in U(B)$ such that $ua \in A$. Thus $u \in a^{-1}A \cap U(B) \subseteq qf(A) \cap U(B) = U(A)$; so $a = u^{-1}(ua) \in A$. Hence $qf(A) \cap B = A$. □

Corollary 3.9. *Let $A \subsetneq B$ be an extension of integral domains and $R = A + XB[X]$. If R is a* WFD*, then A and B are each weakly factorial* GCD*-domains.*

Proof. The proof is similar to the proof of Corollary 3.6. □

It is well known that A is a GCD-domain if and only if A is a PVMD with $Cl_t(A) = 0$ [**14**, Corollary 1.5]. Thus an argument similar to that given in the proof of Corollary 3.7 also proves the following corollary.

Corollary 3.10. *Let A be an integral domain, $\{X_\alpha\}$ a nonempty set of indeterminates over A, $N_v = \{f \in A[\{X_\alpha\}] | (A_f)_v = A\}$, $B = A[\{X_\alpha\}]_{N_v}$, and $R = A + XB[X]$. Then R is a* WFD *if and only if A is a weakly factorial* GCD*-domain.*

Corollary 3.11. *Let A be a subring of a field K and $R = A+XK[X]$. Then the following statements are equivalent.*

(1) *R is a weakly Krull domain.*

(2) *R is a* WFD.

(3) *R is an* AWFD.

(4) *R is a* GWFD.

(5) *A is a field.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) hold for any integral domain R. The implications (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (5) follow from Theorem 3.4 and Theorem 3.8, respectively. □

Remark 3.12. (a) Let $A = \mathbf{Z} \subsetneq \mathbf{Z}[t] = B$. Then A and B are each (almost) weakly factorial GCD-domains. However, $R = A + XB[X]$ is not (almost) weakly factorial by Theorems 3.5 and 3.8. In fact, R is not even a weakly Krull domain by Theorem 3.4. Thus the converses to Corollaries 3.6 and 3.9 are false. However, it is interesting to note that $Cl_t(R) = 0$ [**3**, Theorem 4.9].

(b) Let $A \subsetneq B$ be an extension of integral domains. If B is integral over A, then $B_{A-\{0\}}$ is a field. If A is also integrally closed, then $qf(A) \cap B = A$. So in this case, $R = A + XB[X]$ is weakly Krull if and only if $B[X]$ is weakly Krull by Theorem 3.4. Specifically, let $A = \mathbf{Z} \subsetneq \mathbf{Z}[i] = B$. Then $R = A+XB[X]$ is weakly Krull. However, R

is not an AWFD by Theorem 3.5. In fact, $Cl_t(R) = \mathbf{Z}/2\mathbf{Z} \oplus (\bigoplus_{n=1}^{\infty} \mathbf{Z})$ by [**3**, Example 4.16].

(c) We have already observed that if $B[X]$ is a WFD, then B, and hence $B[X]$, is integrally closed. However, $R = \mathbf{R} + X\mathbf{C}[X]$ is a WFD which is not integrally closed.

4. The subring $D = A + X^2B[X]$ of $R = A + XB[X]$. As in Section 3, let $A \subsetneq B$ be an extension of integral domains, X an indeterminate over B, $R = A + XB[X]$ and $D = A + X^2B[X]$. Then $D = R \cap B[X^2, X^3]$, and hence D is a subring of both R and $B[X^2, X^3]$. The integral domain $B[X^2, X^3]$ has recently been studied by the authors in [**6**]. In this section, we prove that $D = A + X^2B[X]$ is a weakly Krull domain if and only if $B_{A-\{0\}}$ is a field, $qf(A) \cap B = A$, and $B[X]$ is a weakly Krull domain; and that if $\operatorname{char} B \neq 0$, then D is an AWFD if and only if $qf(A) \cap B = A$, $B[X]$ is an AWFD, and for each $0 \neq b \in B$, there is an integer $n = n(b) \geq 1$ and $u \in U(B)$ such that $ub^n \in A$.

Lemma 4.1 (cf. [**6**, Lemma 2.4]). *Let $A \subsetneq B$ be an extension of integral domains, $R = A + XB[X]$ and $D = A + X^2B[X]$. Let P be a prime ideal of D. Then there is a unique prime ideal of R lying over P. Thus the natural map $Spec(R) \to Spec(D)$, given by $Q \to Q \cap D$, is an order-preserving bijection. In particular, $ht_D(X^2B[X]) = ht_R(XB[X])$.*

Proof. Let P be a prime ideal of D. Since $D \subseteq R$ is an integral extension, there is a prime ideal Q of R lying over P.

Case 1. $X^2B[X] \subseteq P$. Then $XB[X] \subseteq Q$. Thus the result follows from the fact that $D/X^2B[X] \cong A \cong R/XB[X]$.

Case 2. $X^2B[X] \nsubseteq P$. Then there is a $b(X) \in B[X]$ such that $X^2b(X) \notin P$. For any $f(X) \in B[X]$, $X^2b(X)f(X) \in D$, and thus $f(X) = (X^2b(X)f(X))/(X^2b(X)) \in D_P$. Hence $R \subseteq B[X] \subseteq D_P$, which implies that $R_{PD_P \cap R} = D_P$. Thus $PD_P \cap R$ is a unique prime ideal of R lying over P. $\square$

Our next lemma is the analog of Lemmas 3.1–3.3.

Lemma 4.2. *Let $A \subsetneq B$ be an extension of integral domains and $D = A + X^2B[X]$. Then*

(1) $(D : X^2B[X]) = B[X]$.

(2) $X^2B[X]$ *is a divisorial prime ideal of* D.

(3) $X^2B[X]$ *is a maximal* t*-ideal of* D *if and only if* $qf(A) \cap B = A$.

Proof. (1) It is clear that $B[X] \subseteq (D : X^2B[X])$. For the converse, let $u \in (D : X^2B[X])$. Then $uX^2B[X] \subseteq D \subseteq B[X^2, X^3]$, and hence $u = (1/X^2)f(X)$ for some $f(X) \in B[X^2, X^3]$. If $f(0) \neq 0$, then $uX^2X = f(X)X \notin B[X^2, X^3]$, and so $uX^2B[X] \nsubseteq D$. Thus $f(0) = 0$, and hence $u \in B[X]$.

(2) By (1), we need only show that $(D : B[X]) = X^2B[X]$. Clearly, $X^2B[X] \subseteq (D : B[X])$, and the converse follows from the fact that $B[X]$ contains the identity.

(3) The proof is similar to that of Lemma 3.3. □

Theorem 4.3. *Let $A \subsetneq B$ be an extension of integral domains and $D = A + X^2B[X]$. Then the following statements are equivalent.*

(1) D *is a weakly Krull domain.*

(2) $X^2B[X]$ *is a height-one maximal* t*-ideal of* D *and* $B[X]$ *is a weakly Krull domain.*

(3) $B_{A-\{0\}}$ *is a field,* $qf(A) \cap B = A$*, and* $B[X]$ *is a weakly Krull domain.*

(4) $R = A + XB[X]$ *is a weakly Krull domain.*

Proof. By a similar argument as given in the proof of (1) $\Rightarrow$ (2) of Theorem 3.4, one can easily prove (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (3) follows from the remarks preceding Theorem 3.4 and Lemmas 4.1 and 4.2. (3) $\Leftrightarrow$ (4) is Theorem 3.4. For (3) $\Rightarrow$ (1), let $S = \{X^n | n = 0, 2, 3, \dots\}$ and $\bar{S} = \{X^n\}_{n \geq 0}$; then $D_S = B[X]_S = B[X]_{\bar{S}}$. Recall from [**7**, Proposition 4.11] and [**6**, Proposition 2.7] that $B[X]$ is weakly Krull if and only if $B[X^2, X^3]$ is weakly Krull. Thus $D_S \cap D_{X^2B[X]} \subseteq B[X^2, X^3]_S \cap B[X^2, X^3]_{X^2B[X]} = B[X^2, X^3]$. The argument given in the proof of (3) $\Rightarrow$ (1) of Theorem 3.4 also shows that $D_S \cap D_{X^2B[X]} =$

D. Thus D is a weakly Krull domain since $D_S = B[X]_S$ is weakly Krull, Lemma 2.1 (2), and $D_{X^2B[X]}$ is one-dimensional quasilocal (and hence weakly Krull) by [**25**, Lemma 2.6] and Lemma 4.1.

Remark 4.4. (a) Let $A \subsetneq B$ be an extension of integral domains, $K = qf(B)$ and $D = A + X^2B[X]$. Assume that D is a weakly Krull domain. Let $f = X+1$ and $Q_f = fK[X] \cap D$. Note that $Q_f \nsubseteq X^2B[X]$, and so $Q_f D_{X^2B[X]} = D_{X^2B[X]}$. Let $S = \{X^n | n = 0, 2, 3, \dots\}$; then $D_S = B[X]_S$ and $Q_f D[X]_S \subsetneq D_S$. Since $A_f = B$, $fK[X] \cap B[X] = fB[X]$, cf. [**16**, Theorem 28.1], and thus $Q_f D_S = fB[X]_S$. In particular, Q_f is t-locally principal. Also, since D is weakly Krull, Q_f is t-invertible [**9**, Lemma 2.2].

Suppose that $Q_f = gD$ for some $g \in D$. Then $fK[X]_S = Q_f K[X]_S = gK[X]_S$; so $g = X^m uf$ for some integer m and $u \in K$, which is contrary to the facts that $uf \notin D$ and both f and g are not divisible by X in $K[X]$. Thus, D cannot be a WFD. A similar argument also shows that if char$B = 0$, then D is not an AWFD.

(b) By [**21**, (2.22) Examples], there is a monomorphism $\varphi : A \to \text{Pic}\,(D) \subseteq Cl_t(D)$ given by $\varphi(a) = [(1+aX, X^2)]$, where A is considered as an abelian group under addition and $Pic(D)$ is the Picard group of D. Thus $Cl_t(D) \neq 0$ and $Cl_t(D)$ is not torsion if char$B = 0$. This observation gives another proof that D is never a WFD, and is not an AWFD if char $B = 0$.

In our final result, we determine when $D = A + X^2B[X]$ is an AWFD when char$B \neq 0$.

Theorem 4.5. *Let $A \subsetneq B$ be an extension of integral domains with char$B = p \neq 0$. Then the following statements are equivalent.*

(1) $D = A + X^2B[X]$ *is an* AWFD.

(2) $X^2B[X]$ *is a* (*height-one*) *maximal t-ideal of D, $B[X]$ is an* AWFD, *and for each $0 \neq b \in B$, there is an integer $n = n(b) \geq 1$ and $a \in A$ such that $aB = b^nB$.*

(3) $qf(A) \cap B = A$, *$B[X]$ is an* AWFD, *and for each $0 \neq b \in B$, there is an integer $n = n(b) \geq 1$ and $u \in U(B)$ such that $ub^n \in A$.*

(4) $R = A + XB[X]$ *is an* AWFD.

Proof. The proof of (1) $\Rightarrow$ (2) is a simple modification of the proof of (1) $\Rightarrow$ (2) of Theorem 3.5, (2) $\Rightarrow$ (3) is Lemma 4.2 (3), and (3) $\Leftrightarrow$ (4) is in Theorem 3.5.

(3) $\Rightarrow$ (1). Note that if $h \in B[X]$, then $h^p \in B[X^2, X^3]$ since $\operatorname{char} B = p$; and that for $0 \neq g \in D$, if $gB[X]$ is primary, then gD is also primary (for the proof, see the first paragraph of the proof of (3) $\Rightarrow$ (1) of Theorem 3.5 and note that if $h = gb(X) \in gD_S \cap D$, then $b(X) \in B[X^2, X^3]$). Let $0 \neq f \in D \subsetneq B[X]$. Since $B[X]$ is an AWFD, there is an integer $n \geq 1$ such that $f^n = X^m f_1 \cdots f_l$ for some integer $m \geq 0$ and primary elements f_i of $B[X]$ with each $f_i(0) \neq 0$. Thus $f^{pn} = X^{pm} f_1^p \cdots f_l^p$ with each $f_i^p \in B[X^2, X^3]$ and $f_i^p(0) \neq 0$. By the same argument given in the second paragraph of the proof of (3) $\Rightarrow$ (1) of Theorem 3.5, one can easily show that there is an integer $e \geq 1$ such that f^{epn} has a primary factorization in D. Thus D is an AWFD. $\square$

Corollary 4.6. *Let A be a subring of a field K and $D = A + X^2K[X]$. Then*

(1) *D is weakly Krull if and only if A is a field.*

(2) *D is never a* WFD.

(3) *D is an* AWFD *if and only if A is a field and* $\operatorname{char} K \neq 0$.

Proof. This follows directly from Theorem 4.3, Remark 4.4 and Theorem 4.5, respectively. It also follows from the results in Section 2 together with the fact that $Cl_t(K[X^2, X^3]) = \operatorname{Pic}(K[X^2, X^3]) = K$ as additive abelian groups [**21**, p. 40]. $\square$

Example 4.7. As observed in Remark 2.7 (a), $R = \mathbf{R} + X\mathbf{C}[X]$ is a WFD. However, $D = \mathbf{R} + X^2\mathbf{C}[X]$ is weakly Krull, but not an AWFD by Corollary 4.6.

Acknowledgments. This research was conducted while the second author visited the University of Tennessee at Knoxville. He would like to thank the University of Tennessee for its financial support and hospitality.

REFERENCES

1. D.F. Anderson, *The class group and local class group of an integral domain*, in *Non-Noetherian ring theory*, Math. Appl., Kluwer Acad. Publ., Dordrecht, 2000, pp. 33–55.

2. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Rings between $D[X]$ and $K[X]$*, Houston J. Math. **17** (1991), 109–129.

3. D.F. Anderson, S.E. Baghdadi and S. Kabbaj, *On the class group of $A+XB[X]$ domains*, Lecture Notes in Pure and Appl. Math., vol. 205, Dekker, New York, 1999, pp. 73–85.

4. ———, *The homogeneous class group of $A + XB[X]$ domains*, Internat. J. Commutative Rings **1** (2002), 11–25.

5. D.F. Anderson, G.W. Chang and J. Park, *Generalized weakly factorial domains*, Houston J. Math. **29** (2003), 1–13.

6. ———, *$D[X^2, X^3]$ over an integral domain D*, Lecture Notes in Pure and Appl. Math., vol. 231, Dekker, New York, 2003, pp. 1–14.

7. D.D. Anderson, E. Houston and M. Zafrullah, *t-linked extensions, the t-class group, and Nagata's theorem*, J. Pure Appl. Algebra **86** (1993), 109–124.

8. D.D. Anderson and L.A. Mahaney, *On primary factorization*, J. Pure Appl. Algebra **54** (1988), 141–154.

9. D.D. Anderson, J. Mott and M. Zafrullah, *Finite character representations for integral domains*, Boll. Un. Mat. Ital. **6** (1992), 613–630.

10. D.F. Anderson and A. Ryckaert, *The class group of $D + M$*, J. Pure Appl. Algebra **52** (1988), 199–212.

11. D.D. Anderson and M. Zafrullah, *Weakly factorial domains and groups of divisibility*, Proc. Amer. Math. Soc. **109** (1990), 907–913.

12. ———, *Almost Bezout domains*, J. Algebra **142** (1991), 285–309.

13. E. Bastida and R. Gilmer, *Overrings and divisorial ideals of the form $D+M$*, Michigan Math. J. **20** (1973), 79–95.

14. A. Bouvier and M. Zafrullah, *On some class groups of an integral domain*, Bull. Soc. Math. Gréce (N.S.) **29** (1988), 45–59.

15. J. Brewer and E. Rutter, *$D + M$ constructions with general overrings*, Michigan Math. J. **23** (1976), 33–42.

16. R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972.

17. E. Houston and M. Zafrullah, *On t-invertibility*, II, Comm. Algebra **17** (1989), 1955–1969.

18. H.C. Hutchins, *Examples of commutative rings*, Polygonal Publishing House, Passaic, NJ, 1981.

19. B.G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra **123** (1989), 151–170.

20. I. Kaplansky, *Commutative rings*, rev. ed., Univ. of Chicago Press, Chicago, 1974.

21. T.Y. Lam, *Lectures on modules and rings*, Springer, New York, 1999.

22. T.G. Lucas, *Examples built with $D + M, A + XB[X]$, and other pullback constructions*, in *Non-Noetherian ring theory*, Math. Appl., Kluwer Acad. Publ., Dordrecht, 2000, pp. 341–368.

23. M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. **51** (1985), 29–62.

24. ———, *Putting t-invertibility to use*, in *Non-Noetherian ring theory*, Math. Appl., Kluwer Acad. Publ., Dordrecht, 2000, pp. 429–457.

25. ———, *Facets of Rings between $D[X]$ and $K[X]$*, Lecture Notes in Pure and Appl. Math., vol. 231, Dekker, New York, 2003, pp. 445–460.

Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300
E-mail address: anderson@math.utk.edu

Department of Mathematics, University of Incheon, Incheon, Korea 402-749
E-mail address: whan@incheon.ac.kr

Department of Mathematics, Inha University, Incheon, Korea 402-751
E-mail address: jnpark@inha.ac.kr