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FULLY AND STRONGLY ALMOST SUMMING MULTILINEAR MAPPINGS

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ABSTRACT. In this paper we generalize a theorem of Kwapień which asserts that a linear operator T is absolutely $(1;1)$-summing whenever T^* is absolutely $(q;q)$-summing for some $q \geq 1$. We also introduce the classes of strongly and fully almost summing multilinear mappings and investigate structural properties such as a Dvoretzky-Rogers type theorem and connections with other classes of absolutely summing mappings.

1. Introduction. The success of the theory of absolutely summing linear operators has motivated the investigation of new classes of multilinear mappings and polynomials between Banach spaces. The first possible directions of a multilinear theory of absolutely summing multilinear mappings were outlined by Pietsch [**15**] and several related concepts have been exhaustively studied by several authors. Recently a question of Pietsch about Hilbert-Schmidt multilinear mappings was answered by Matos in [**8**] and this work motivated the study of a new class of multilinear mappings, called the space of fully absolutely summing multilinear mappings, see [**9, 16, 17**].

The concept of almost summing operators was first considered for the multilinear and polynomial cases by Botelho [**3**] and Botelho-Braunss-Junek [**4**]. In [**12**] and [**13**] it is proved that whenever $n \geq 2$ and $E_1, \dots, E_n$ are $\mathcal{L}_\infty$-spaces, every continuous n-linear mapping from $E_1 \times \cdots \times E_n$ into any Banach space F is almost 2-summing, showing that coincidence results for almost summing mappings are much more common than was known. These coincidental results motivate the study of other natural directions for extending the concepts of almost summing linear operators to polynomial and multilinear mappings. Our first definition leads us to the space of strongly almost summing

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mappings which is strictly contained in the space of almost summing mappings. Among other results, we will show that every continuous scalar valued bilinear mapping defined on $\mathcal{L}_\infty$-spaces is strongly almost 2-summing, generalizing a result of Botelho [**3**] about almost summing bilinear mappings. The second definition we will work with, inspired in [**9**], creates the space of fully almost summing mappings and furnishes some new other interesting results.

Throughout, $E, E_1, \dots, E_n, F$ will stand for (real or complex) Banach spaces. If $2 \le q \le \infty$ and $(r_j)_{j=1}^\infty$ are the Rademacher functions, we say that E has cotype q if there exists $C \ge 0$ such that for any $k \in \mathbf{N}$ and $x_1, \dots, x_k \in E$,

$$\left(\sum_{j=1}^k \|x_j\|^q \right)^{1/q} \le C \left(\int_0^1 \Big\| \sum_{j=1}^k r_j(t) x_j \Big\|^2 dt \right)^{1/2}.$$

The infimum of the C is denoted by $C_q(E)$. To cover the case $q = \infty$ we replace $(\sum_{j=1}^k \|x_j\|^q)^{1/q}$ by $\max_{j \le k} \|x_j\|$. If $1 \le q \le 2$, we say that E has type q if there exists $C \ge 0$ such that for any $k \in \mathbf{N}$ and $x_1, \dots, x_k \in E$,

$$\left(\int_0^1 \Big\| \sum_{j=1}^k r_j(t) x_j \Big\|^2 dt \right)^{1/2} \le C \left(\sum_{j=1}^k \|x_j\|^q \right)^{1/q}$$

The infimum of the C is denoted by $T_q(E)$.

For $p \in \,]0, \infty[$, the space of all $(x_j)_{j=1}^\infty$ in E such that $(\langle \varphi, x_j \rangle)_{j=1}^\infty \in l_p$ for every continuous linear functional $\varphi : E \to \mathbf{K}$ will be denoted by $l_p^w(E)$. We define $\|.\|_{w,p}$ in $l_p^w(E)$ by $\|(x_j)_{j \in \mathbf{N}}\|_{w,p} = \sup_{\varphi \in B_{E'}} \|(\langle \varphi, x_j \rangle)_{j \in \mathbf{N}}\|_p$.

The following concept of absolutely summing multilinear mappings is a natural generalization of the definition of absolutely summing operators and has been explored by several authors, cf. [**2, 7, 10–12**].

Definition 1 (Alencar-Matos [**1**]). If $p, q_1, \dots, q_n \in \,]0, \infty[$, a continuous multilinear mapping $T : E_1 \times \cdots \times E_n \to F$ is absolutely $(p; q_1, \dots, q_n)$-summing, or $(p; q_1, \dots, q_n)$-summing, if there exists $C > 0$ such that

$$\left(\sum_{j=1}^{\infty} \|T(x_j^{(1)}, \dots, x_j^{(n)})\|^p\right)^{1/p} \leq C \prod_{r=1}^{n} \|(x_j^{(r)})_{j=1}^{\infty}\|_{w,q_r} \quad \forall\, (x_j^{(k)})_{j=1}^{\infty} \in l_{q_k}^w(E_k). \tag{1.1}$$

In order to avoid trivialities we assume that $1/p \leq 1/q_1 + \dots + 1/q_n$. Henceforth we will denote the space of all absolutely $(p; q_1, \dots, q_n)$-summing n-linear mappings from $E_1 \times \dots \times E_n$ into F by the symbol $\mathcal{L}_{as(p;q_1,\dots,q_n)}(E_1, \dots, E_n; F)$.

The infimum of the $C > 0$ for which (1.1) holds defines a norm (p norm if $p < 1$) on the space of all absolutely $(p; q_1, \dots, q_n)$-summing multilinear mappings. This norm is denoted by $\|.\|_{as(p;q_1,\dots,q_n)}$ and $\mathcal{L}_{as(p;q_1,\dots,q_n)}(E_1, \dots, E_n; F)$ endowed with this norm is a Banach space. When $q_1 = \dots = q_n = q$, we write $\mathcal{L}_{as(p;q)}(E_1, \dots, E_n; F)$. If $1/p = 1/q_1 + \dots + 1/q_n$, we denote the $(p; q_1, \dots, q_n)$-summing multilinear mappings by $\mathcal{L}_{d(q_1,\dots,q_n)}(E_1, \dots, E_n; F)$, and these mappings are called p-dominated and constitute an important particular case due to the strong analogy with the linear case.

2. A multilinear version for a theorem of Kwapień. In this section we present an interesting generalization of the following theorem, due to Kwapień.

Theorem 1 (Kwapień [**6**]). *Let X be a Banach space and H a Hilbert space. If $u \in \mathcal{L}(X; H)$ is such that u^* is q-summing for some $1 \leq q < \infty$, then u is 1-summing and $\|u\|_{as,1} \leq A_1^{-1} B_q \|u^*\|_{as,q}$, where A_1 and B_q are the constants of Khinchin's inequality.*

The adjoint of $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is defined by

$$T^* : F^* \longrightarrow \mathcal{L}(E_1, \dots, E_n; \mathbf{K})$$
$$\varphi \longrightarrow T^*\varphi : E_1 \times \dots \times E_n \longrightarrow \mathbf{K}$$

with $(T^*\varphi)(x_1, \dots, x_n) = \varphi(T(x_1, \dots, x_n))$.

Theorem 2. *If $E_1, \dots, E_N$ are Banach spaces, H is a Hilbert space and*

$$T \in \mathcal{L}(E_1, \dots, E_N; H)$$

is such that T^* is almost 2-summing, then T is absolutely $(1;1)$-summing and

$$\|T\|_{as(1;1)} \leq A_1^{-1} \|T^*\|_{al,2}.$$

Proof. We first consider the case of an operator $T: E_1 \times \cdots \times E_N \to l_2^n$, $n \in \mathbf{N}$.

Consider $x^{(k,1)}, \dots, x^{(k,m)} \in E_k$, $1 \leq k \leq N$. Call on Khinchin's inequality, see [**5**, Theorem 1.10], to obtain

$$\begin{aligned}
&\sum_{j=1}^{m} \left\| T(x^{(1,j)}, \dots, x^{(N,j)}) \right\| \\
&\qquad = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \left| \left\langle T(x^{(1,j)}, \dots, x^{(N,j)}), e_k \right\rangle \right|^2 \right)^{1/2} \\
&\qquad = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \left| \left\langle (x^{(1,j)}, \dots, x^{(N,j)}), T^* e_k \right\rangle \right|^2 \right)^{1/2} \\
&\qquad \leq \sum_{j=1}^{m} \left[A_1^{-1} \left(\int_0^1 \left| \sum_{k=1}^{n} \left\langle (x^{(1,j)}, \dots, x^{(N,j)}), T^* e_k \right\rangle r_k(t) \right| dt \right) \right] \\
&\qquad = A_1^{-1} \int_0^1 \sum_{j=1}^{m} \left| \left\langle (x^{(1,j)}, \dots, x^{(N,j)}), \sum_{k=1}^{n} r_k(t) T^* e_k \right\rangle \right| dt \\
&\qquad \leq A_1^{-1} \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|_{as(1;1)} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1} dt.
\end{aligned}$$

Thus, since $\mathcal{L}(E_1, \dots, E_N; \mathbf{K}) = \mathcal{L}_{as(1;1)}(E_1, \dots, E_N; \mathbf{K})$ holds isometrically, we also obtain

$$\begin{aligned}
\sum_{j=1}^{m} \left\| T(x^{(1,j)}, \dots, x^{(N,j)}) \right\| &\leq A_1^{-1} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1} \\
&\qquad \times \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\| dt
\end{aligned} \tag{2.1}$$

and, on the other hand, since T^* is almost summing we have
(2.2)

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)T^*e_k\right\|^2 dt\right)^{1/2} \leq \|T^*\|_{al,2}\,\|(e_k)_{k=1}^n\|_{w,2} = \|T^*\|_{al,2}\,.$$

Now consider $T \in \mathcal{L}(E_1,\dots,E_N;H)$ whose adjoint T^* is almost summing.

If $x^{(k,1)},\dots,x^{(k,m)} \in E_k$, $1 \leq k \leq N$, identify the span of the $T(x^{(1,j)},\dots,x^{(N,j)})'s$, $j=1,\dots,m$ with l_2^n for an appropriate n and define by Ψ such identification. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto this span. We have $P^*=P$ and by (2.1) and (2.2), we obtain

$$\begin{aligned}\sum_{j=1}^m &\left\|T(x^{(1,j)},\dots,x^{(N,j)})\right\|\\ &= \sum_{j=1}^m \left\|\Psi\circ P\circ T(x^{(1,j)},\dots,x^{(N,j)})\right\|\\ &\leq A_1^{-1}\,\|T^*\circ P^*\circ\Psi^*\|_{al,2}\prod_{i=1}^N \left\|(x^{(i,j)})_{j=1}^m\right\|_{w,1}\\ &\leq A_1^{-1}\,\|T^*\|_{al,2}\,\|P^*\|\,\|\Psi^*\|\prod_{i=1}^N \left\|(x^{(i,j)})_{j=1}^m\right\|_{w,1}\\ &\leq A_1^{-1}\,\|T^*\|_{al,2}\,\|P\|\,\|\Psi\|\prod_{i=1}^N \left\|(x^{(i,j)})_{j=1}^m\right\|_{w,1}\\ &= A_1^{-1}\,\|T^*\|_{al,2}\prod_{i=1}^N \left\|(x^{(i,j)})_{j=1}^m\right\|_{w,1}.\end{aligned}$$

Therefore, T is absolutely $(1;1)$-summing and $\|T\|_{as(1;1,\dots,1)} \leq A_1^{-1}\,\|T^*\|_{al,2}$. □

3. Almost and strongly almost summing multilinear mappings. The first attempts to a concept of almost summability for polynomials and multilinear mappings are due to Botelho [**3**] and Botelho-Braunss-Junek [**4**].

Definition 2 (Botelho-Braunss-Junek [**4**]). If $p_1, \dots , p_n \geq 1$, a continuous n-linear mapping $T : E_1 \times \cdots \times E_n \to F$ is said to be almost $(p_1, \dots , p_n)$-summing if there exists $C \geq 0$ such that

$$\left(\int_0^1 \| \sum_{j=1}^{k} T(x_j^{(1)}, \dots , x_j^{(n)}) r_j(t) \|^2 \, dt \right)^{1/2} \leq C \prod_{r=1}^{n} \| (x_j^{(r)})_{j=1}^{k} \|_{w,p_r}$$

for every k and any $x_j^{(l)}$ in $E_l, l = 1, \dots , n$ and $j = 1, \dots , k$.

The space of all almost $(p_1, \dots , p_n)$-summing multilinear mappings from $E_1 \times \cdots \times E_n$ into F will be denoted by $\mathcal{L}_{al(p_1,\dots,p_n)}(E_1, \dots , E_n; F)$. When $p_1 = \cdots = p_n = p$ we write $\mathcal{L}_{al,p}(E_1, \dots , E_n; F)$.

The infimum of the $C > 0$ for which last inequality holds defines a norm and turns the space of all almost $(p_1, \dots , p_n)$-summing multilinear mappings a Banach space.

The first nontrivial coincidence result for almost summing mappings is due to Botelho [**3**] and asserts that every scalar valued bilinear mapping defined on $\mathcal{L}_\infty$-spaces is almost 2-summing. Further recent work of the first named author showed other coincidence situations:

Theorem 3 (Pellegrino [**12, 13**]). *If $n \geq 2$ and E is an $\mathcal{L}_\infty$-space, then*

$$\mathcal{L}(^n E; F) = \mathcal{L}_{al,2}(^n E; F),$$

regardless of the Banach space F.

As we have mentioned, motivated by these several coincidence theorems, we will give a more restrictive concept, related to the definition of almost summing mappings and next we will show that we still have nontrivial coincidence results in this new situation.

Definition 3. A continuous n-linear mapping is said to be strongly almost $(q_1, \ldots, q_n)$-summing if there exists $C > 0$ such that

$$\left(\int_0^1 \left\| \sum_{j_1,\ldots,j_n=1}^{k} T(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}) r_{\pi(j_1,\ldots j_n)}(t) \right\|^2 dt \right)^{1/2} \leq C \prod_{r=1}^{n} \left\| (x_j^{(r)})_{j=1}^{k} \right\|_{w,q_r}$$

for every k, where π is any injection from $\mathbf{N} \times \cdots \times \mathbf{N}$ into $\mathbf{N}$.

It is important to observe that the particular choice of π is irrelevant. The linear space composed by the n-linear strongly almost $(q_1, \ldots, q_n)$-summing mappings from $E_1 \times \cdots \times E_n$ into F is denoted by $\mathcal{L}_{sal(q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)$. When $q_1 = \cdots = q_n = q$ we denote by $\mathcal{L}_{sal,q}(E_1, \ldots, E_n; F)$.

One can verify some analogy with the linear definition of almost summing operators since, as in the linear case, if $p > 2$ the only n-linear mapping which is strongly almost p-summing is the trivial mapping.

Proposition 1. *If $p > 2$, the unique multilinear mapping which is strongly almost $(p, \ldots, p)$-summing is the null mapping.*

Proof. If $T \in \mathcal{L}_{sal,p}(^m E; F)$, then

$$\begin{aligned}
\left(\int_0^1 \left\| \sum_{j_1,\ldots,j_m=1}^{n} T(x, \ldots, x) r_{\pi(j_1,\ldots,j_m)}(t) \right\|^2 dt \right)^{1/2}
&= \|T(x, \ldots, x)\| \left(\int_0^1 \left| \sum_{j_1 \ldots j_m=1}^{n} r_{\pi(j_1,\ldots,j_m)}(t) \right|^2 dt \right)^{1/2} \\
&= \|T(x, \ldots, x)\| \left(\int_0^1 \left| \sum_{j=1}^{n^m} r_j(t) \right|^2 dt \right)^{1/2} \\
&\geq C_2(\mathbf{K})^{-1} \|T(x, \ldots, x)\| \left(\left| \sum_{j=1}^{n^m} 1^2 \right| \right)^{1/2} \\
&= C_2(\mathbf{K})^{-1} \|T(x, \ldots, x)\| \, n^{m/2}.
\end{aligned}$$

Thus, since T is strongly almost $(p, \dots, p)$-summing, we will be able to find $C > 0$ such that

$$n^{m/2} \|T(x, \dots, x)\| \leq C \left\|(x)_{j=1}^{n}\right\|_{w,p}^{m} = C \|x\|^{m} n^{m/p}.$$

Therefore

$$\|T\| \leq C n^{m/p - m/2} \quad \forall\, n \in \mathbf{N}.$$

Making $n \to \infty$, we have $\|T\| = 0$, whenever $p > 2$. □

One can also check that if $\dim E < \infty$ and $p \leq 2$, then $\mathcal{L}_{sal,p}({}^{n}E; E) = \mathcal{L}({}^{n}E; E)$, and this fact is a first indication that one can expect a Dvoretzky-Rogers Theorem for strongly almost summing mappings.

As immediate outcome of the contraction principle, see [**5**], we can justify the denomination "strongly" in our definition by observing that every strongly almost $(q_1, \dots, q_n)$-summing n-linear mapping is almost $(q_1, \dots, q_n)$-summing. Since the random variables on Definition 3 are still independent and symmetric, we can invoke the concepts of type and cotype and obtain some natural connections. Firstly, we need some definitions.

Definition 4 (Matos [**9**]). A continuous n-linear mapping $T : E_1 \times \cdots \times E_n \to F$ is said to be fully $(p; q_1, \dots, q_n)$-summing if there exists $C \geq 0$ such that

$$\left(\sum_{j_1, \dots, j_n = 1}^{\infty} \left\| T(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)}) \right\|^{p} \right)^{1/p} \leq C \prod_{l=1}^{n} \|(x_j^{(l)})_{j=1}^{\infty}\|_{w,q_l}$$
$$\forall\, (x_k^{(l)})_{k=1}^{\infty} \in l_{q_l}^{w}(E_l).$$

In this case we write $T \in \mathcal{L}_{fas(p; q_1, \dots, q_n)}(E_1, \dots, E_n; F)$. The infimum of the C is denoted by $\|.\|_{fas(p; q_1, \dots, q_n)}$.

Several results about fully summing mappings can be found in [**9, 16, 17**]. Now the same standard reasoning used for almost summing mappings, see [**3**], can be analogously used in order to obtain the following proposition:

Proposition 2. *If F has finite cotype q, then every F-valued strongly almost $(p_1,\dots,p_n)$-summing multilinear mapping is fully $(q;p_1,\dots,p_n)$-summing. On the other hand, if F has type q, then every fully $(q;p_1,\dots,p_n)$-summing multilinear mapping is strongly almost $(p_1,\dots,p_n)$-summing. In particular, if F is a Hilbert space, then*

$$\mathcal{L}_{fas(2;2,\dots,2)}(^nE;F) = \mathcal{L}_{sal,2}(^nE;F). \tag{3.1}$$

The next corollary is a generalization of Theorem 7.1 of [**3**].

Corollary 1. *If E is an $\mathcal{L}_\infty$-space, then $\mathcal{L}_{sal,2}(^2E;\mathbf{K}) = \mathcal{L}(^2E;\mathbf{K})$.*

Proof. Since every scalar valued continuous bilinear mapping defined on $\mathcal{L}_\infty$-spaces is 2-dominated, see [**2**], and since $\mathcal{L}_{d(2,2)}(^2E;\mathbf{K}) \subset \mathcal{L}_{fas(2;2,2)}(^2E;\mathbf{K})$, see [**13**], then, by (3.1), $\mathcal{L}_{sal,2}(^2E;\mathbf{K}) = \mathcal{L}(^2E;\mathbf{K})$. □

We also have some structural properties, such as:

Proposition 3. *If every continuous n-linear mapping $T : E_1 \times \cdots \times E_n \to F$ is strongly almost $(q_1,\dots,q_n)$-summing, then every continuous r-linear $T : E_{j_1} \times \cdots \times E_{j_r} \to F$ is strongly almost $(q_{j_1},\dots,q_{j_r})$-summing, where $1 \le r \le n$, $j_1,\dots,j_r \in \{1,\dots,n\}$ and $j_t \neq j_s$ if $t \neq s$.*

If $p > 1$ and $\dim E = \infty$, we know that $\mathcal{L}_{al,p}(E;E) \neq \mathcal{L}(E;E)$ [**4**, Example 4]. As a corollary of this result and Proposition 3, we have a Dvoretzky-Rogers theorem for strongly almost summing mappings.

Corollary 2. *If $1 < p \le 2$ we have $\mathcal{L}_{sal,p}(^nE;E) = \mathcal{L}(^nE;E) \Leftrightarrow \dim E < \infty$.*

We can observe that, despite the fact that the definition of strongly almost summing mappings is restrictive, we do not have to look further to find examples of such mappings. A simple computation asserts, for example, that if $u : E \to F$ is an almost p-summing linear mapping and φ is a continuous linear functional, then

$$T : E \times E \longrightarrow F : T(x, y) = u(x)\varphi(y)$$

is strongly almost $(p, 2)$-summing.

4. Fully almost summing mappings. The next concept, suggested by Matos, is also natural and furnishes various interesting consequences.

Definition 5. A continuous n-linear mapping T is fully almost $(p; p_1, \dots, p_n)$-summing if there exists $C > 0$ such that

$$\left(\int_I \left\| \sum_{j_1,\dots j_n=1}^{k} T(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)}) \prod_{s=1}^{n} r_{j_s}(t_s) \right\|^p d\lambda \right)^{1/p} \leq C \prod_{r=1}^{n} \left\| (x_j^{(r)})_{j=1}^k \right\|_{w,p_r}$$

for every natural k, where λ is the Lebesgue measure over the Borel sets of $I = [0, 1]^n$.

The linear space of all fully almost $(p; p_1, \dots, p_n)$-summing n-linear mappings from $E_1 \times \cdots \times E_n$ into F will be represented by the symbol $\mathcal{L}_{fal(p;p_1,\dots,p_n)}(E_1, \dots, E_n; F)$. The infimum of the constants C is denoted by $\|.\|_{fal(p;p_1,\dots,p_n)}$. In the case $p_1 = \cdots = p_n = q$ we write $\mathcal{L}_{fal(p;q)}(E_1, \dots, E_n; F)$ and $\|.\|_{fal(p;q)}$.

It must be mentioned that we are no longer able to explore type and cotype as we did in last section, since we do not have independent random variables anymore. In fact it is not hard to see that the random variables

$$r_{jk} : [0, 1]^2 \longrightarrow [0, 1] : r_{jk}(t, s) = r_j(t) r_k(s)$$

are not independent since $\lambda(r_{11}^{-1}(1) \cap r_{12}^{-1}(1) \cap r_{21}^{-1}(1) \cap r_{22}^{-1}(-1)) = 0$, whereas

$$\lambda(r_{11}^{-1}(1)).\lambda(r_{12}^{-1}(1)).\lambda(r_{21}^{-1}(1)).\lambda(r_{22}^{-1}(-1)) = 1/8,$$

where λ denotes the Lebesgue measure over the Borel sets of $[0, 1]^2$.

In order to obtain nontrivial examples of n-linear fully almost $(p;q)$-summing mappings we must have $q \leq 2$, since it can be proved that if $q > 2$ and $T \in \mathcal{L}_{fal(p;q)}(^nE;F)$ for some $n \in \mathbf{N}$, then $T = 0$.

The following property shows more similarity with the definition of strongly almost summing mappings.

Proposition 4. *If* $\mathcal{L}_{fal(p;p_1,\dots,p_n)}(E_1,\dots,E_n;F) = \mathcal{L}(E_1,\dots,E_n;F)$, *then*

$$\mathcal{L}_{fal(p;p_{k_1},\dots,p_{k_j})}\left(E_{k_1},\dots,E_{k_j};F\right) = \mathcal{L}\left(E_{k_1},\dots,E_{k_j};F\right)$$

whenever $k_r \in \{1,\dots,n\}$, *with* $1 \leq r \leq n$ *and* $k_r \neq k_s$ *if* $r \neq s$.

Proof. The case $n = 2$ is illustrative. If $T \in \mathcal{L}(E_1;F)$, we must show that $T \in \mathcal{L}_{fal(p;p_1)}(E_1;F)$. Let us consider $\varphi \in E'$ and $a \in E$ such that $\varphi(a) = 1$. Define

$$\begin{gathered} R : E_1 \times E_2 \longrightarrow F \\ (x,y) \longrightarrow R(x,y) = T(x)\,\varphi(y)\,. \end{gathered}$$

Since $R \in \mathcal{L}(E_1,E_2;F)$, then by hypothesis $R \in \mathcal{L}_{fal(p;p_1,p_2)}(E_1,E_2;F)$ and, making $y_1 = a$, $y_2 = y_3 = \cdots = 0$, we get

$$\begin{aligned} &\int_0^1 \left\| \sum\nolimits_{j=1}^m r_j(t) T(x_j) \right\|^p dt \\ &\qquad = \int_0^1 \left\| \sum\nolimits_{j=1}^m r_j(t) R(x_j,a) \right\|^p dt \\ &\qquad = \int_0^1 \int_0^1 \left\| \sum\nolimits_{j,k=1}^m r_j(t) R(x_j,y_k) \right\|^p dt\,d\theta \\ &\qquad = \int_0^{1/2} \int_0^1 \left\| \sum\nolimits_{j,k=1}^m r_j(t) r_k(\theta)\, R(x_j,y_k) \right\|^p dt\,d\theta \\ &\qquad\quad + \int_{1/2}^1 \int_0^1 \left\| \sum\nolimits_{j,k=1}^m r_j(t) r_k(\theta)\, R(x_j,y_k) \right\|^p dt\,d\theta \\ &\qquad = \int_0^1 \int_0^1 \left\| \sum\nolimits_{j,k=1}^m r_j(t) r_k(\theta)\, R(x_j,y_k) \right\|^p dt\,d\theta \\ &\qquad \leq \|R\|_{fal(p;p_1,p_2)}^p \left\| (x_j)_{j=1}^m \right\|_{w,p_1}^p \|a\|^p\,. \end{aligned}$$

This shows that $T \in \mathcal{L}_{fal(p;p_1)}(E_1;F)$. The same reasoning furnishes

$$\mathcal{L}_{fal(p;p_2)}(E_2;F) = \mathcal{L}(E_2;F). \qquad \square$$

It also can be checked that every finite type multilinear mapping is fully almost $(p;2)$-summing and an adequate use of the Rademacher functions furnishes

$$\mathcal{L}(E_1,\dots,E_n;F) = \mathcal{L}_{fal(p;1,\dots,1)}(E_1,\dots,E_n;F),$$

for every $0 < p < \infty$.

The next theorem asserts that, similarly to a result for almost summing mappings (see [**3**, Theorem 4.1]), we have an inclusion theorem concerning dominated mappings and fully almost summing mappings.

Theorem 4. *Let $E_1,\dots,E_n$, F be Banach spaces. If $r, r_1,\dots,r_n \in]0,\infty]$, with $1/r = 1/r_1 + \cdots + 1/r_n$, then*

$$\mathcal{L}_{d(r_1,\dots,r_n)}(E_1,\dots,E_n;F) \subset \mathcal{L}_{fal(r;2)}(E_1,\dots,E_n;F).$$

Proof. Let us consider $T \in \mathcal{L}_{d(r_1,\dots,r_n)}(E_1,\dots,E_n;F)$ and $x_j^k \in E_k$, with $j = 1,\dots,m$ and $k = 1,\dots,n$. Denoting $\|T\|_{d(r_1,\dots,r_n)} = \|T\|_d$, and by applying the Grothendieck-Pietsch domination theorem for multilinear mappings, we obtain:

$$\begin{aligned}
&\left(\int_I \left\|\sum\nolimits_{j_1,\cdots,j_n=1}^m r_{j_1}(t_1)\dots r_{j_n}(t_n)T(x_{j_1}^1,\dots,x_{j_n}^n)\right\|^r d\lambda\right)^{1/r} \\
&\quad = \left(\int_I \left\|T\left(\sum\nolimits_{j_1=1}^m r_{j_1}(t_1)x_{j_1}^1,\dots,\sum\nolimits_{j_n=1}^m r_{j_n}(t_n)x_{j_n}^n\right)\right\|^r d\lambda\right)^{1/r} \\
&\quad \le \left\{\int_I \left\|T\right\|_d^r \prod\nolimits_{k=1}^n \left[\int_{B_{E'_k}} \left|\varphi_k\left(\sum\nolimits_{j_k=1}^m r_{j_k}(t_k)x_{j_k}^k\right)\right|^{r_k}\right.\right. \\
&\qquad\qquad\qquad \left.\left. \cdot\, d\mu_k(\varphi_k)\right]^{r/r_k} d\lambda\right\}^{1/r} \\
&\quad = \|T\|_d \left\{\int_I \prod\nolimits_{k=1}^n \left[\int_{B_{E'_k}} \left|\varphi_k\left(\sum\nolimits_{j_k=1}^m r_{j_k}(t_k)x_{j_k}^k\right)\right|^{r_k}\right.\right. \\
&\qquad\qquad\qquad \left.\left. \cdot\, d\mu_k(\varphi_k)\right]^{r/r_k} d\lambda\right\}^{1/r} = \bigotimes
\end{aligned}$$

Since $1/r = 1/r_1 + \cdots + 1/r_n$, denoting

$$C_k(t_k) = \left[\int_{B_{E'_k}} \left| \varphi_k \left(\sum\nolimits_{j_k=1}^{m} r_{j_k}(t_k) x_{j_k}^k \right) \right|^{r_k} d\mu_k \right]^{r/r_k}$$

and invoking the Hölder inequality, we have

$$\begin{aligned}
&\int_I \prod\nolimits_{k=1}^{n} C_k(t_k)\, d\lambda \\
&\qquad \leq \prod\nolimits_{k=1}^{n} \left[\int_I |C_k(t_k)|^{r_k/r}\, d\lambda \right]^{r/r_k} \\
&\qquad = \prod\nolimits_{k=1}^{n} \left[\int_I \int_{B_{E'_k}} \left| \varphi_k \left(\sum\nolimits_{j_k=1}^{m} r_{j_k}(t_k) x_{j_k}^k \right) \right|^{r_k} d\mu_k\, d\lambda \right]^{r/r_k} \\
&\qquad = \prod\nolimits_{k=1}^{n} \left[\int_{B_{E'_k}} \int_0^1 \left| \varphi_k \left(\sum\nolimits_{j_k=1}^{m} r_{j_k}(t_k) x_{j_k}^k \right) \right|^{r_k} dt_k\, d\mu_k \right]^{r/r_k}.
\end{aligned}$$

Replacing the above inequality in $\bigotimes$, and by applying Khinchin's inequality, we have

$$\begin{aligned}
\bigotimes &\leq \|T\|_d \prod\nolimits_{k=1}^{n} \left[\int_{B_{E'_k}} \int_0^1 \left| \varphi_k \left(\sum\nolimits_{j_k=1}^{m} r_{j_k}(t_k) x_{j_k}^k \right) \right|^{r_k} dt_k\, d\mu_k \right]^{1/r_k} \\
&\leq \|T\|_d \prod\nolimits_{k=1}^{n} (B_{r_k}) \left\{ \int_{B_{E'_k}} \left(\sum\nolimits_{j_k=1}^{m} \left| \langle \varphi_k, x_{j_k}^k \rangle \right|^2 \right)^{r_k/2} d\mu_k \right\}^{1/r_k} \\
&\leq \|T\|_d \prod\nolimits_{k=1}^{n} (B_{r_k}) \|(x_j^k)_{j=1}^m\|_{w,2}.
\end{aligned}$$

Hence, $T \in \mathcal{L}_{fal(r;2)}(E_1, \ldots, E_n; F)$ and $\|T\|_{fal(r;2)} \leq \|T\|_d \prod_{k=1}^{n} (B_{r_k})$, where the B_{r_k} are the constants of Khinchin's inequality. □

Proposition 5. *If $1 \leq p \leq 2$ and F has type p, then*

$$\mathcal{L}_{fas(p;q_1,\ldots,q_n)}(E_1, \ldots, E_n; F) \subset \mathcal{L}_{fal(p;q_1,\ldots,q_n)}(E_1, \ldots, E_n; F).$$

Proof. The case $n = 2$ is illustrative. The other cases are analogous.

Given $T \in \mathcal{L}_{fas(p;q_1,q_2)}(E_1, E_2; F)$, $x_j^k \in E_k$, $j = 1, \ldots, m$ and $k = 1, 2$, we have

$$\begin{aligned}
&\left(\int_0^1 \int_0^1 \left\| \sum\nolimits_{j_1,j_2=1}^m r_{j_1}(t_1) r_{j_2}(t_2) T(x_{j_1}^1, x_{j_2}^2) \right\|^p dt_1\, dt_2 \right)^{1/p} \\
&\leq \left[\int_0^1 \left(\int_0^1 \left\| \sum\nolimits_{j_1=1}^m r_{j_1}(t_1) \sum\nolimits_{j_2=1}^m r_{j_2}(t_2) T(x_{j_1}^1, x_{j_2}^2) \right\|^2 dt_1 \right)^{p/2} dt_2 \right]^{1/p} \\
&\leq T_p(F) \left[\int_0^1 \left(\sum\nolimits_{j_1=1}^m \left\| \sum\nolimits_{j_2=1}^m r_{j_2}(t_2) T(x_{j_1}^1, x_{j_2}^2) \right\|^p \right)^{p/p} dt_2 \right]^{1/p} \\
&= T_p(F) \left[\sum\nolimits_{j_1=1}^m \int_0^1 \left\| \sum\nolimits_{j_2=1}^m r_{j_2}(t_2) T(x_{j_1}^1, x_{j_2}^2) \right\|^p dt_2 \right]^{1/p} \\
&\leq T_p(F) \left[\sum\nolimits_{j_1=1}^m \left(\int_0^1 \left\| \sum\nolimits_{j_2=1}^m r_{j_2}(t_2) T(x_{j_1}^1, x_{j_2}^2) \right\|^2 dt_2 \right)^{p/2} \right]^{1/p} \\
&\leq T_p(F)^2 \left[\sum\nolimits_{j_1=1}^m \left(\sum\nolimits_{j_2=1}^m \left\| T(x_{j_1}^1, x_{j_2}^2) \right\|^p \right)^{p/p} \right]^{1/p} \\
&\leq T_p(F)^2 \, \|T\|_{fas(p;q_1,q_2)} \, \|(x_j^1)_{j=1}^m\|_{w,q_1} \|(x_j^2)_{j=1}^m\|_{w,q_2}. \qquad \square
\end{aligned}$$

Corollary 3. *If E_j is an $\mathcal{L}_\infty$ space, $j = 1, \ldots, n$ and F has type 2 and cotype 2, then*

$$\mathcal{L}(E_1, \ldots, E_n; F) = \mathcal{L}_{fal(2;2)}(E_1, \ldots, E_n; F)$$

Proof. It suffices to use the last proposition and observe that

$$\mathcal{L}(E_1, \ldots, E_n; F) = \mathcal{L}_{fas(2;2)}(E_1, \ldots, E_n; F)$$

for such spaces, see [**14**]. $\square$

Proceeding as in the proof of the last proposition, one can also obtain:

Proposition 6. *If F has finite cotype p, then*

$$\mathcal{L}_{fal(p;q_1,\ldots,q_n)}(E_1, \ldots, E_n; F) \subset \mathcal{L}_{fas(p;q_1,\ldots,q_n)}(E_1, \ldots, E_n; F).$$

Observe that a direct consequence of Theorem 4 and Proposition 6 give us the following result, which generalizes Theorem 3.15 of [**5**]:

Corollary 4. *Let* $E_1, \dots, E_n$ *and* F *be Banach spaces. If* F *has cotype* $q < \infty$, $q_1, \dots, q_n \in]0, \infty]$ *and* $1/q = 1/q_1 + \dots + 1/q_n$, *then*

$$\mathcal{L}_{d(q_1,\dots,q_n)}(E_1, \dots, E_n; F) \subset \mathcal{L}_{fas(q;2)}(E_1, \dots, E_n; F).$$

This result was obtained recently, and independently, by Pérez-Garcia and Villanueva in [**14**, Theorem 3.10].

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