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ASYMPTOTIC BEHAVIOR OF PERIODIC COMPETITION DIFFUSION SYSTEM

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ABSTRACT. In this paper, we consider the existence and attraction of positive periodic solution of a competition diffusion system. We first construct a pair of upper and lower solutions, then use the periodic comparison existence theorem to get a pair of T-periodic solutions $(\overline{u}, \underline{v})$ and $(\underline{u}, \overline{v})$. Finally we give a sufficient condition of $(\overline{u}, \underline{v}) = (\underline{u}, \overline{v})$ to answer the open question described by Ahmad and Lazer.

1. Introduction. The periodic competition diffusion system with no-flux boundary conditions

(1.1)

$$\begin{aligned} &u_t = \Delta u + u[a(x,t) - b(x,t)u - c(x,t)v], \\ &v_t = \Delta v + v[d(x,t) - e(x,t)u - f(x,t)v], \end{aligned} \qquad (x,t) \in \Omega \times [0,+\infty),$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times [0,+\infty),$$

models the two species competition diffusion phenomena in a periodic changing environment, the coefficients $a(x,t), b(x,t), \dots, f(x,t)$ are sufficiently smooth functions defined on a cylinder $\Omega \times [0,+\infty)$, where Ω is a smooth bounded domain in R^n. We assume that $a(x,t), \dots, f(x,t)$ are strictly positive and periodic in the time variable t with period $T > 0$, and set

$$a_L = \min_{\overline{\Omega}\times[0,T]} a(x,t), \qquad a_M = \max_{\overline{\Omega}\times[0,T]} a(x,t), \dots,$$
$$f_L = \min_{\overline{\Omega}\times[0,T]} f(x,t), \qquad f_M = \max_{\overline{\Omega}\times[0,T]} f(x,t).$$

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Recently there have been investigations [**1**–**8**] concerned with the periodic boundary value problem (1.1) with the periodic conditions

$$u(x,t+T)=u(x,t), \qquad v(x,t+T)=v(x,t), \quad (x,t)\in\Omega\times(0,+\infty); \tag{1.2}$$

and the initial boundary value problem (1.1) with the initial conditions

$$u(x,0)=\varphi(x), \qquad v(x,0)=\psi(x), \quad x\in\overline{\Omega}. \tag{1.3}$$

If the coefficients $a,b,\dots,f$ are independent of the space variable x, Tineo and Rivero [**7**] used an iterative monotone scheme to prove that there exists a spatially homogeneous positive periodic solution of the problem (1.1)–(1.2), and this solution is a global attractor of the problem (1.1), (1.3) for the nonnegative nontrivial initial data.

Ahmad and Lazer [**1**] proved that the conditions

$$a_L>\frac{c_M d_M}{f_L}, \qquad d_L>\frac{e_M a_M}{b_L} \tag{1.4}$$

imply the existence of the positive periodic solutions $(\overline{u},\underline{v})$ and $(\underline{u},\overline{v})$ of the problem (1.1)–(1.2), and the sector $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ is a global attractor of the problem (1.1), (1.3) for any nonnegative nontrivial initial data.

In this paper, using the mixed quasimonotone properties in this system, we first construct a pair of upper and lower solutions, then use the periodic comparison existence theorem developed in [**1, 6, 8**] to get a pair of T-periodic solutions $(\overline{u},\underline{v})$ and $(\underline{u},\overline{v})$. Finally we note the relation between the stability of equilibrium for a dynamic system and the criterion of negative definite quadric form to give a sufficient condition of $(\overline{u},\underline{v})=(\underline{u},\overline{v})$ to answer the open question described by Ahmad and Lazer in [**1**].

2. Preliminaries.

Denote

$$\begin{aligned} f(u,v)&=u[a(x,t)-b(x,t)u-c(x,t)v],\\ g(u,v)&=v[d(x,t)-e(x,t)u-f(x,t)v]. \end{aligned}$$

Assume that (1.4) holds so that $f(u,v)$ and $g(u,v)$ have the relation shown in Figure 1. In that case it is possible to choose $a^*\in$

$[a_L, a_M], b^* \in [b_L, b_M], \ldots, f^* \in [f_L, f_M]$, so that the following inequality holds:

$$\frac{c^*}{f^*} < \frac{a^*}{d^*} < \frac{b^*}{e^*}. \tag{2.1}$$

Next, (2.1) is considered as the basic assumption in this paper. The intersection point (u^*, v^*) of the system

$$\begin{aligned} f^*(u,v) &= u[a^* - b^*u - c^*v] = 0, \\ g^*(u,v) &= v[d^* - e^*u - f^*v] = 0, \end{aligned} \tag{2.2}$$

which has positive components is determined by

$$\begin{aligned} u^* &= (a^*f^* - c^*d^*)/(b^*f^* - c^*e^*), \\ v^* &= (b^*d^* - a^*e^*)/(b^*f^* - c^*e^*). \end{aligned} \tag{2.3}$$

Remark. The condition (2.1) implies that (u^*, v^*) is a stable equilibrium of the following system

$$\frac{du}{dt} = f^*(u,v), \qquad \frac{dv}{dt} = g^*(u,v).$$

In the $(u,v)-$ plane, we consider the following lines

$$\begin{aligned} &L_1 : a_M - b_L u - c_L v = 0; \qquad L_2 : d_L - e_M u - f_M v = 0; \\ &L_3 : a_L - b_M u - c_M v = 0; \qquad L_4 : d_M - e_L u - f_L v = 0. \end{aligned}$$

$R(\overline{u}^*, \underline{v}^*)$ is the intersection point of L_1 and L_2, and $Q(\underline{u}^*, \overline{v}^*)$ is that of L_3 and L_4. Denote

$$A\left(\frac{a_M}{b_L}, 0\right), \quad B\left(\frac{d_L}{e_M}, 0\right), \quad C\left(0, \frac{a_L}{c_M}\right), \quad D\left(0, \frac{d_M}{f_L}\right)$$

in Figure 1.

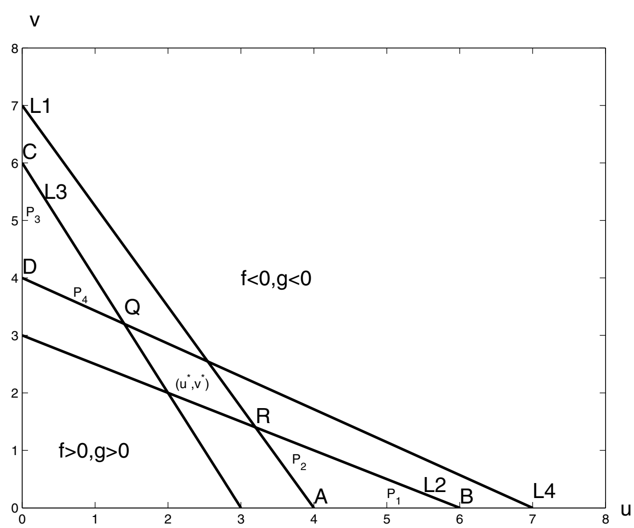

FIGURE 1. Graph of the functions $f(u,v)$ and $g(u,v)$.

3. Main result.

Theorem 3.1. *Suppose that* (1.4) *holds so that it is possible to choose* $a^*, \ldots, f^*$ *satisfying* (2.1), *and that* $a_M - a_L, b_M - b_L, \ldots, f_M - f_L$ *are sufficiently small. Then the periodic competition diffusion system* (1.1)–(1.2) *has a unique stable positive periodic solution. Moreover, this positive periodic solution is a global attractor to the problem* (1.1), (1.3) *with nonnegative nontrivial initial functions.*

Proof. Choose $a^* \in [a_L, a_M], b^* \in [b_L, b_M], \ldots, f^* \in [f_L, f_M]$, so that (2.1) holds. We first construct the upper and lower solutions. From Figure 1, if (u,v) is in the interior of the triangle ΔRAB, then $f(u,v) < 0,\ g(u,v) > 0$. Similarly, if (u,v) is in the interior of the triangle ΔQCD, then $f(u,v) > 0,\ g(u,v) < 0$. We choose $P_1(u_1,v_1), P_2(u_2,v_2)$ in ΔRAB such that P_1 is close to the point B and P_2 close to R; however, $u_1 \geq u_2,\ v_1 \leq v_2$. Set

$$\widetilde{u}(t) = u_1 + (u_2 - u_1)(1 - e^{-\varepsilon t}), \qquad \widehat{v}(t) = v_1 + (v_2 - v_1)(1 - e^{-\varepsilon t}).$$

It is obvious that $(\widetilde{u}(t), \widehat{v}(t))$ is on the line P_1P_2 as $t \in [0, +\infty)$ and

$(\widetilde{u}(0), \widehat{v}(0)) = (u_1, v_1)$, $\lim_{t\to+\infty}(\widetilde{u}(t), \widehat{v}(t)) = (u_2, v_2)$. Because the line P_1P_2 is included strictly in the interior of ΔRAB, there is a constant $\delta > 0$ such that

$$f(\widetilde{u}(t), \widehat{v}(t)) \le -\delta < 0, \qquad g(\widetilde{u}(t), \widehat{v}(t)) \ge \delta > 0. \tag{3.1}$$

Noting that

$$\begin{aligned} \widetilde{u}_t - \Delta\widetilde{u} &= \varepsilon(u_2 - u_1)e^{-\varepsilon t} \ge -\varepsilon(u_1 - u_2), \\ \widehat{v}_t - \Delta\widehat{v} &= \varepsilon(v_2 - v_1)e^{-\varepsilon t} \le \varepsilon(v_2 - v_1). \end{aligned} \tag{3.2}$$

Choosing

$$\varepsilon \le \min\left\{\frac{\delta}{u_1 - u_2}, \frac{\delta}{v_2 - v_1}\right\},$$

then we have

$$\widetilde{u}_t - \Delta\widetilde{u} \ge f(\widetilde{u}, \widehat{v}), \qquad \widehat{v}_t - \Delta\widehat{v} \le g(\widetilde{u}, \widehat{v}). \tag{3.3}$$

Similarly, we set

$$\begin{aligned} \widehat{u}(t) &= u_3 + (u_4 - u_3)(1 - e^{-\varepsilon_1 t}), \\ \widetilde{v}(t) &= v_3 + (v_4 - v_3)(1 - e^{-\varepsilon_1 t}), \end{aligned} \tag{3.4}$$

where $P_3(u_3, v_3), P_4(u_4, v_4)$ in the interior of ΔQCD and P_3, P_4 are close to the points C and Q, respectively. It is obvious that $u_3 \ge u_4$, $v_3 \le v_4$. Therefore, the differential inequalities

$$\widehat{u}_t - \Delta\widehat{u} \le f(\widehat{u}, \widetilde{v}), \quad \widetilde{v}_t - \Delta\widetilde{v} \ge g(\widehat{u}, \widetilde{v}), \tag{3.5}$$

hold for suitable $\varepsilon_1 > 0$.

According to the definition in [**6**], the pair $(\widetilde{u}, \widetilde{v})$ and $(\widehat{u}, \widehat{v})$ becomes a suitable T-upper and lower solutions for the periodic boundary value problem (1.1)–(1.2), therefore, there are two periodic solutions $(\overline{u}(x,t), \underline{v}(x,t))$ and $(\underline{u}(x,t), \overline{v}(x,t))$ to the problem (1.1)–(1.2) by the monotone periodic convergence theorem developed in [**1, 6, 8**], and

$$\widehat{u}(t) \le \underline{u}(x,t) \le \overline{u}(x,t) \le \widetilde{u}(t), \qquad \widehat{v}(t) \le \underline{v}(x,t) \le \overline{v}(x,t) \le \widetilde{v}(t). \tag{3.6}$$

Using these upper and lower solutions we can control the asymptotic behavior of periodic solutions. By a simple argument, the expressions of $\widetilde{u}(t), \widehat{u}(t), \widetilde{v}(t), \widehat{v}(t)$ imply that

$$u_4 \le \underline{u}(x,t) \le \overline{u}(x,t) \le u_2, \qquad v_2 \le \underline{v}(x,t) \le \overline{v}(x,t) \le v_4. \tag{3.7}$$

Next we may choose u_2 to be close to the u coordinate of R as we want; then we can prove that $\underline{u}(x,t) = \overline{u}(x,t)$, $\underline{v}(x,t) = \overline{v}(x,t)$. By the direct calculations and the mean-value theorem, we get

$$\begin{aligned}(\overline{u}-\underline{u})_t - \Delta(\overline{u}-\underline{u}) &= f(\overline{u},\underline{v}) - f(\underline{u},\overline{v}) \\ &= f_u(\xi,\underline{v})(\overline{u}-\underline{u}) - f_v(\underline{u},\eta)(\overline{v}-\underline{v}),\end{aligned} \tag{3.8}$$

$$\begin{aligned}(\overline{v}-\underline{v})_t - \Delta(\overline{v}-\underline{v}) &= g(\underline{u},\overline{v}) - g(\overline{u},\underline{v}) \\ &= -g_u(\xi_1,\overline{v})(\overline{u}-\underline{u}) + g_v(\overline{u},\eta_1)(\overline{v}-\underline{v}),\end{aligned} \tag{3.9}$$

where

$$\underline{u} < \xi, \quad \xi_1 < \overline{u}, \qquad \underline{v} < \eta, \quad \eta_1 < \overline{v}. \tag{3.10}$$

From the periodicity of $(\overline{u}(x,t), \underline{v}(x,t))$ and $(\underline{u}(x,t), \overline{v}(x,t))$ in t, we have

$$\begin{aligned}&\int_0^T \int_\Omega [|\nabla(\overline{u}-\underline{u})|^2 + |\nabla(\overline{v}-\underline{v})|^2]\,dx\,dt \\ &\quad = \int_0^T \int_\Omega [f_u(\xi,\underline{v})(\overline{u}-\underline{u})^2 - (f_v(\underline{u},\eta) + g_u(\xi_1,\overline{v}))(\overline{u}-\underline{u})(\overline{v}-\underline{v}) \\ &\qquad\qquad + g_v(\overline{u},\eta_1)(\overline{v}-\underline{v})^2]\,dx\,dt.\end{aligned} \tag{3.11}$$

Note that the integrand of the right-hand side is a quadratic form whose corresponding matrix is

$$\begin{bmatrix} f_u & -(f_v+g_u)/2 \\ -(f_v+g_u)/2 & g_v \end{bmatrix}.$$

If this matrix is evaluated at $(u,v) = (u^*,v^*)$, it follows this matrix is negative definite. In fact

$$\begin{gathered} f_u^*(u^*,v^*) = -b^*u^*, \ f_v^*(u^*,v^*) = -c^*u^*, \ g_u^*(u^*,v^*) = -e^*v^*, \\ g_v^*(u^*,v^*) = -f^*v^*, \\ (f_u^*g_v^* - f_v^*g_u^*)\,|_{(u^*,v^*)} = (b^*f^* - c^*e^*)u^*v^*, \end{gathered}$$

the condition (2.1) implies that

$$f_u^*(u^*,v^*) = -b^*u^* < 0, \qquad g_v^*(u^*,v^*) = -f^*v^* < 0,$$
$$(f_u^*g_v^* - f_v^*g_u^*)\,|_{(u^*,v^*)} = (b^*f^* - c^*e^*)u^*v^* > 0.$$

By means of the continuity of the function $f_u(u,v), f_v(u,v), g_u(u,v)$, $g_v(u,v)$ and the Jacobian determinant $J(u,v) = f_u g_v - f_v g_u$, there certainly exists a sufficient small neighborhood K of the point (u^*,v^*) such that for any

$$(u,v) \quad \text{and} \quad (u_i,v_i) \in K \quad \text{for} \quad i = 1,2,3,$$
$$f_u(u,v) < 0, \qquad g_v(u,v) < 0,$$
$$f_u(u,v)g_v(u_1,v_1) - f_v(u_2,v_2)g_u(u_3,v_3) > 0.$$

Now, according to the conditions that $a_M - a_L, b_M - b_L, \dots, f_M - f_L$ are sufficiently small, we have $(u_2,v_2) \approx (u^*,v^*)$ and $(u_4,v_4) \approx (u^*,v^*)$; then (3.7) and (3.10) imply that

$$(\xi,\underline{v}),(\underline{u},\eta),(\xi_1,\overline{v}),(\overline{u},\eta_1) \in K,$$

so we have

$$f_u(\xi,\underline{v}) < 0, \qquad g_v(\overline{u},\eta_1) < 0,$$
$$f_u(\xi,\underline{v})g_v(\overline{u},\eta_1) - f_v(\underline{u},\eta) + g_u(\xi_1,\overline{v}) > 0.$$

Therefore, the integrand in the right-hand side of (3.11) is a negative quadratic form by the Hurwitz criterion. We know that the left-hand side of (3.11) is nonnegative, so we have, $\overline{u}(x,t) = \underline{u}(x,t)$, $\overline{v}(x,t) = \underline{v}(x,t)$. That is, $(\overline{u},\underline{v}) = (\underline{u},\overline{v})$ is the unique positive periodic solution of the problem (1.1)–(1.2).

Finally, we discuss the attraction of the periodic solution in relation to the nonnegative solutions of (1.1), (1.3). By the same argument as that in [**1**, Theorem 4.1, p. 279], details are omitted. If (u,v) is a solution of the initial boundary value problem (1.1), (1.3), then given $\varepsilon > 0$ there exists $t_1 = t_1(\varepsilon) > 0$ such that if $t \geq t_1$,

$$\underline{u}(x,t) - \varepsilon < u(x,t) < \overline{u}(x,t) + \varepsilon; \qquad \underline{v}(x,t) - \varepsilon < v(x,t) < \overline{v}(x,t) + \varepsilon$$

for all $x \in \overline{\Omega}$, where $\varphi, \psi \in C^1(\overline{\Omega})$, $\varphi(x) \geq 0$, $\psi(x) \geq 0$, $\varphi(x) \not\equiv 0$, $\psi(x) \not\equiv 0$ and

$$\frac{\partial \varphi}{\partial n}\,|_{\partial\Omega} = \frac{\partial \psi}{\partial n}\,|_{\partial\Omega} = 0.$$

According to the previous step, we have

$$\overline{u}(x,t) - \varepsilon < u(x,t) < \overline{u}(x,t) + \varepsilon; \qquad \overline{v}(x,t) - \varepsilon < v(x,t) < \overline{v}(x,t) + \varepsilon.$$

This means that $(\overline{u}, \underline{v}) = (\underline{u}, \overline{v})$ is a global attractor of the problem (1.1), (1.3) for the nonnegative nontrivial initial functions. This completes the proof. □

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